- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Local Confluence
- Orthogonality
- Beyond Orthogonality

Confluence

Local Confluence

Critical Pair Lemma

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

We decide WCR as follows:

For every $s \leftarrow \rtimes \rightarrow t \in \mathrm{CP}(\mathcal{R})$ we check $s \downarrow t$ by computing:

- the set of reducts of s: $\rightarrow(s) = \{s' \mid s \rightarrow^* s'\}$, and
- the set of reducts of t: $\rightarrow(t) = \{t' \mid t \rightarrow^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Then $s \downarrow t$ if and only if $\rightarrow(s) \cap \rightarrow(t) \neq \emptyset$.

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Assume that there exists a non-convergent critical pair $s \leftarrow \rtimes \rightarrow t$.

Then $s \leftarrow \cdot \rightarrow t$, and not $s \downarrow t$. This would contradict WCR. ∎

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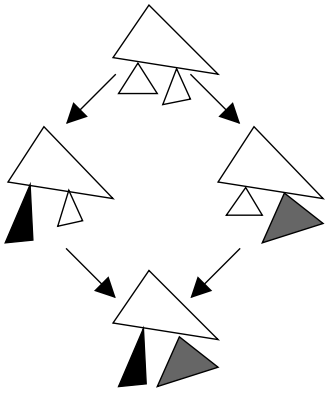

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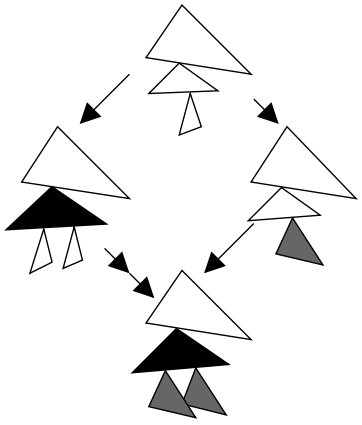

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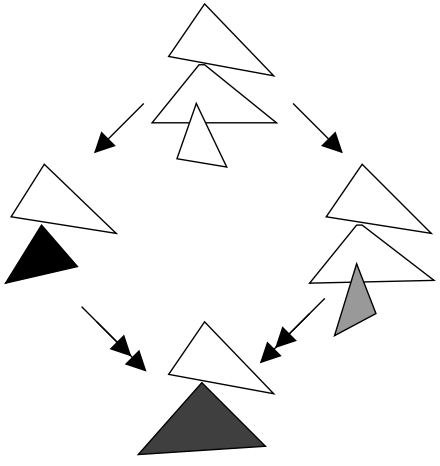

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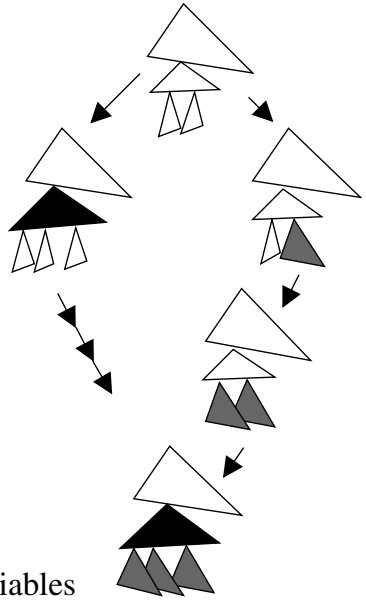

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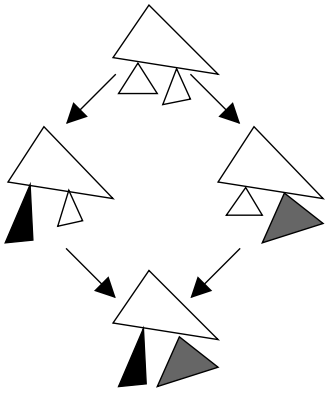

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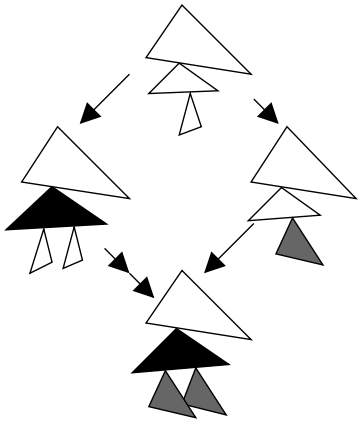

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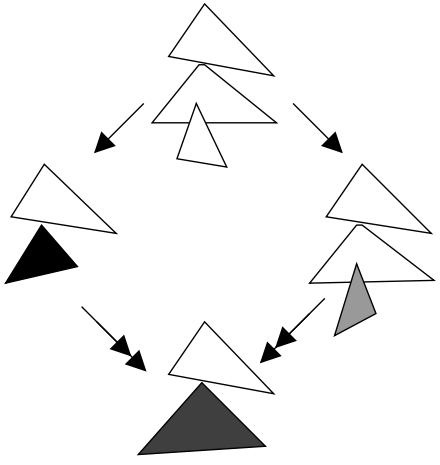

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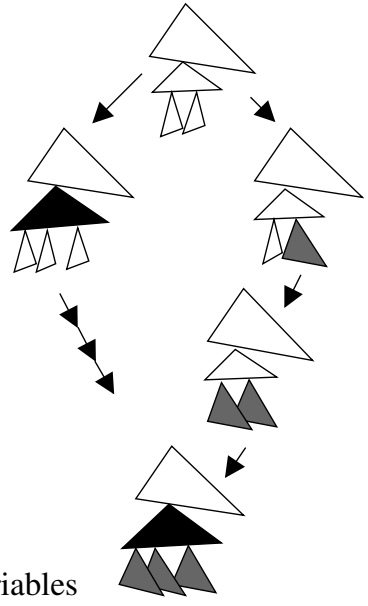

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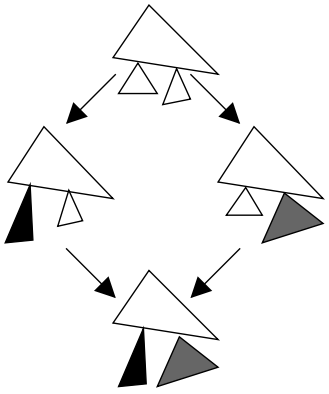

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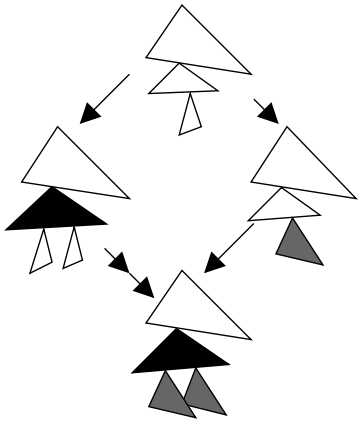

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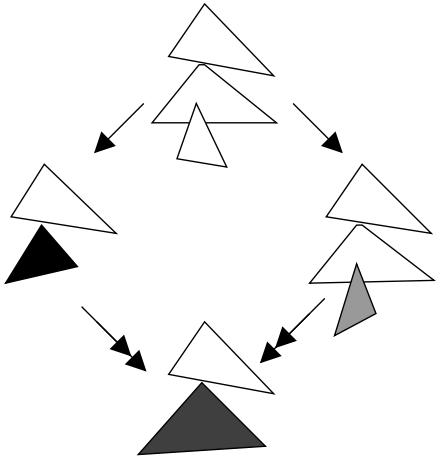

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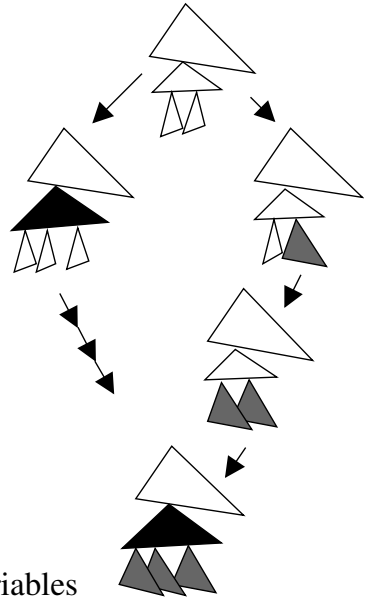

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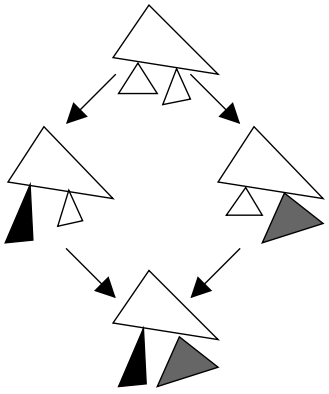

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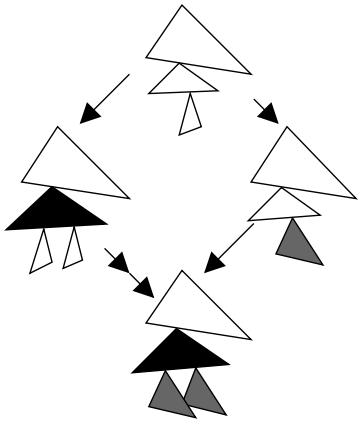

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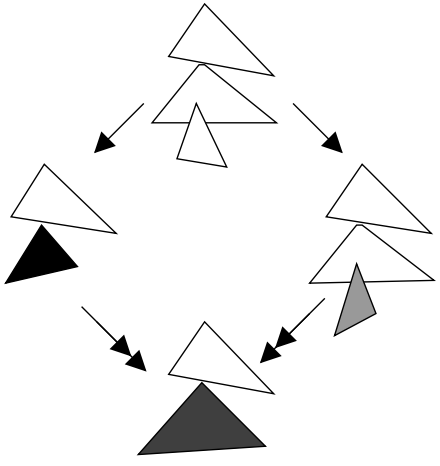

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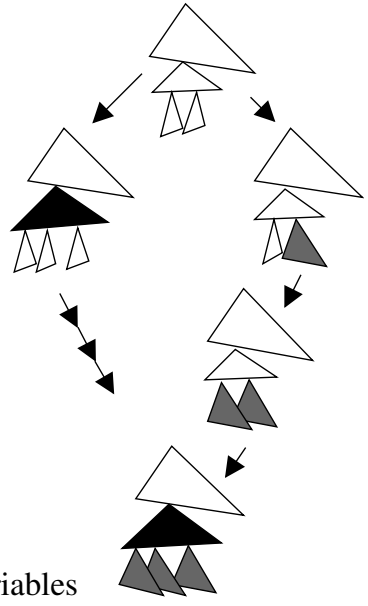

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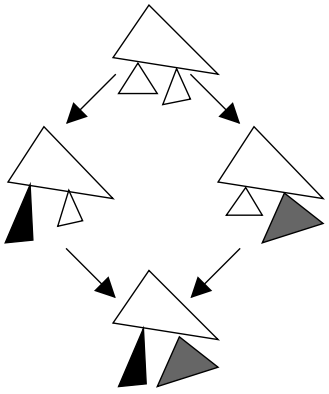

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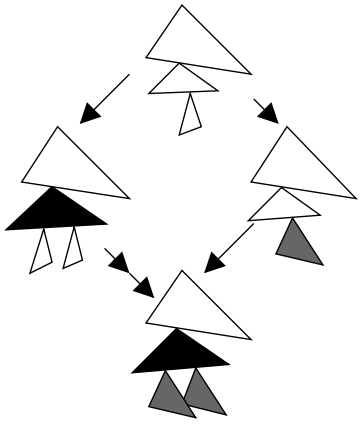

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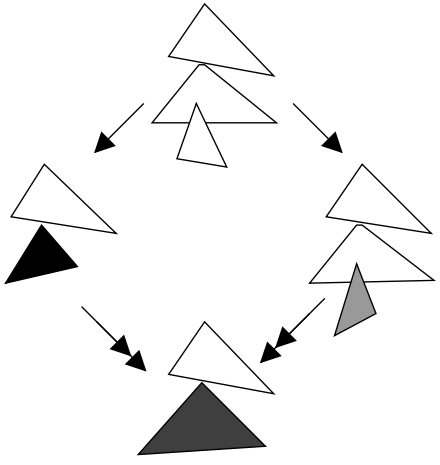

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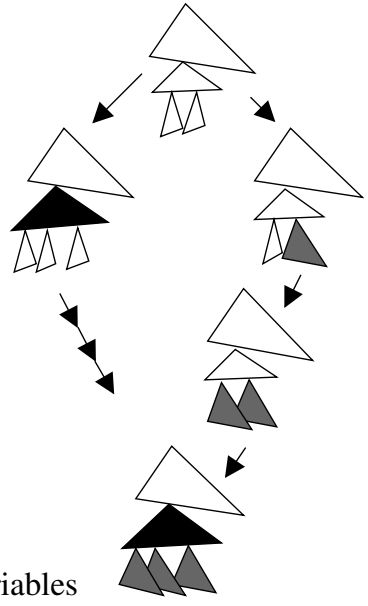

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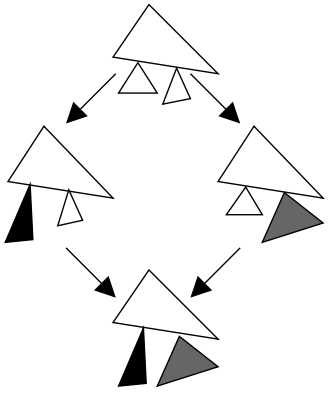

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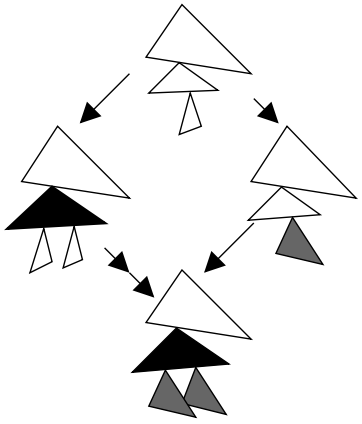

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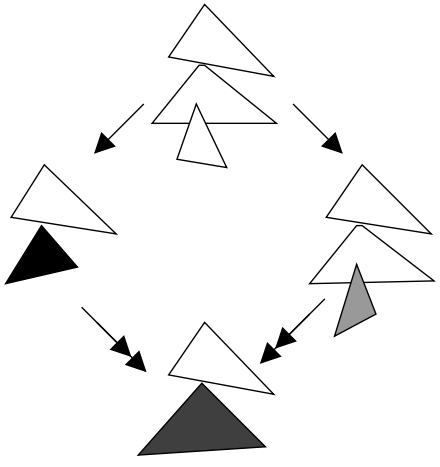

