- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Decidability

Decidability

Definition

A Turing machine is a tuple $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ where:

- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $F \subseteq Q$ is the set of final states,
- Σ is a finite alphabet,
- $\square \in \Sigma$ is the blank symbol,
- $\delta : (Q \setminus F) \times \Sigma \to Q \times \Sigma \times \{L, R\}$ is the transition function.

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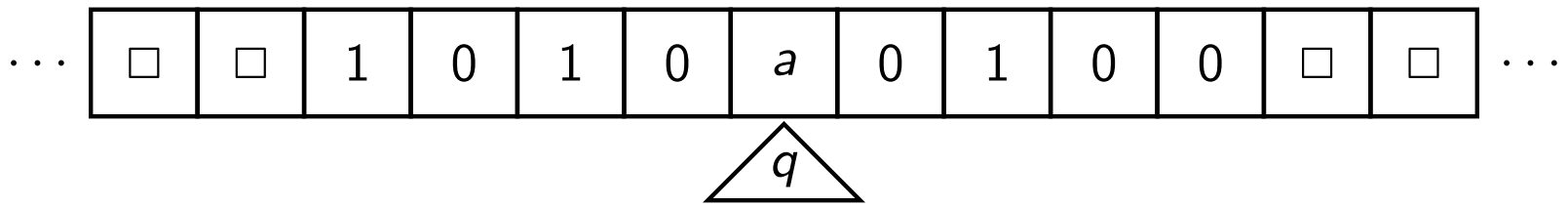
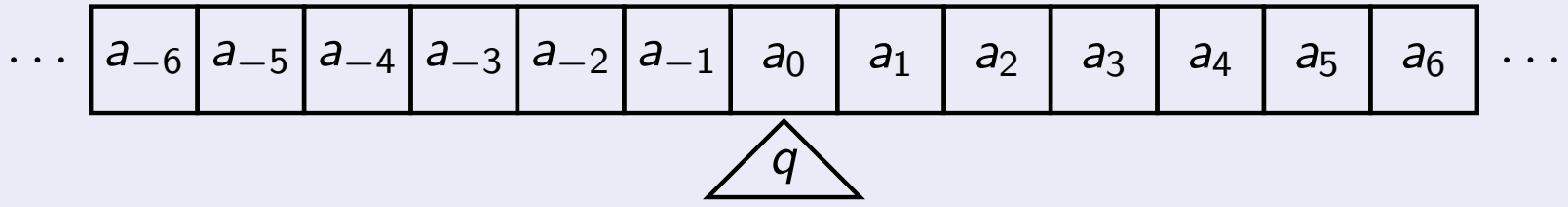
Turing machines work on a two-sided infinite tape with one read/write head:

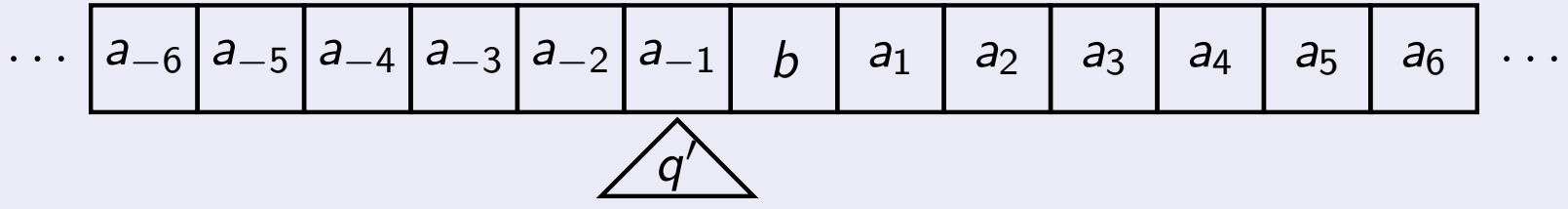


Definition (State Transition)

The current state:



If $\delta(q, a_0) = (b, q', L)$, then the following state is:



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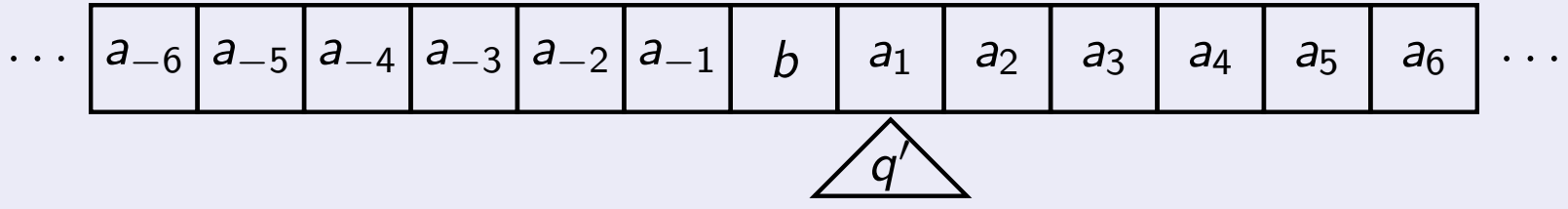
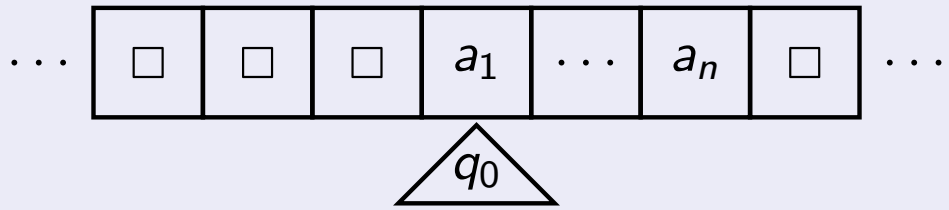


Definition

A Turing machine $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ is said to halt on $w = a_1 a_2 \dots a_n \in \Sigma^*$ if it reaches a final state when started on the configuration:



The tape content upon reaching the final state is called output.

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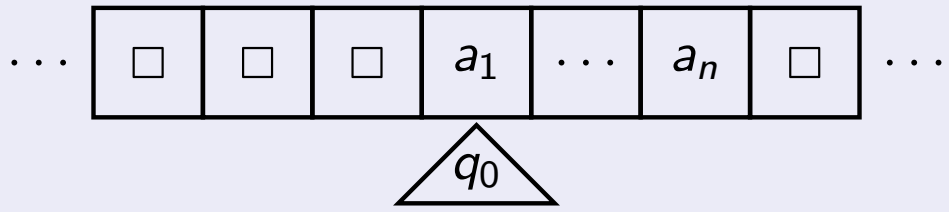

The tape content upon reaching the final state is called output.

Definition (Decidable Problems)

A predicate $P \subseteq \mathbb{N}^d$ is decidable if there is a Turing machine M such that:

- M terminates on all inputs $s^{n_1}0 \cdots s^{n_d}0$ for $\langle n_1, \dots, n_d \rangle \in \mathbb{N}^d$, and
- M halts with output 0 if and only if $\langle n_1, \dots, n_d \rangle \in P$.

Problem (Halting Problem)

Input: Turing machine M, word $w \in \Sigma^*$
Question: Does M halt on w?

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Problem (Uniform Halting Problem)