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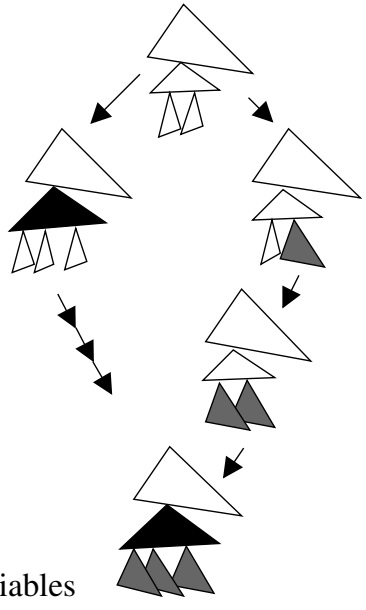

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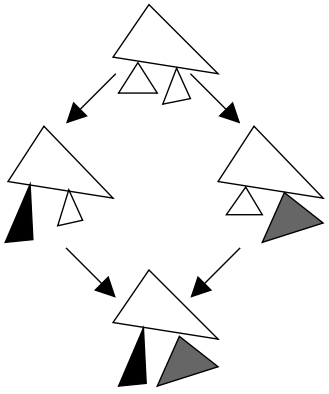

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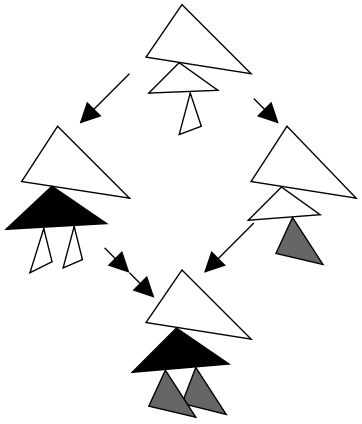

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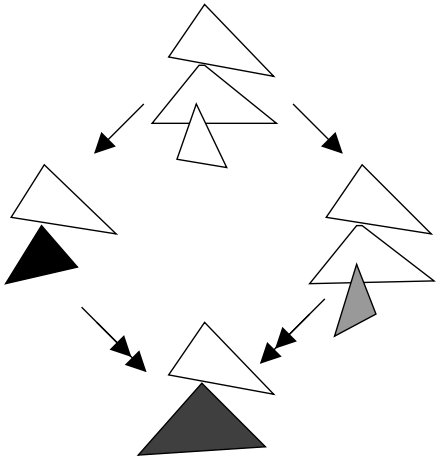

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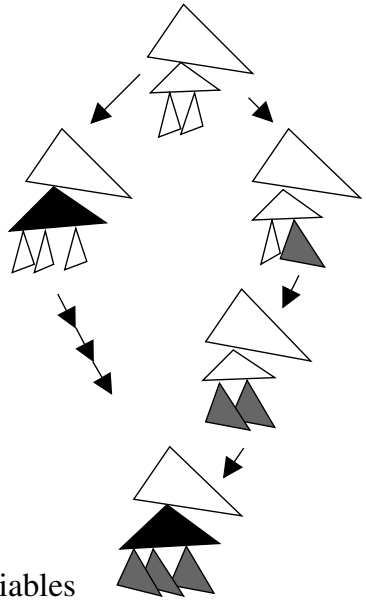

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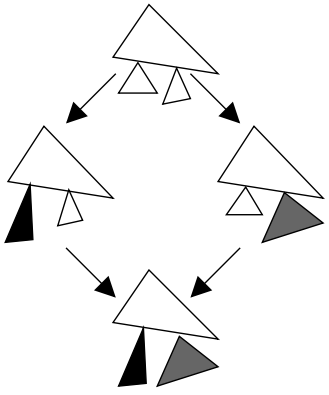

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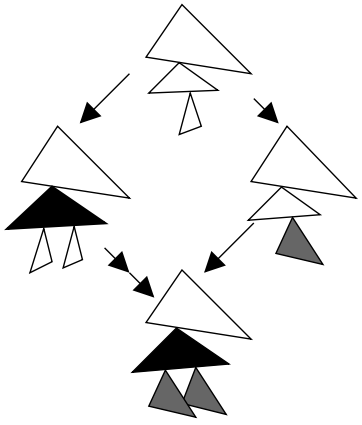

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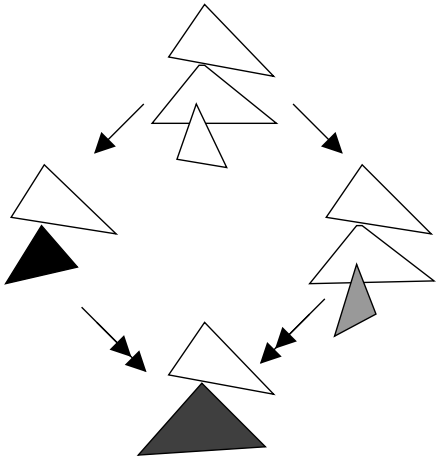

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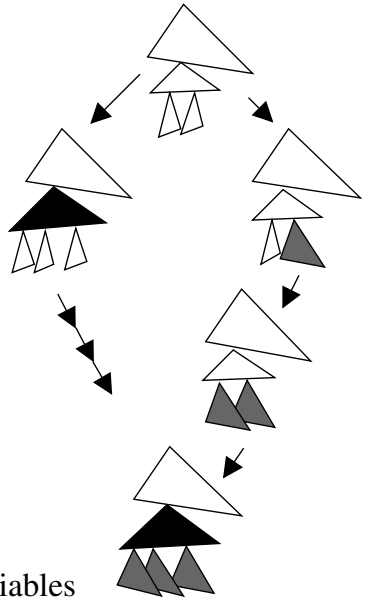

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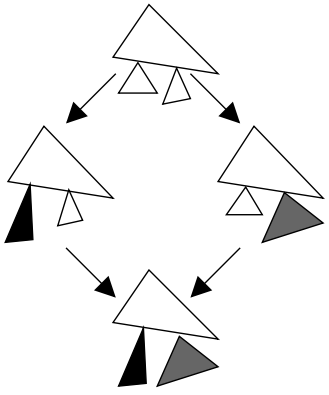

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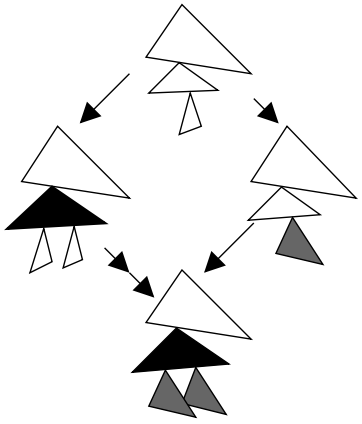

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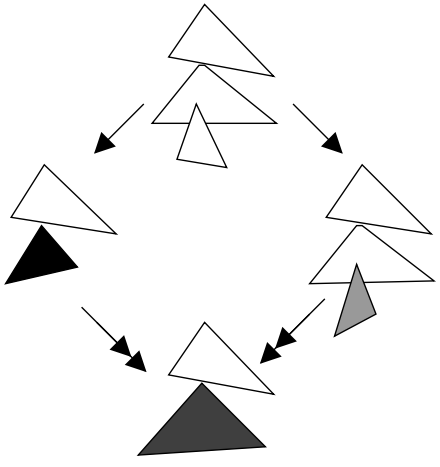

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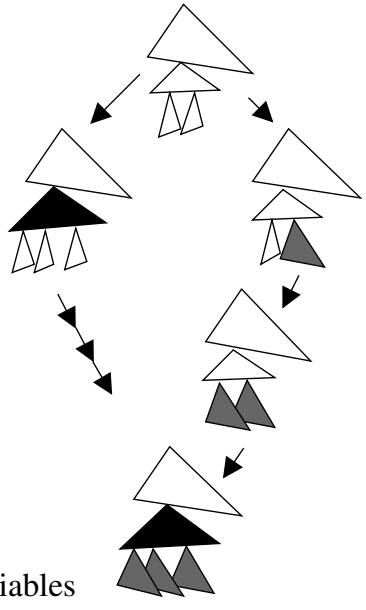

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Let $C = t[\Box]_{p_0}$, then for some σ:

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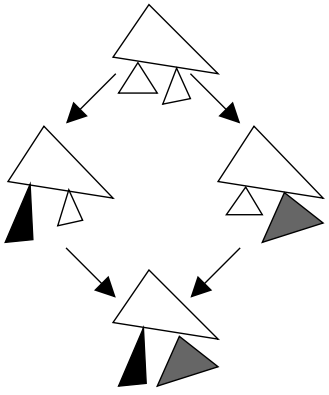

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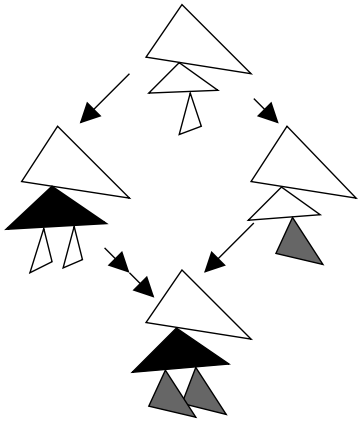

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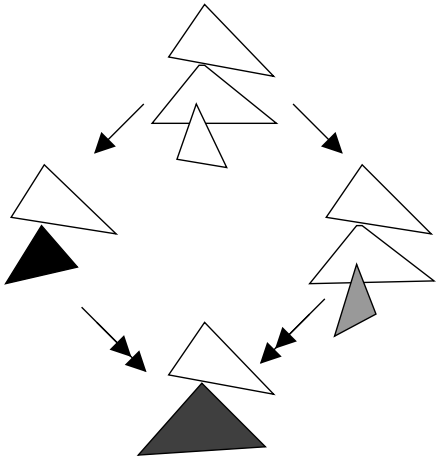

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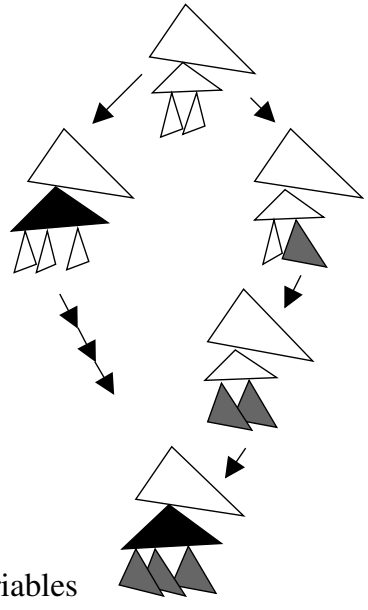

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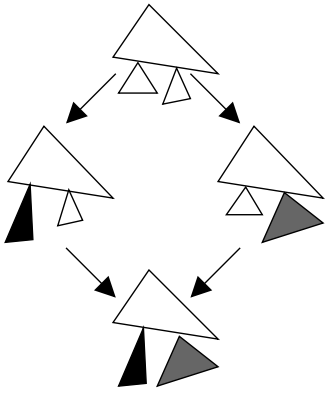

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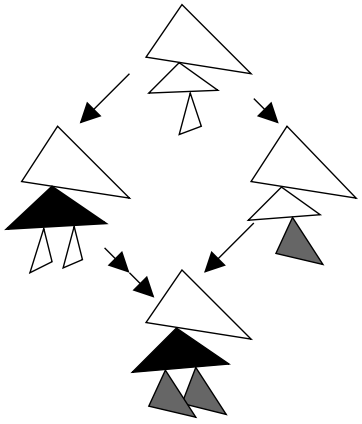

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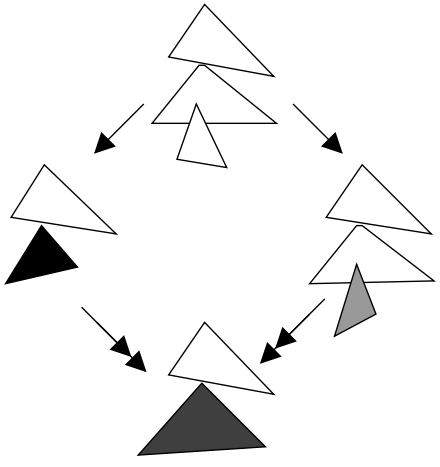

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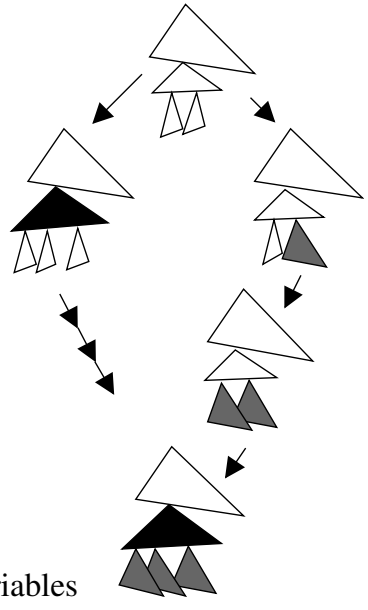

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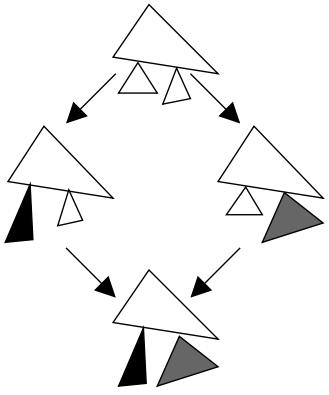

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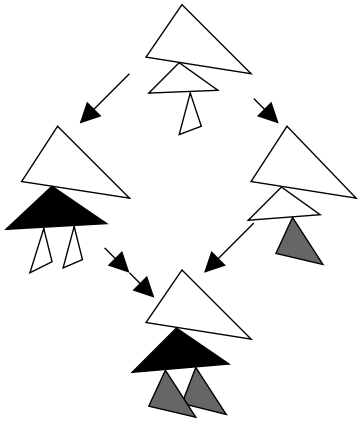

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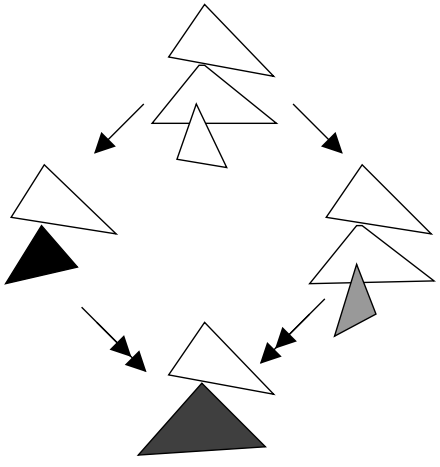

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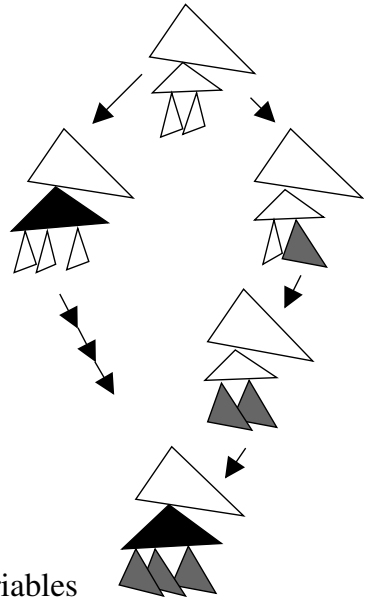

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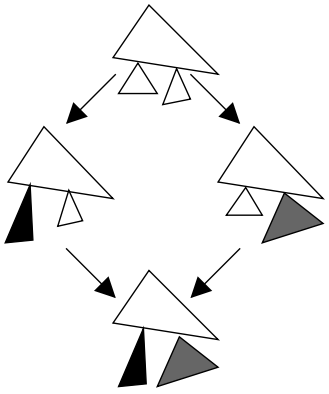

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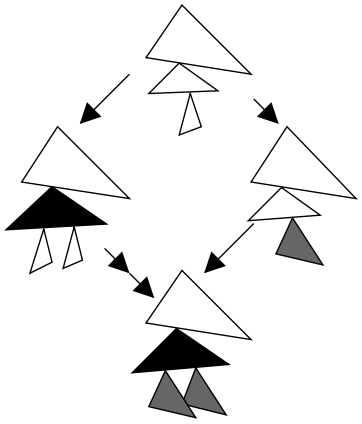

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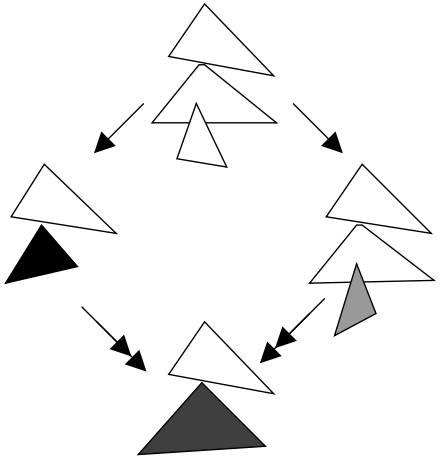

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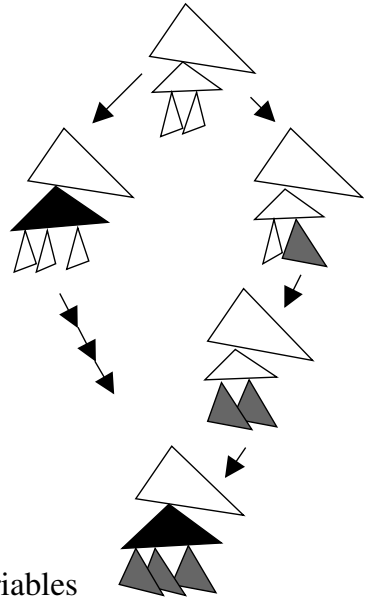

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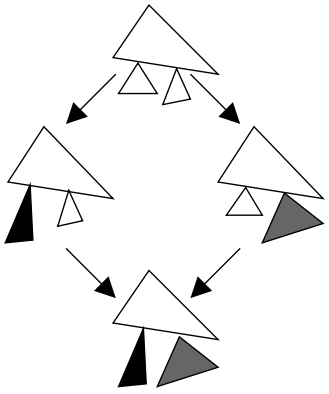

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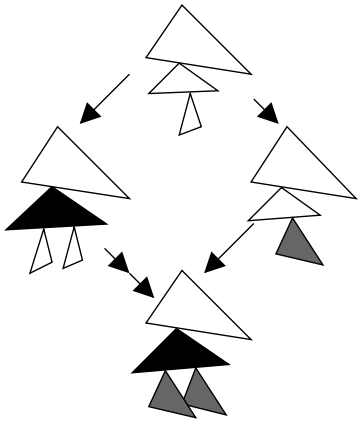

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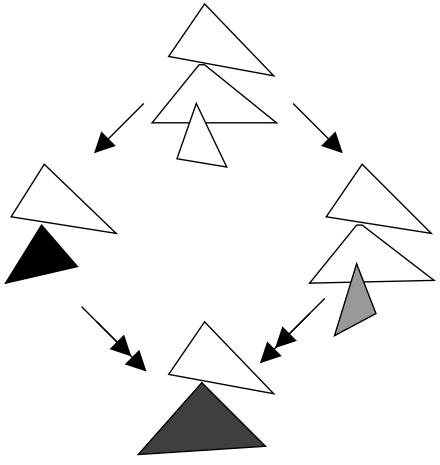

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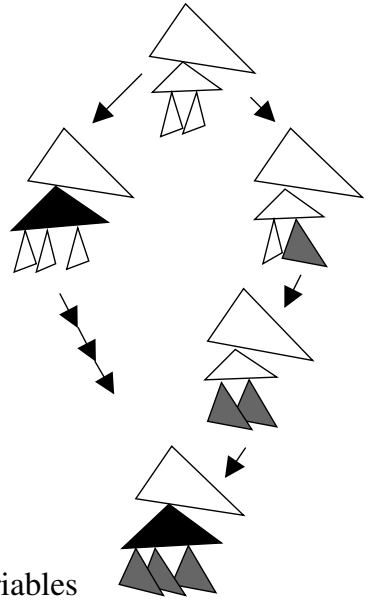

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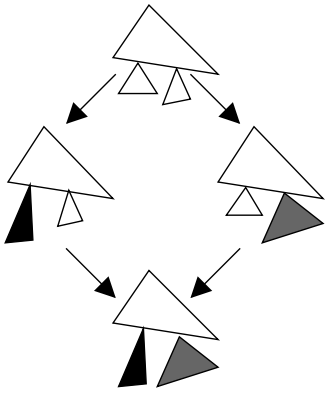

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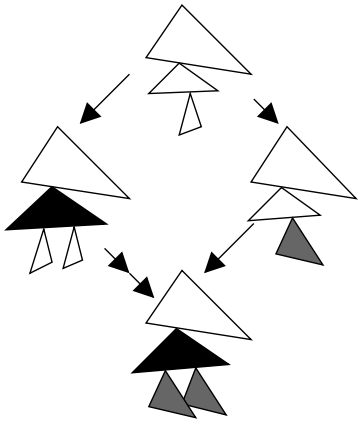

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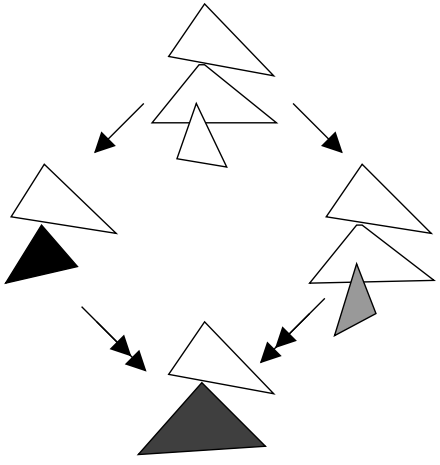

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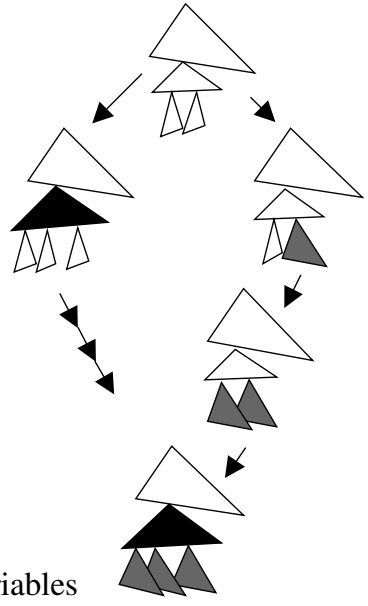

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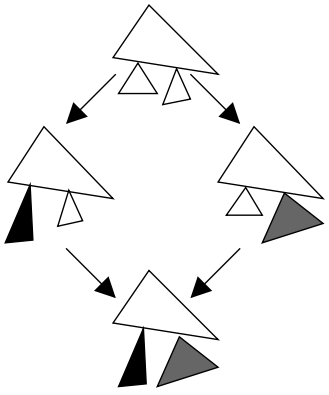

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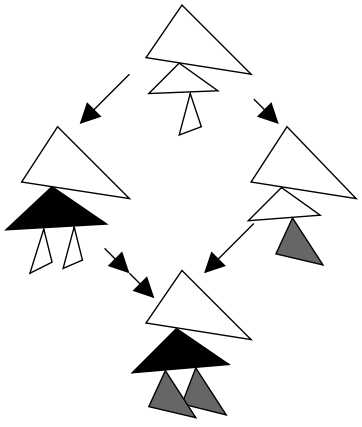

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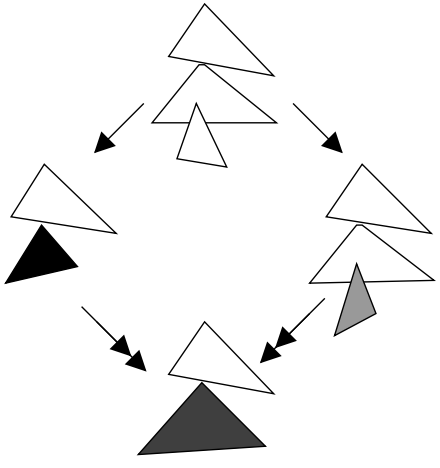

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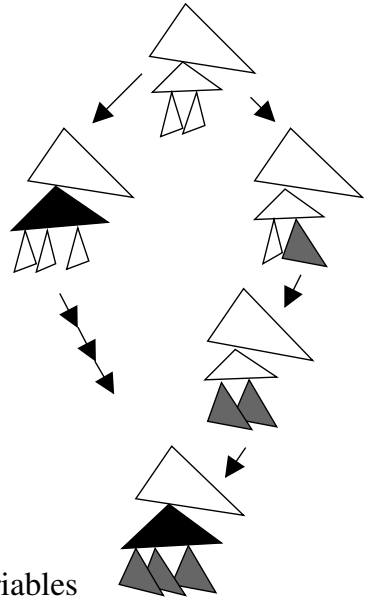

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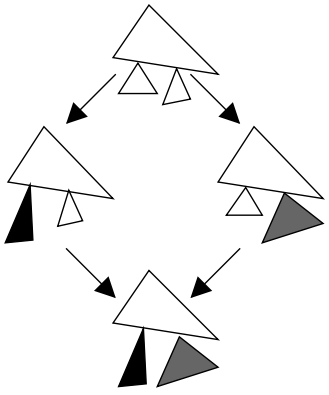

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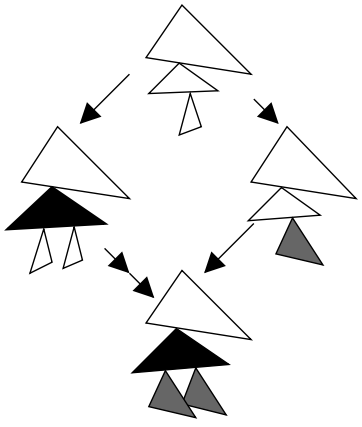

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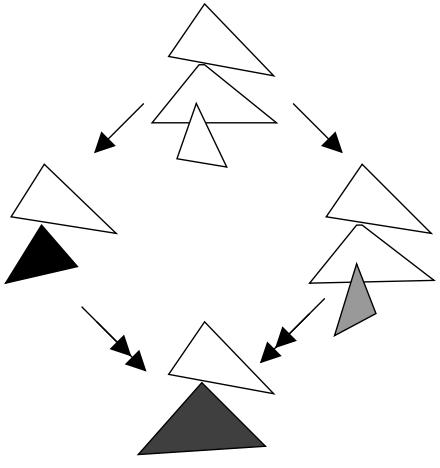

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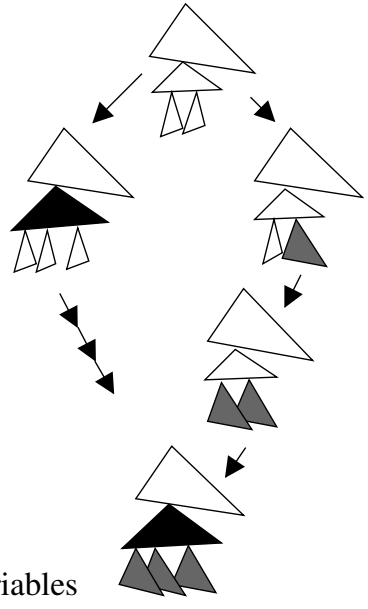

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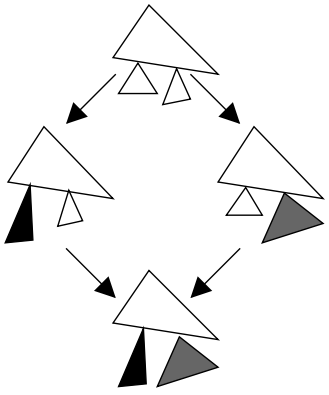

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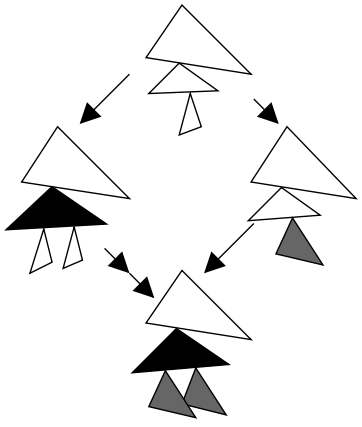