Input: *Turing machine M*
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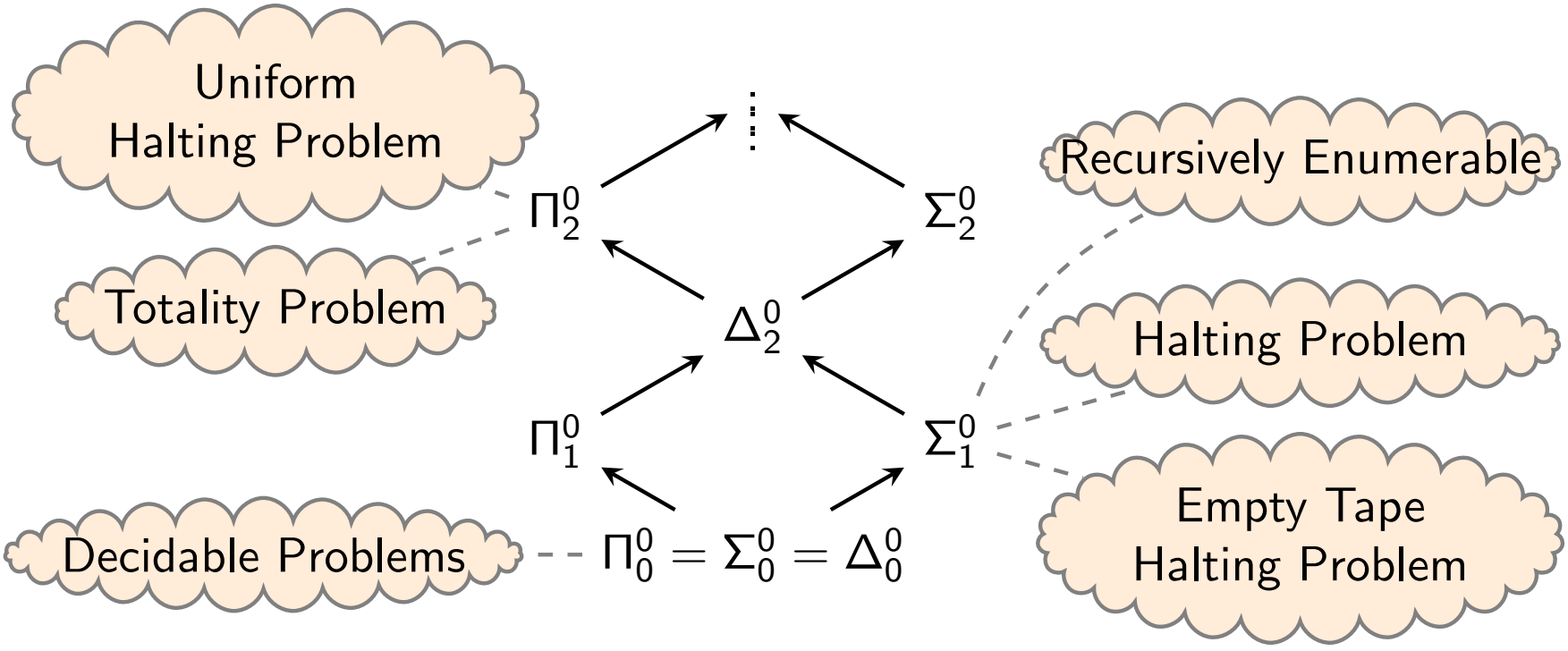
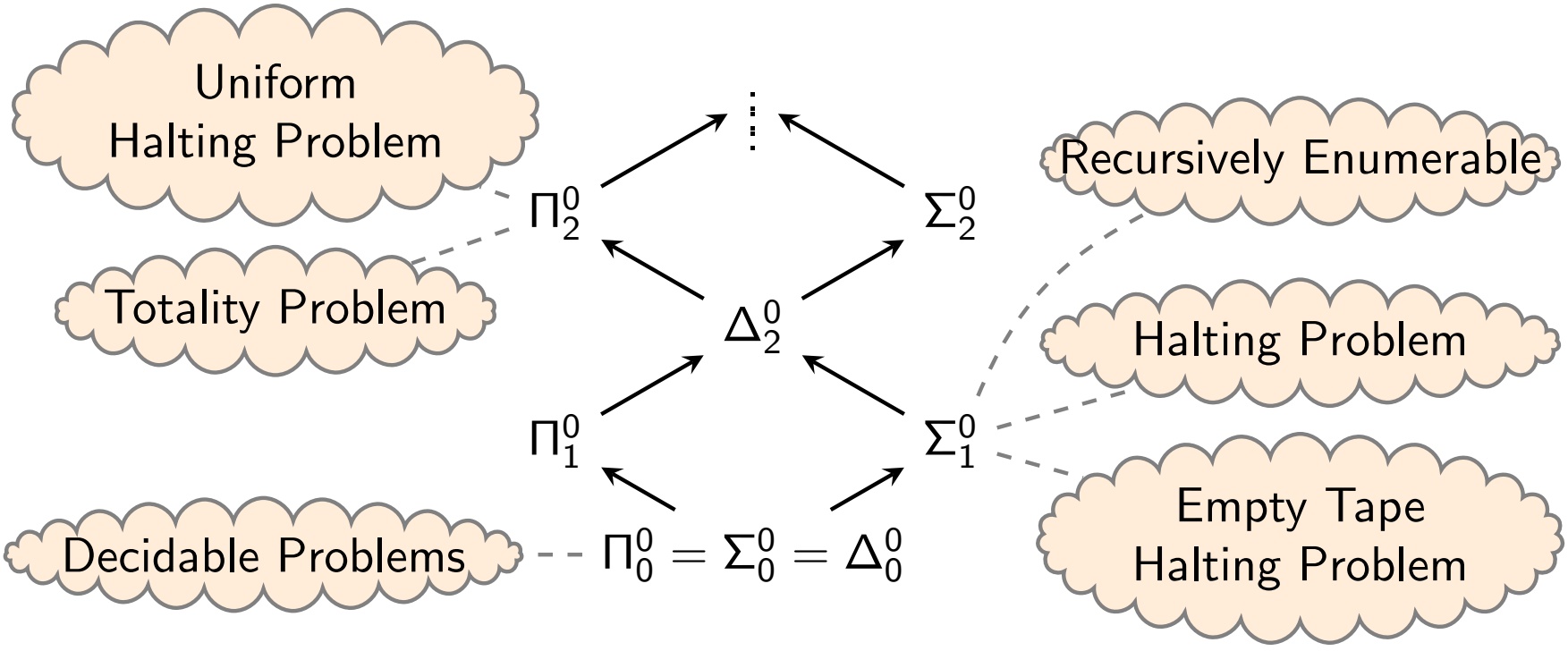
Problem (Totality Problem)