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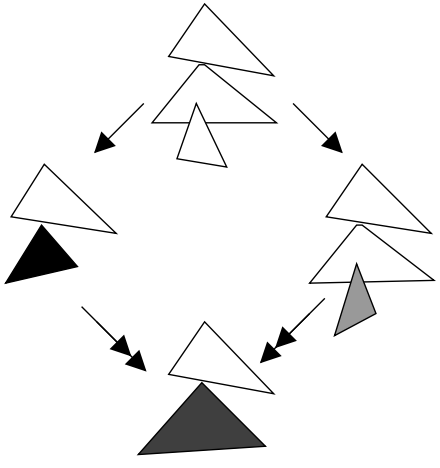

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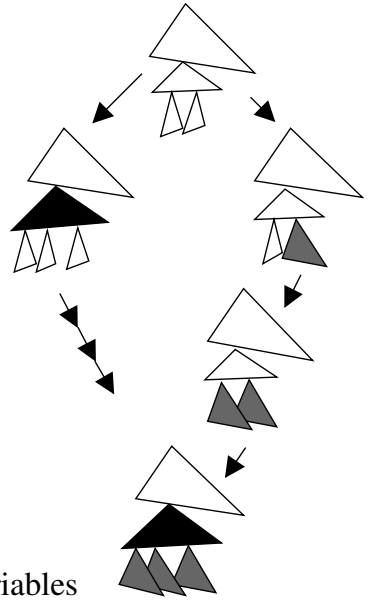

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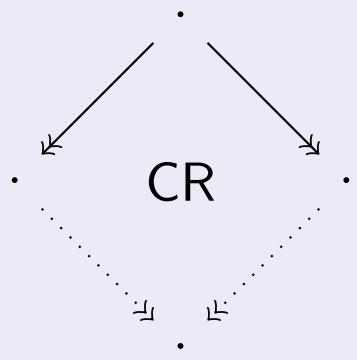
every two coinitial rewrite sequences can be joined



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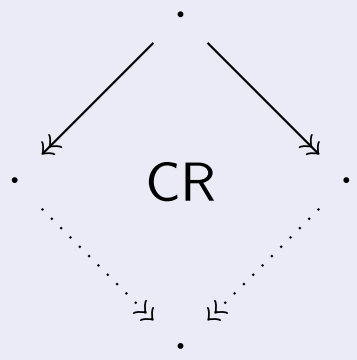

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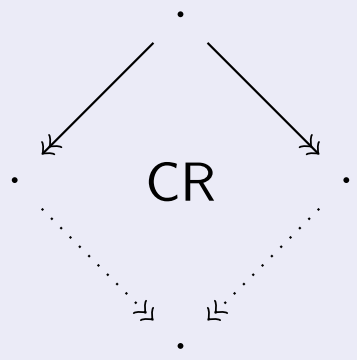

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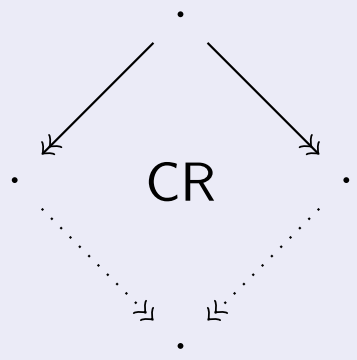

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- . . . what about nonterminating TRSs ?

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$$c \to g(c)$$

- no critical pairs but no confluence (Huet 1980)

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Observe from preceding examples:

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$$\mathsf{I} \cdot x \to x$$

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Examples

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$$(\mathsf{K} \cdot x) \cdot y \to x$$

$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

$$\mathsf{ack}(0, y) \to \mathsf{s}(y)$$

$$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$

$$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

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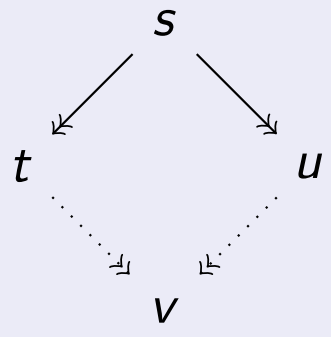
$\forall s, t, u$



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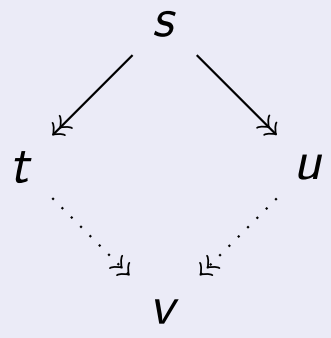

$\exists v$

For orthogonal TRSs there is a canonical way to compute common reduct...

Outline

- Overview
- Local Confluence
- Orthogonality
  - Definitions
  - Descendants
  - Parallel Moves Lemma
- Beyond Orthogonality

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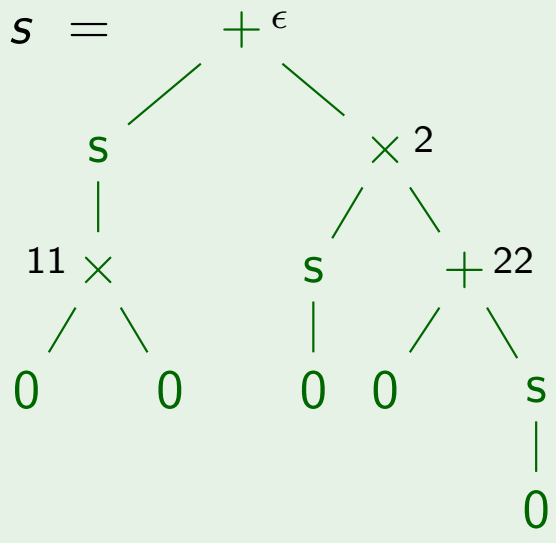
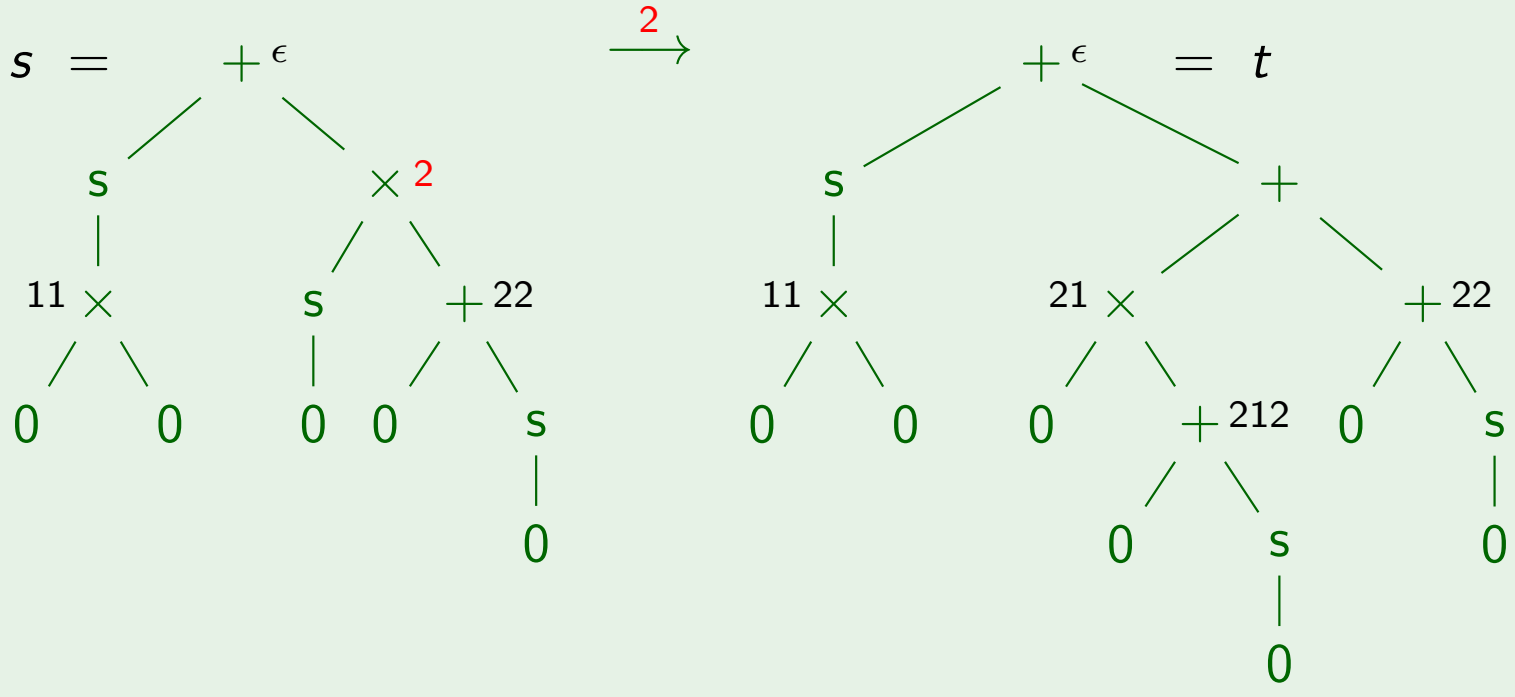
$0 + y \to y \quad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad 0 \times y \to 0 \quad \mathsf{s}(x) \times y \to (x \times y) + y$



redex positions in s: $\epsilon \mid 11 \mid 2 \mid 22 \mid$

Example

$0 + y \to y \quad s(x) + y \to s(x + y) \quad 0 \times y \to 0 \quad s(x) \times y \to (x \times y) + y$



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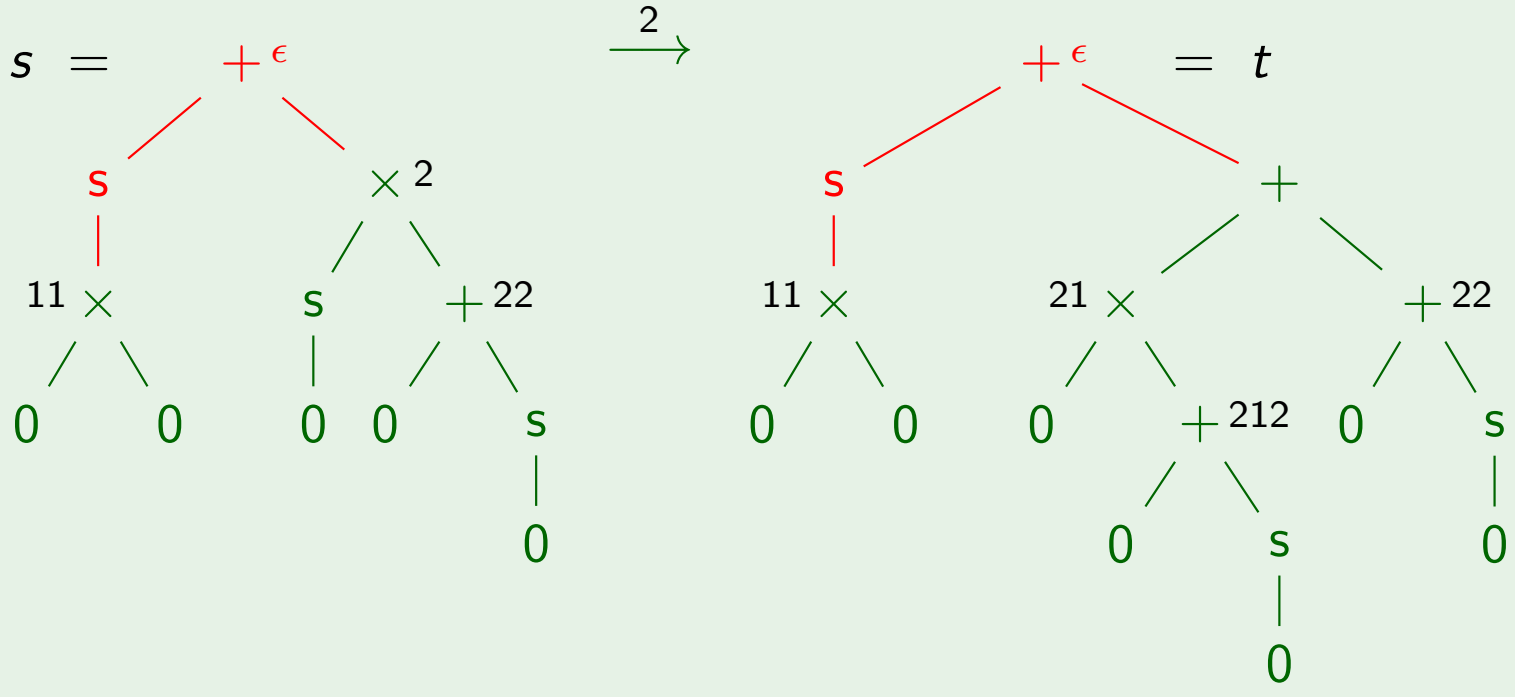

redex positions in s:	ϵ	11	2	22	
redex positions in t:	ϵ				

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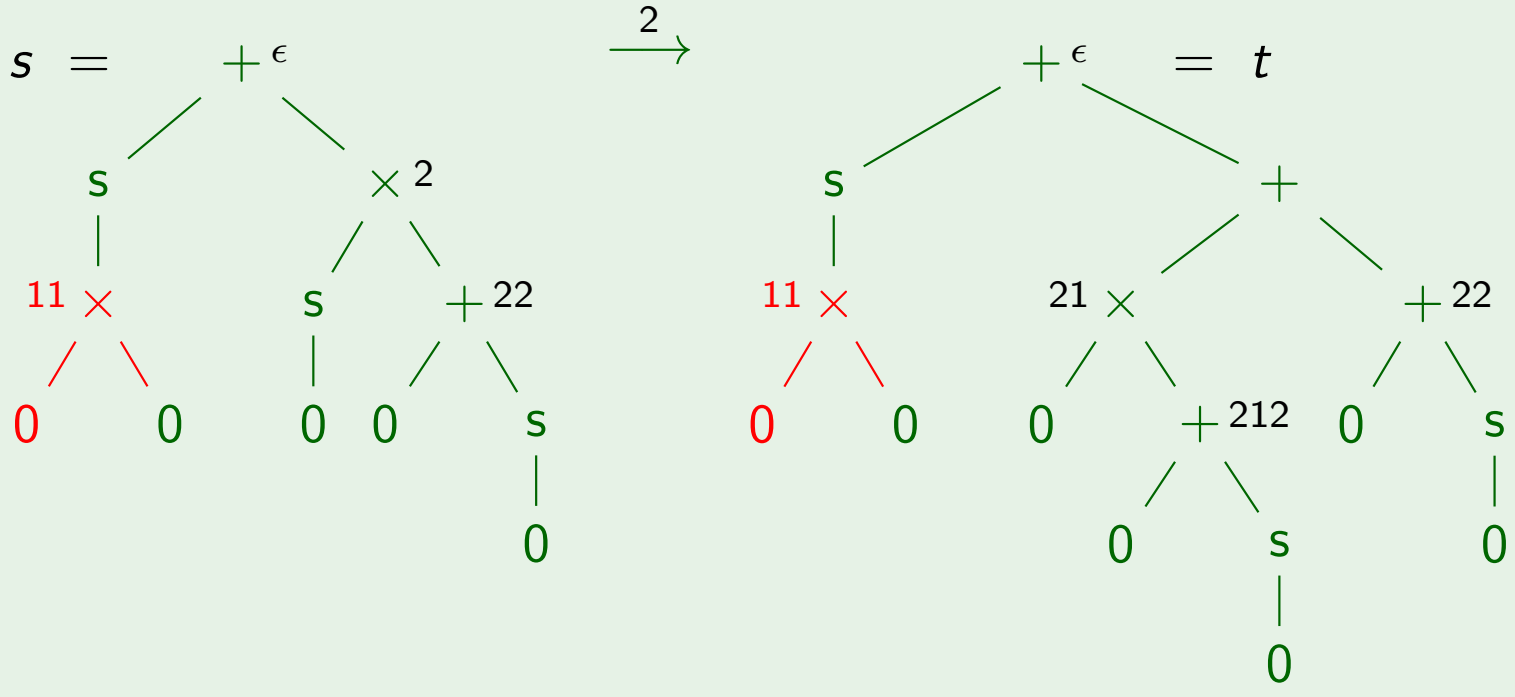

redex positions in s: $\epsilon \mid 11 \mid 2 \mid 22 \mid$

redex positions in t: $\epsilon \mid 11 \mid \mid$

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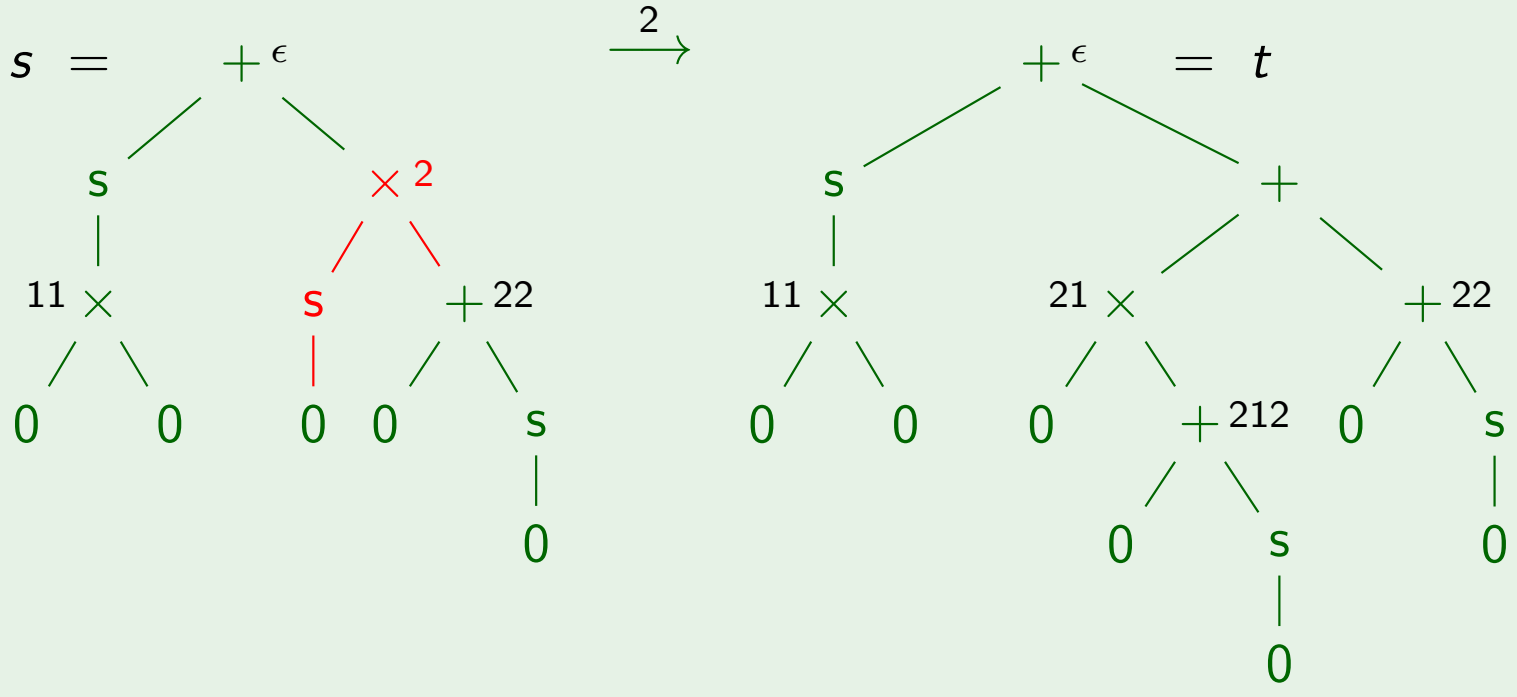

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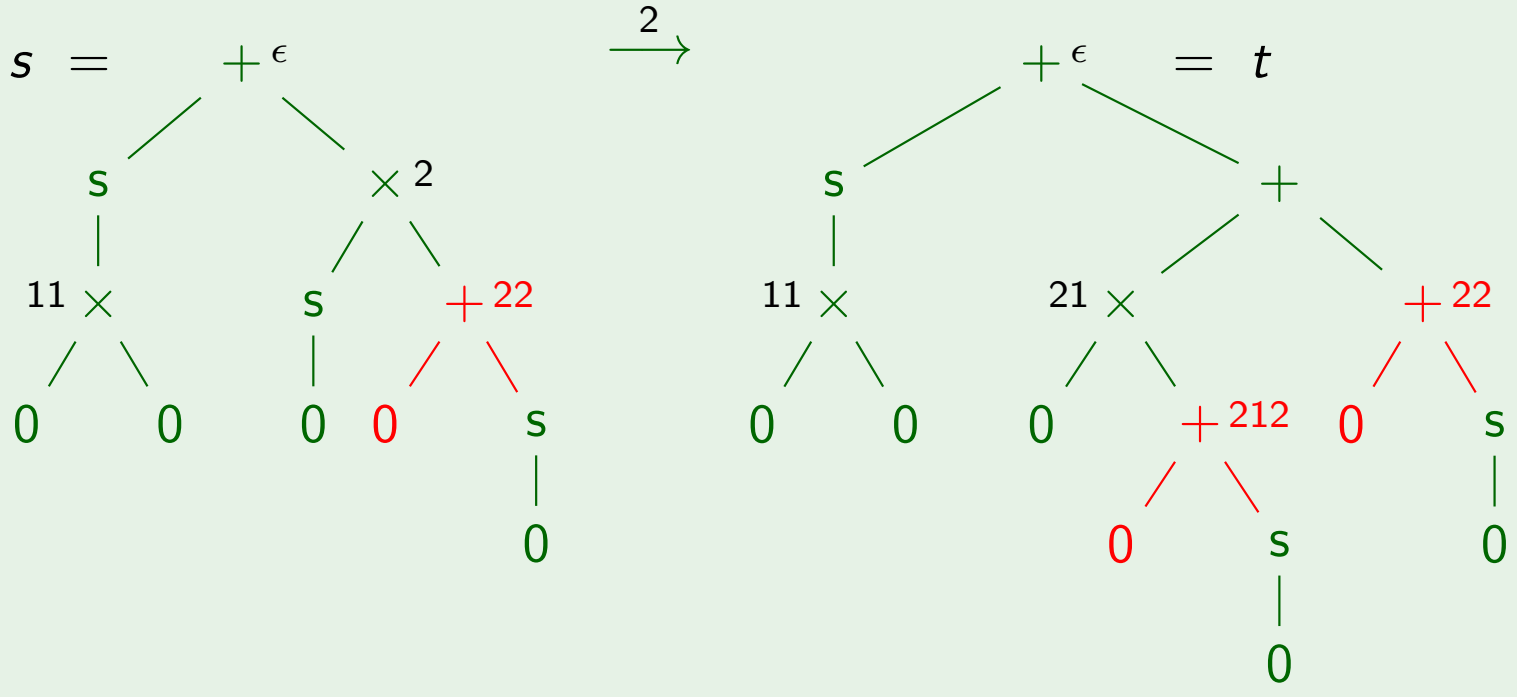


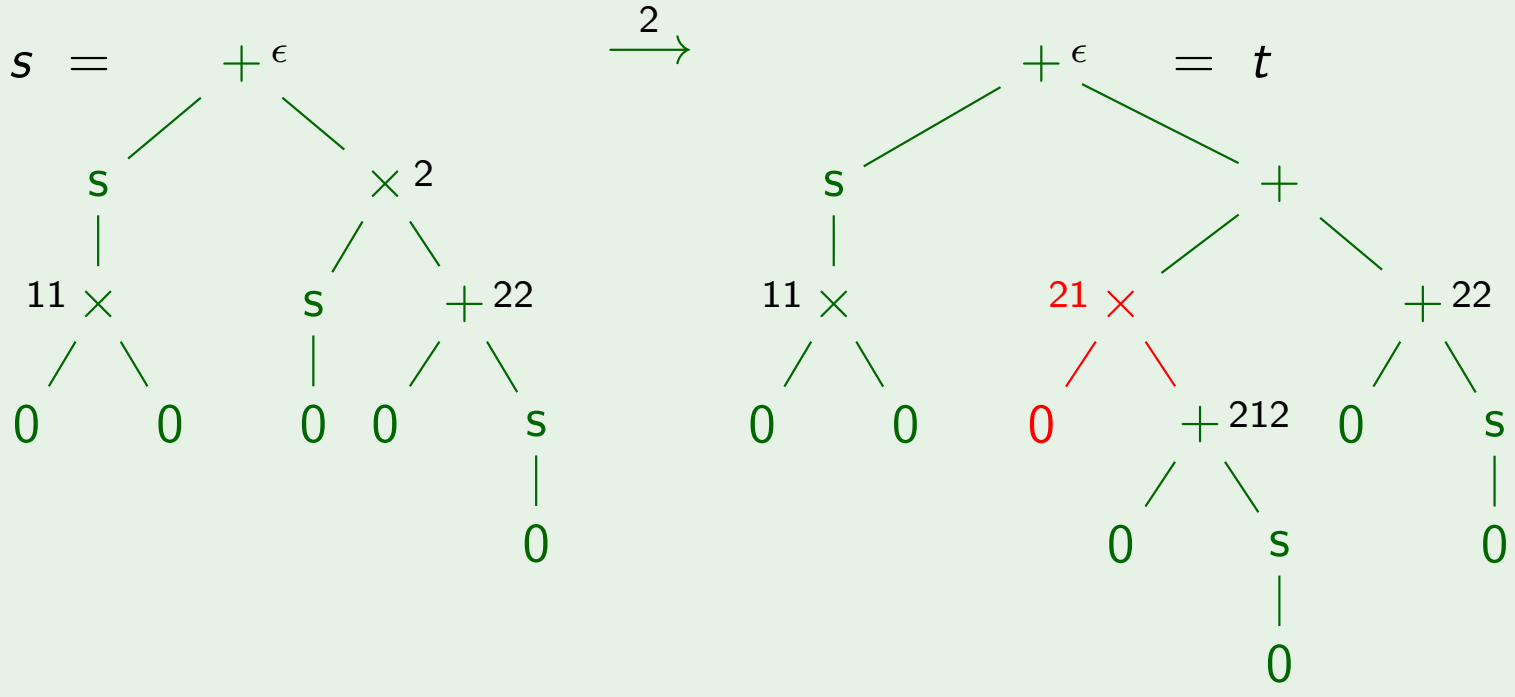
redex positions in s: $\epsilon \mid 11 \mid 2 \mid 22$

redex positions in t: $\epsilon \mid 11 \mid \mid 212 \mid 22$

duplicated

Example

$0 + y \to y$ $\quad s(x) + y \to s(x + y)$ $\quad 0 \times y \to 0$ $\quad s(x) \times y \to (x \times y) + y$