Input: *Turing machine M*
Question: *Does M halt on all natural numbers $\{s^n 0 \mid n \in \mathbb{N}\}$?*


Uniform
Halting Problem
Totality Problem
Decidable Problems
Π_2^0
Σ_2^0
Δ_2^0
Π_1^0
Σ_1^0
$\Pi_0^0 = \Sigma_0^0 = \Delta_0^0$
Recursively Enumerable
Halting Problem
Empty Tape
Halting Problem

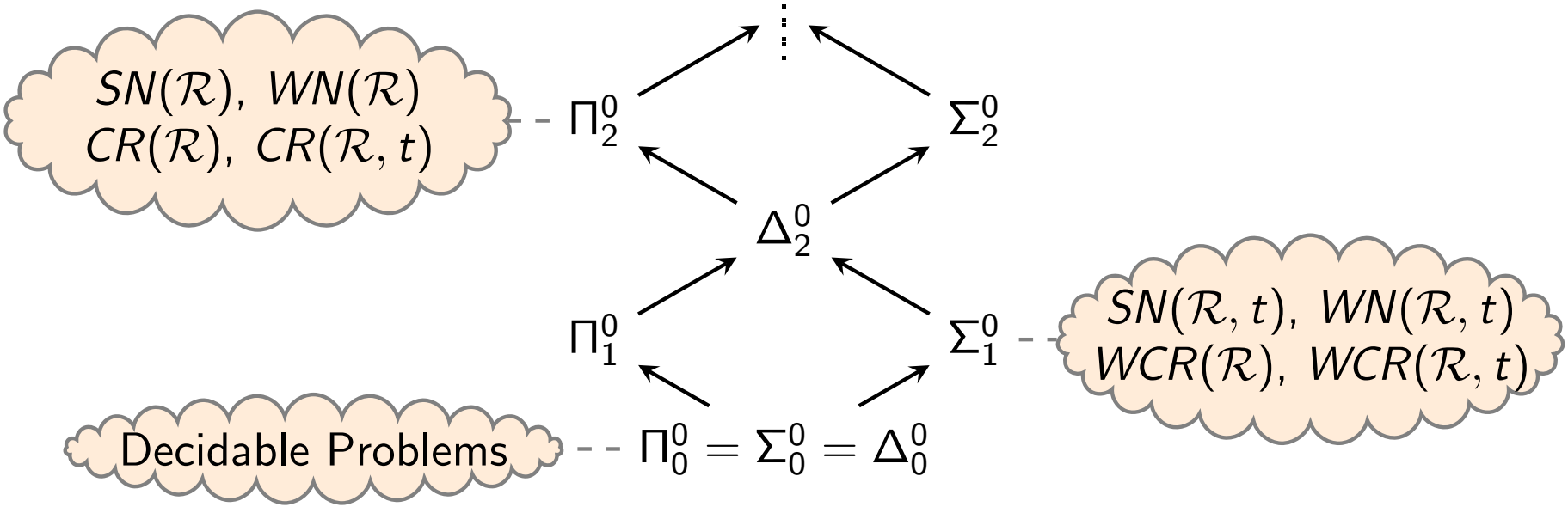


Definition

A problem $A \subseteq \mathbb{N}^d$ is:

- in Σ_1^0 if $\vec{x} \in A \iff \exists y.\ P(\vec{x}, y)$,
- in Π_2^0 if $\vec{x} \in A \iff \forall y.\ \exists z.\ P(\vec{x}, y, z)$,

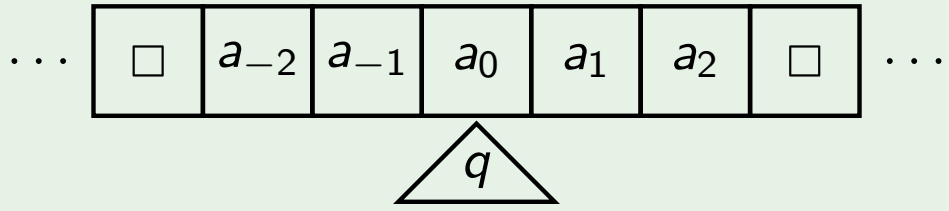
for some decidable predicate P.



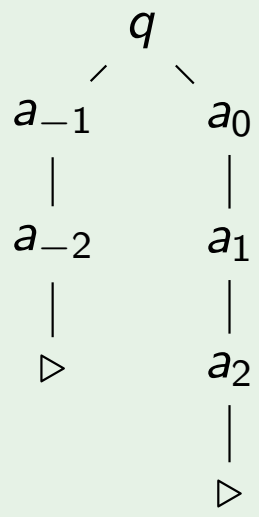
Theorem

- *$SN(\mathcal{R})$ and $WN(\mathcal{R})$ are Π_2^0-complete.*
- *$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ_1^0-complete.*
- *$WCR(\mathcal{R})$ and $WCR(\mathcal{R}, t)$ are Σ_1^0-complete.*
- *$CR(\mathcal{R})$ and $CR(\mathcal{R}, t)$ are Π_2^0-complete.*

Example (From Turing Machine Configurations to Terms)



becomes $q(a_{-1}(a_{-2}(\triangleright)), a_0(a_1(a_2(\triangleright))))$:



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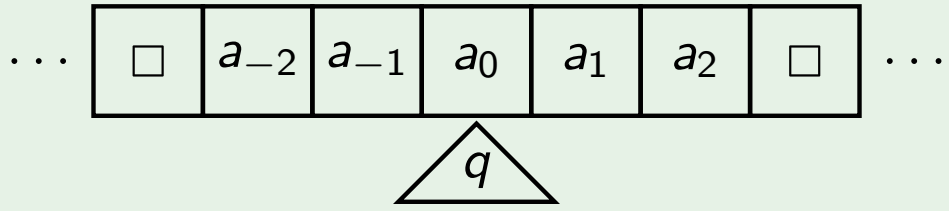

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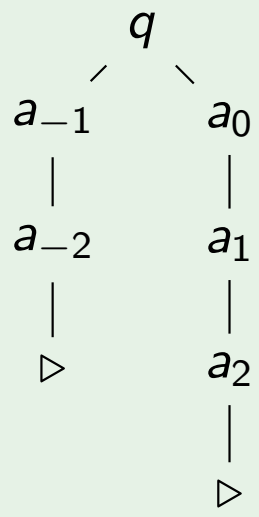

$$q(x, f(y)) \to q'(f'(x), y) \qquad \text{for every } \delta(q, f) = \langle q', f', R\rangle$$
$$q(g(x), f(y)) \to q'(x, g(f'(y))) \qquad \text{for every } \delta(q, f) = \langle q', f', L\rangle$$

Definition (From Turing Machines to Term Rewriting Systems)

For a Turing machine $M = \langle Q, q_0, F, \Sigma, \square, \delta \rangle$ we define a TRS $\mathcal{R}_M$ as follows:

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together with four rules for 'extending the tape':

$$q(\triangleright, f(y)) \to q'(\triangleright, \square(f'(y))) \qquad \text{for every } \delta(q, f) = \langle q', f', L \rangle$$
$$q(x, \triangleright) \to q'(f'(x), \triangleright) \qquad \text{for every } \delta(q, \square) = \langle q', f', R \rangle$$
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Theorem

A Turing machine M halts on all configurations if and only if $\mathcal{R}_M$ terminates.