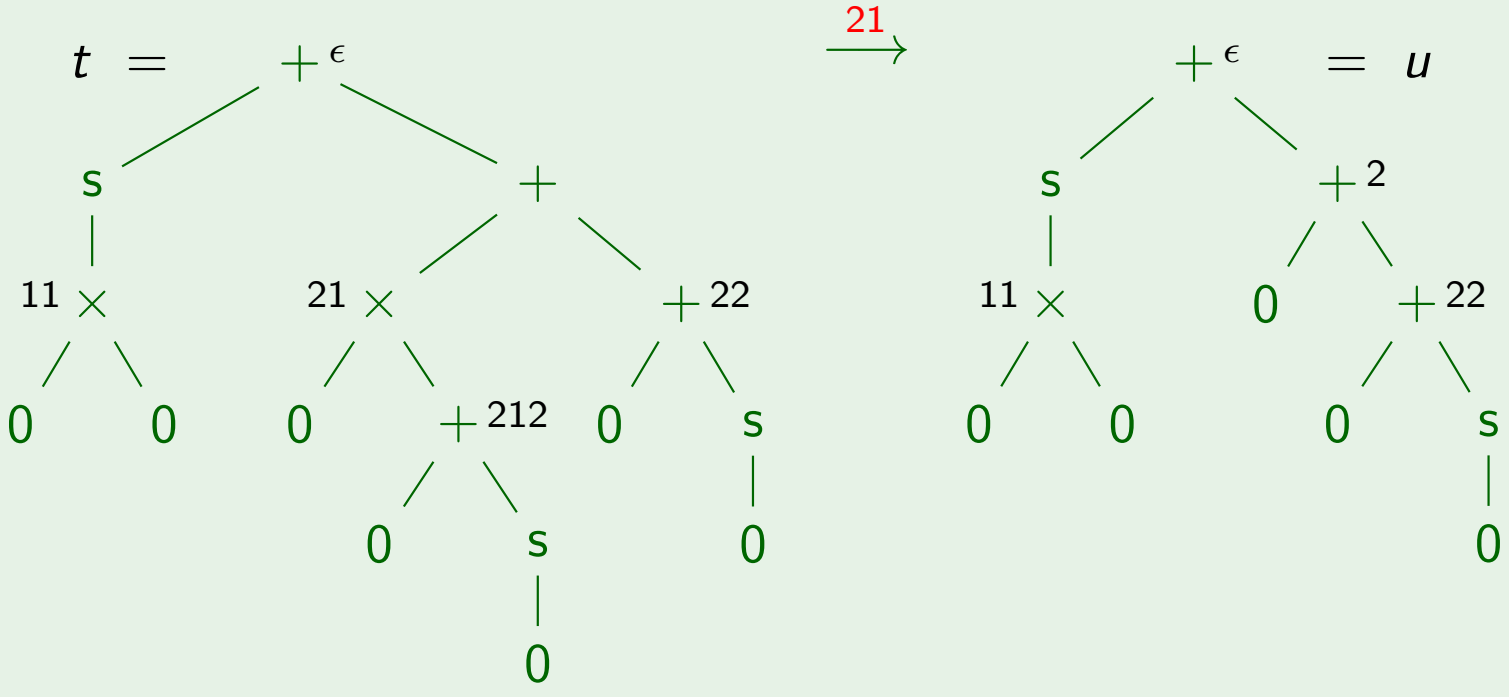
redex positions in s:	ϵ	11	2	22			
redex positions in t:	ϵ	11		212	22	21	

redex at position 22 is duplicated

redex at position 21 is created

Example

$0 + y \to y \quad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad 0 \times y \to 0 \quad \mathsf{s}(x) \times y \to (x \times y) + y$



redex positions in s: $\epsilon \mid 11 \mid 2 \mid 22$

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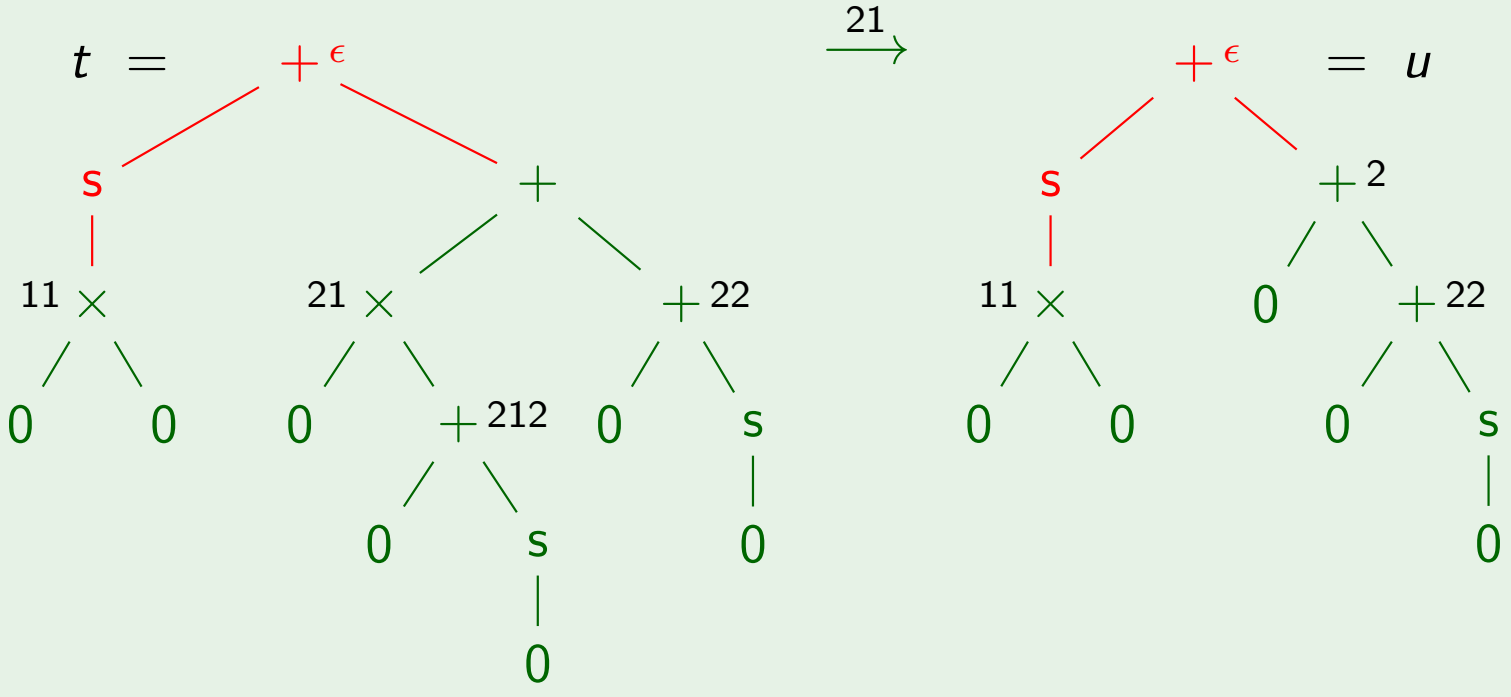

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
redex positions in u:	ϵ					

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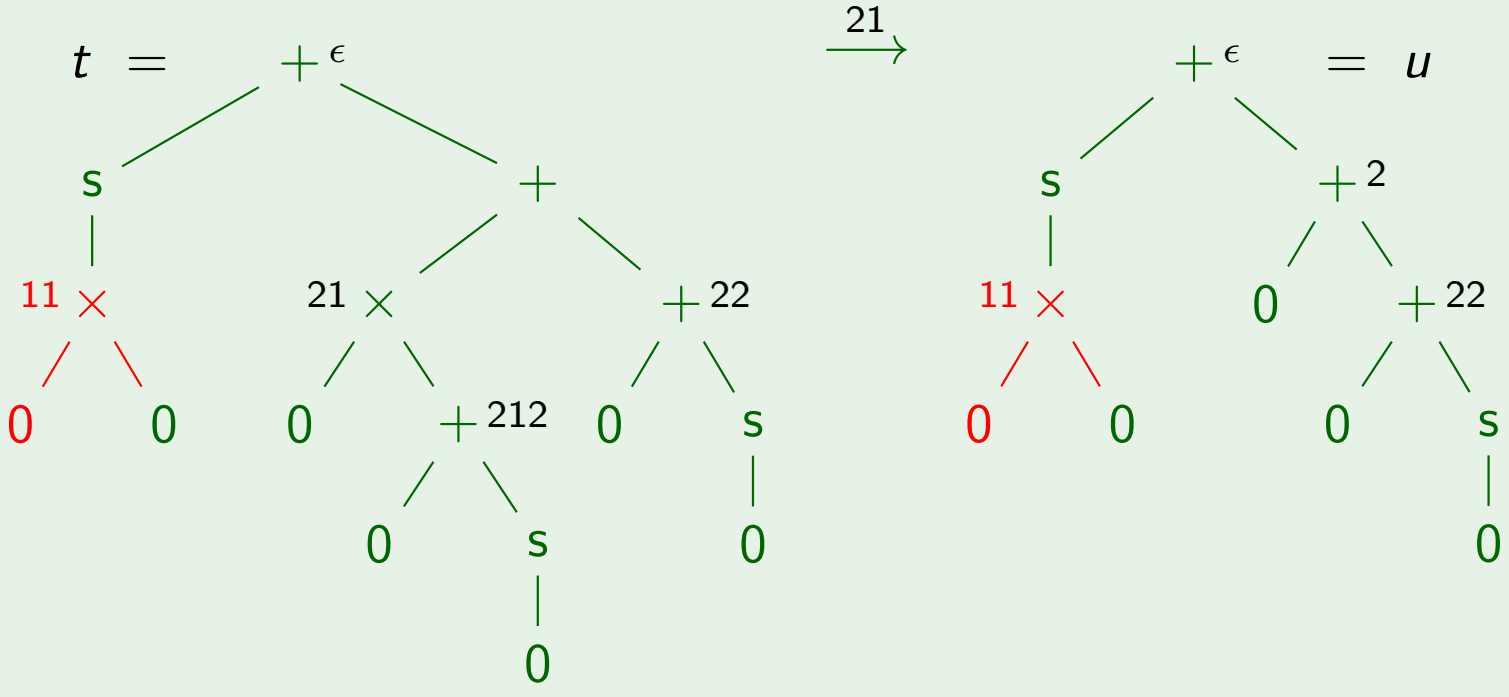

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redex positions in t:	ϵ	11		212	22	21
redex positions in u:	ϵ	11				

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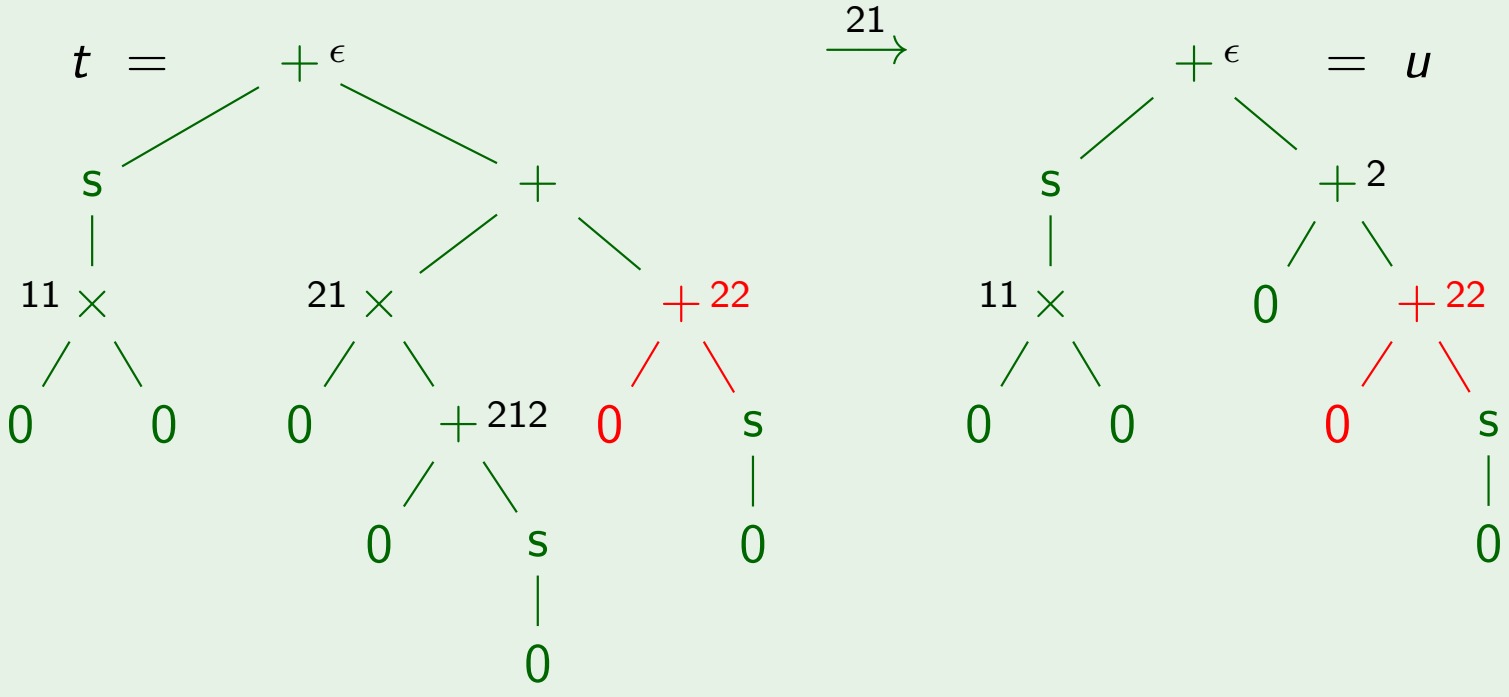

redex positions in s:	ϵ	11	2	22				
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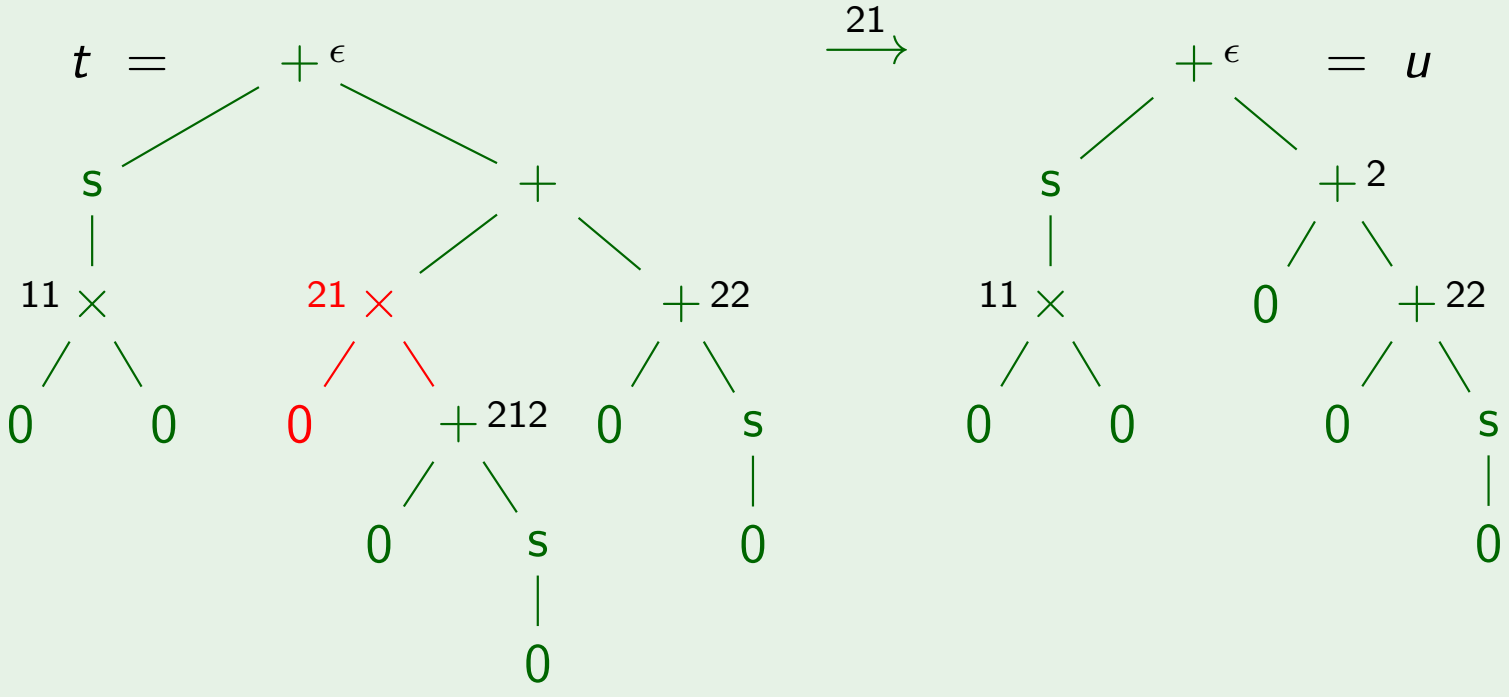

redex positions in s:	ϵ	11	2	22				
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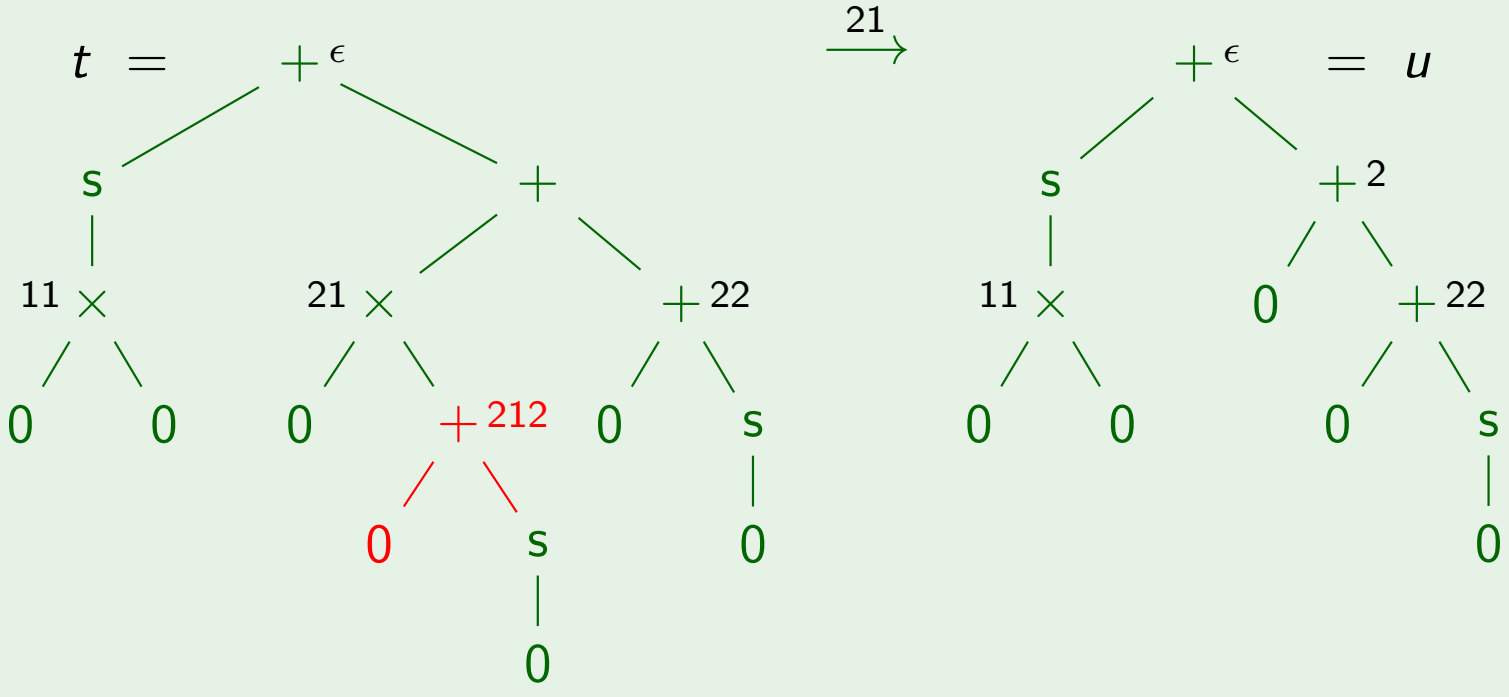

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erased

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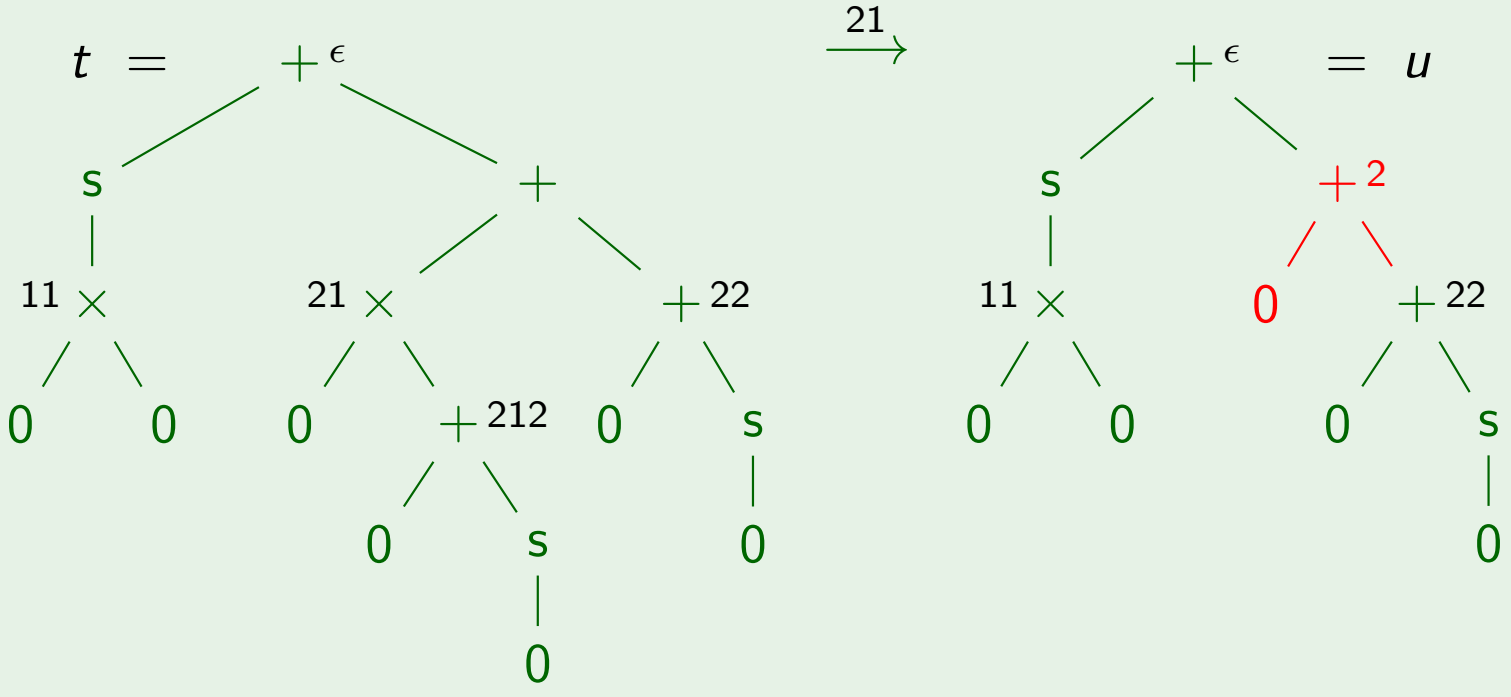

redex positions in s:	ϵ	11	2	22				
redex positions in t:	ϵ	11		212	22	21		
redex positions in u:	ϵ	11			22		2	

redex at position 212 is erased

redex at position 2 is created

rewrite step $A\colon s \xrightarrow[\ell \to r]{p} t$ at position $p \in \mathcal{P}\mathrm{os}(s)$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Step)

- The descendants of q after A in t

$$q/A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}\mathrm{os}_{\mathcal{X}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

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- The descendants of Q after A in t are

$$Q/A = \bigcup_{q \in Q} q/A$$

Remark

- Information about position is needed to determine descendants:

$$\mathrm{f}(x) \to x \qquad\qquad A : \mathrm{f}(\mathrm{f}(x)) \xrightarrow[\mathrm{f}(x)\to x]{?} \mathrm{f}(x)$$

Is $1/A = \epsilon$ or $1/A = \varnothing$?

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- Information about rewrite rule is needed to determine descendants:

$$\begin{array}{l}\mathsf{f}(x) \to \mathsf{f}(x)\\ \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a})\end{array} \qquad\qquad A : \mathsf{f}(\mathsf{a}) \xrightarrow[?]{\epsilon} \mathsf{f}(\mathsf{a})$$

  Is $1/A = 1$ or $1/A = \varnothing$?

rewrite sequence $A\colon s \to^* t$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Sequence)

The descendants of Q after A in t

$$Q/A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q/A_1)/A_2 & \text{If } A = A_1; A_2 \text{ with } A_1 : s \to u \text{ and } A_2 : u \to^* t \end{cases}$$

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Lemma

For left-linear TRSs: if Q is parallel ($\forall\, p \neq q \in Q \colon p \parallel q$) then so is Q/A.

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Lemma

For orthogonal TRSs: *if Q is set of redex positions then so is Q/A.*

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Lemma

For orthogonal TRSs: *if Q is set of redex positions then so is Q/A.*

Terminology

A descendant of redex is called residual.

Remark

- In non-left-linear TRS descendant of redex is not necessarily redex

$$\mathsf{a} \to \mathsf{b} \qquad\qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{f}(\mathsf{b},\mathsf{a})$$
$$\mathsf{f}(x,x) \to \mathsf{b}$$

Remark

- In non-left-linear TRS descendant of redex is not necessarily redex

$$\begin{aligned} a &\to b \\ f(x,x) &\to b \end{aligned} \qquad\qquad f(a,a) \to f(b,a)$$

- In TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{aligned} a &\to b \\ f(a) &\to b \end{aligned} \qquad\qquad f(a) \to f(b)$$

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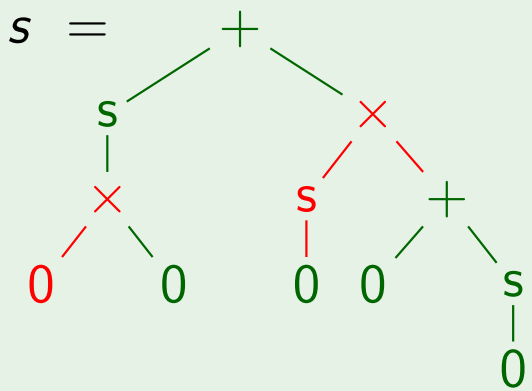
Example

$0 + y \to y \quad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad 0 \times y \to 0 \quad \mathsf{s}(x) \times y \to (x \times y) + y$

$s =$



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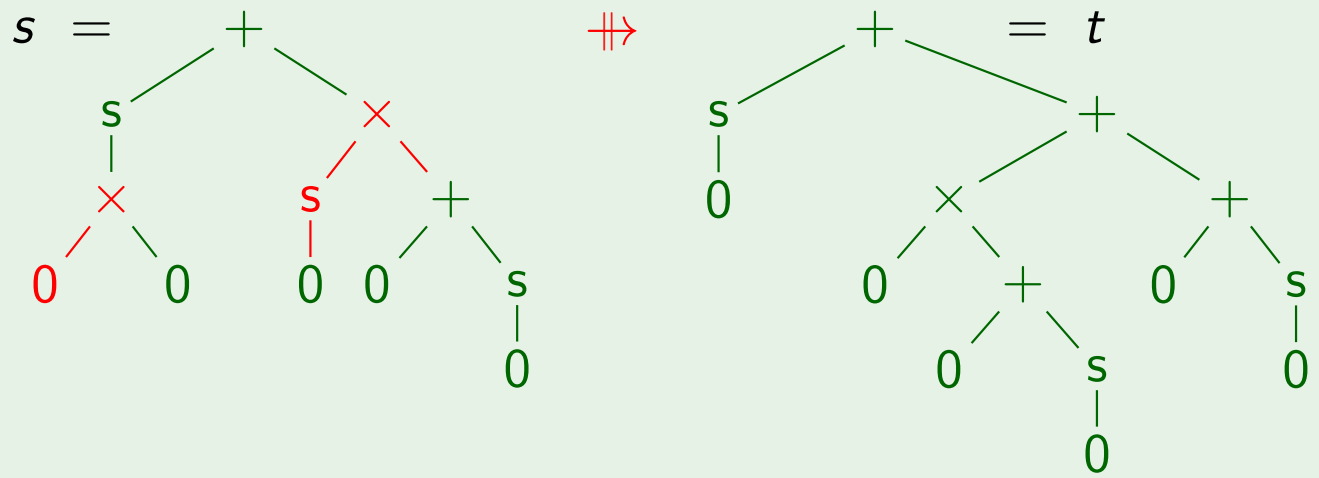
Example

$0 + y \to y \quad s(x) + y \to s(x + y) \quad 0 \times y \to 0 \quad s(x) \times y \to (x \times y) + y$



Definition (Parallel Rewriting)

1. $= \subseteq \Mapsto$
2. $\xrightarrow{\epsilon} \subseteq \Mapsto$
3. $f(s_1, \ldots, s_n) \Mapsto f(t_1, \ldots, t_n)$ if $s_i \Mapsto t_i$ for each $1 \leqslant i \leqslant n$