Complexity of (Weak) Normalization

As a consequence we obtain:

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$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ_1^0-complete.

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The term $t = q_0(\triangleright, \triangleright)$ is normalizing in $\mathcal{R}_M \iff M$ halts on the blank tape.
Moreover $SN(t) \iff WN(t)$ since every reduct contains at most one redex. ■

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For a Turing machine M we define H_M to be $\mathcal{R}_M$ extended with:

$$q(x, f(y)) \to T \qquad \text{for every } f \in \Sigma, q \in Q \text{ with } \delta(q, f) \text{ is undefined}$$

$$q(x, \triangleright) \to T \qquad \text{for every } q \in Q \text{ with } \delta(q, \square) \text{ is undefined}$$

Then $\langle T, q_0(\triangleright, \triangleright)\rangle$ and $\langle q_0(\triangleright, \triangleright), T\rangle$ are the only critical pairs.

Hence $WCR(\mathcal{R})$ and $WCR(\mathcal{R}, S)$ hold

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Hence $CR(\mathcal{R}, S(0(\triangleright)))$ holds

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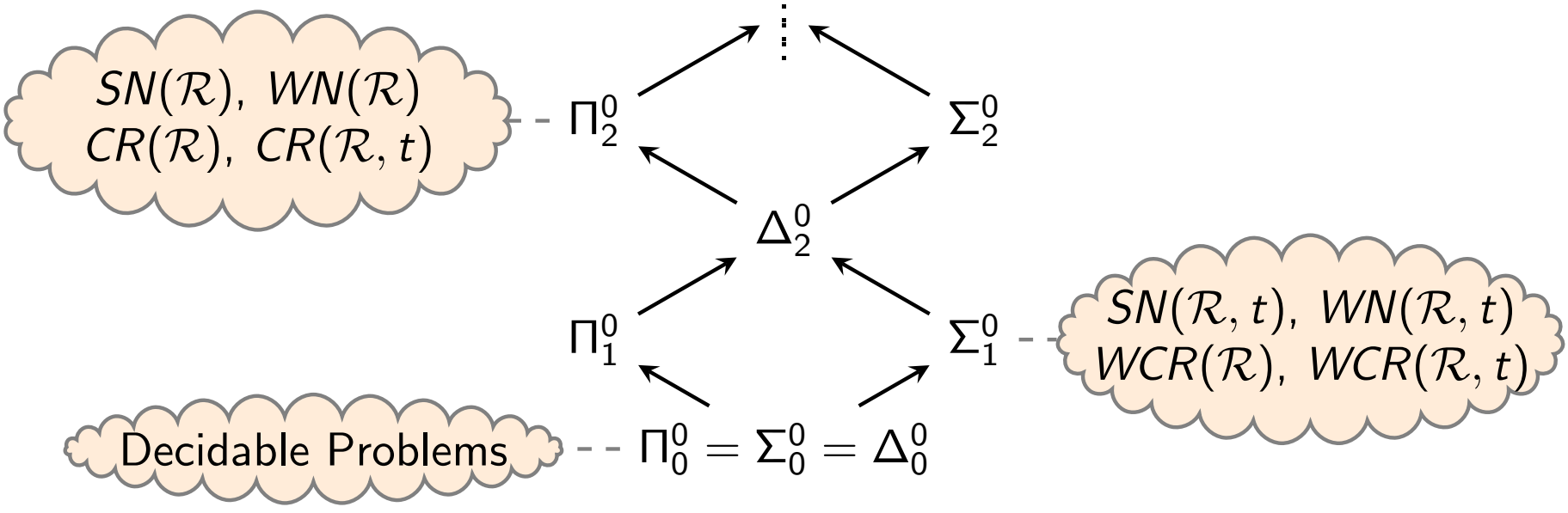
$\iff q_0(\triangleright, s^n(\triangleright)) \to^* T$ for all $n \in \mathbb{N}$

$\iff M$ is total.



Theorem

- *$SN(\mathcal{R})$ and $WN(\mathcal{R})$ are Π_2^0-complete.*
- *$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ_1^0-complete.*
- *$WCR(\mathcal{R})$ and $WCR(\mathcal{R}, t)$ are Σ_1^0-complete.*
- *$CR(\mathcal{R})$ and $CR(\mathcal{R}, t)$ are Π_2^0-complete.*