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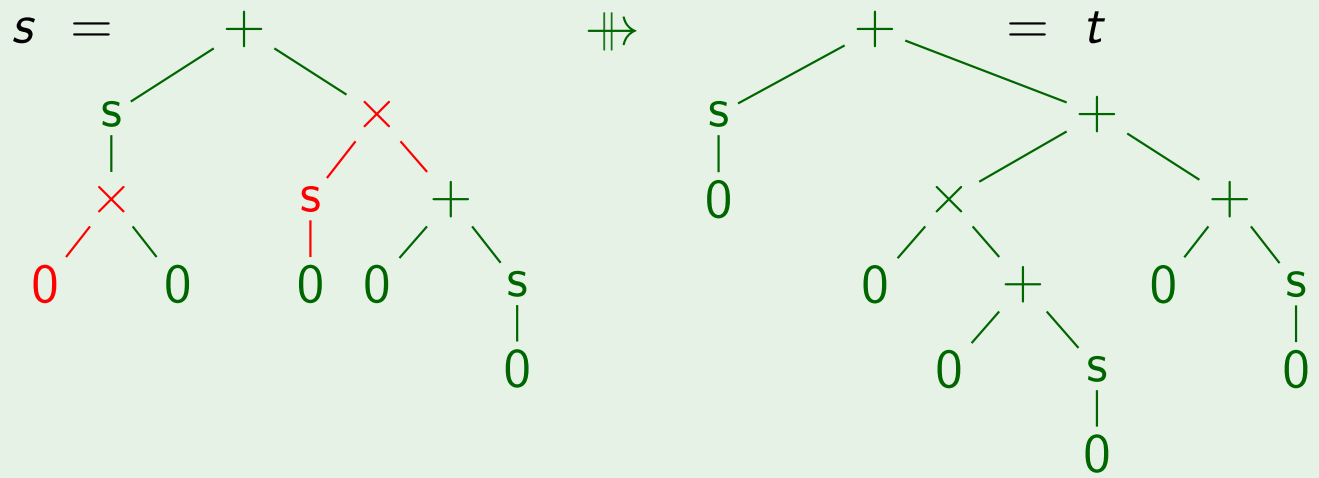

Lemma

$s \Mapsto t \iff s \to^* t$ *by contracting redexes at pairwise parallel positions in* s

Definition (Projection)

A: $s \Vdash t_1$ by contracting redexes at positions in P_A

B: $s \Vdash t_2$ by contracting redexes at positions in P_B

Definition (Projection)

A: $s \twoheadrightarrow\!\!\!\!\!+\; t_1$ by contracting redexes at positions in P_A

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- B/A: $t_1 \twoheadrightarrow\!\!\!\!\!+\; u_1$ by contracting redexes at positions in P_B/A

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- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

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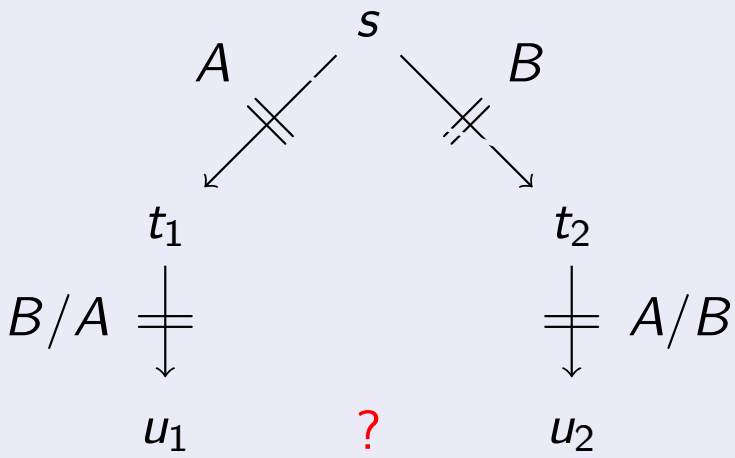
Parallel Moves Lemma

For every orthogonal TRS:



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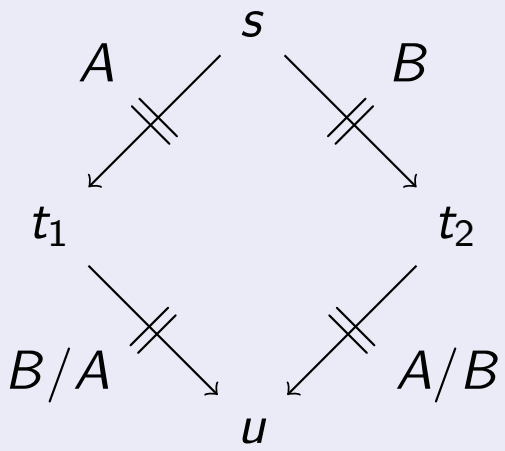

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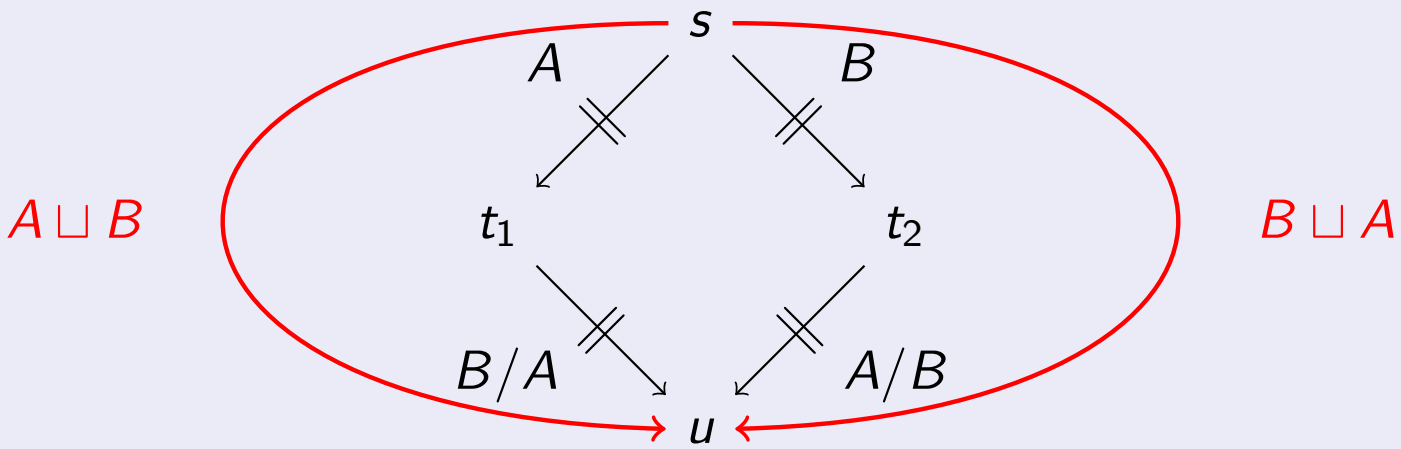

Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

1. $s_1 = s_2$
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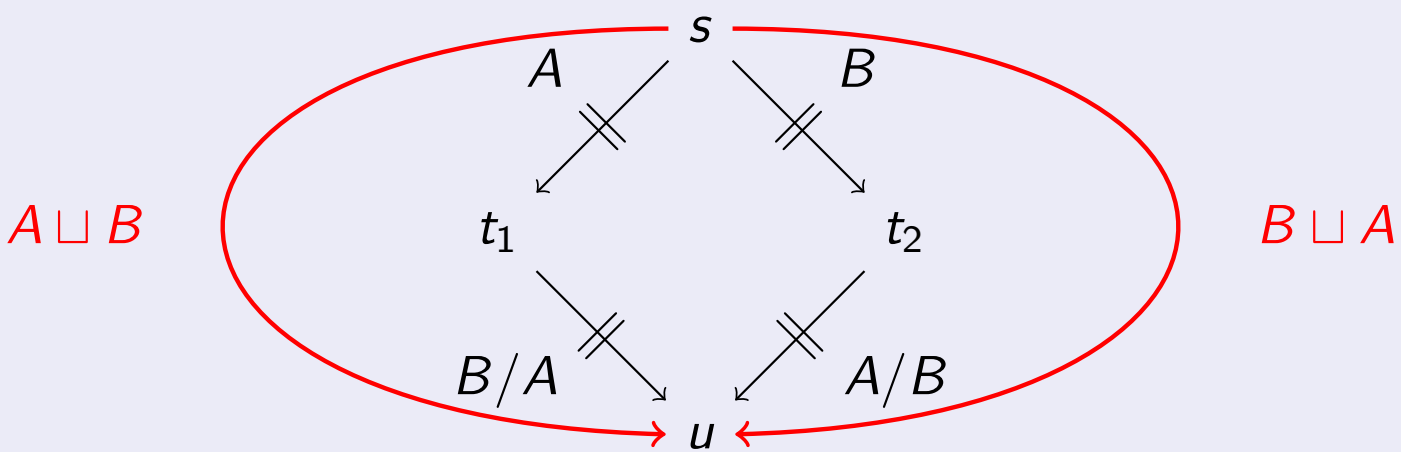
Parallel Moves Lemma (with Permutation Equivalence)



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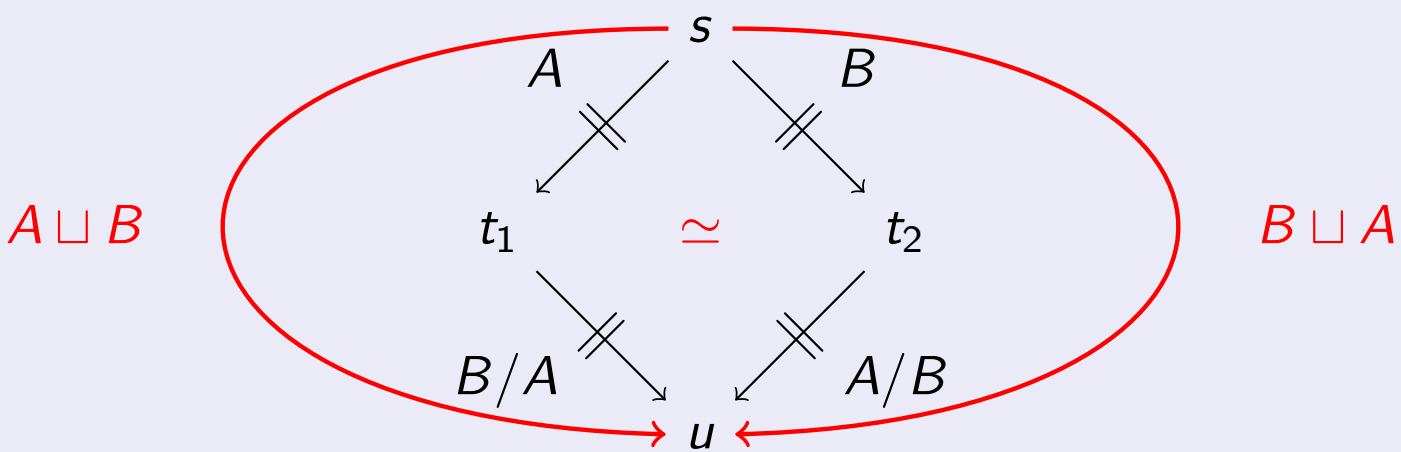
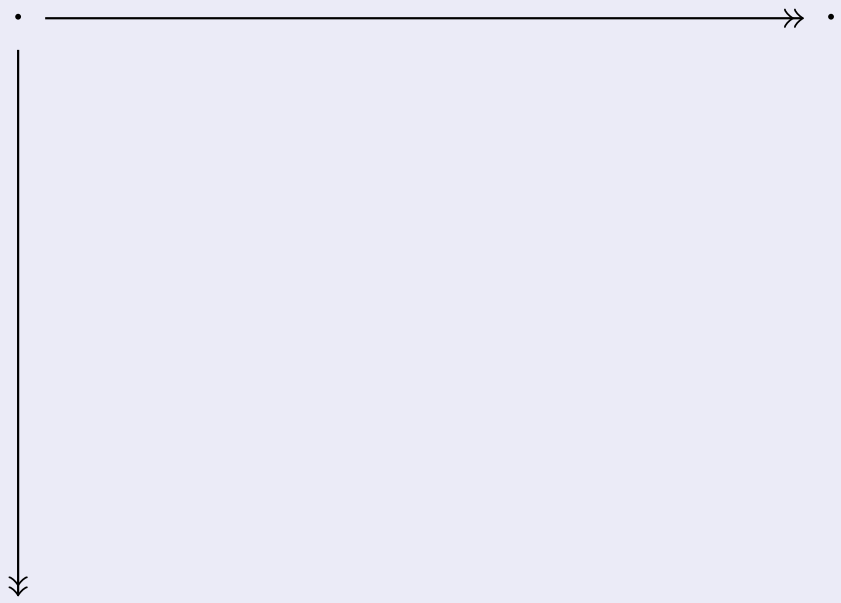


Corollary

Every orthogonal TRSs is confluent.

Proof.



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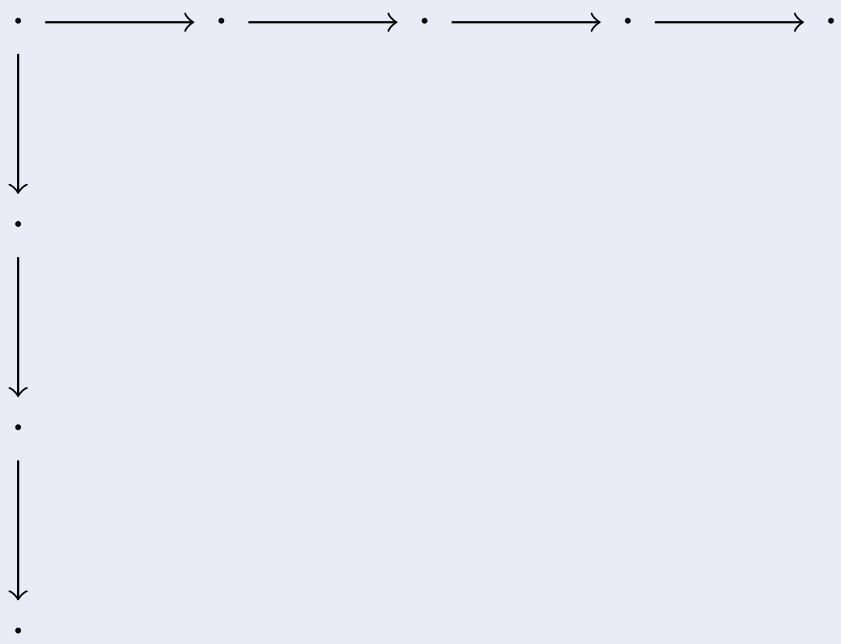

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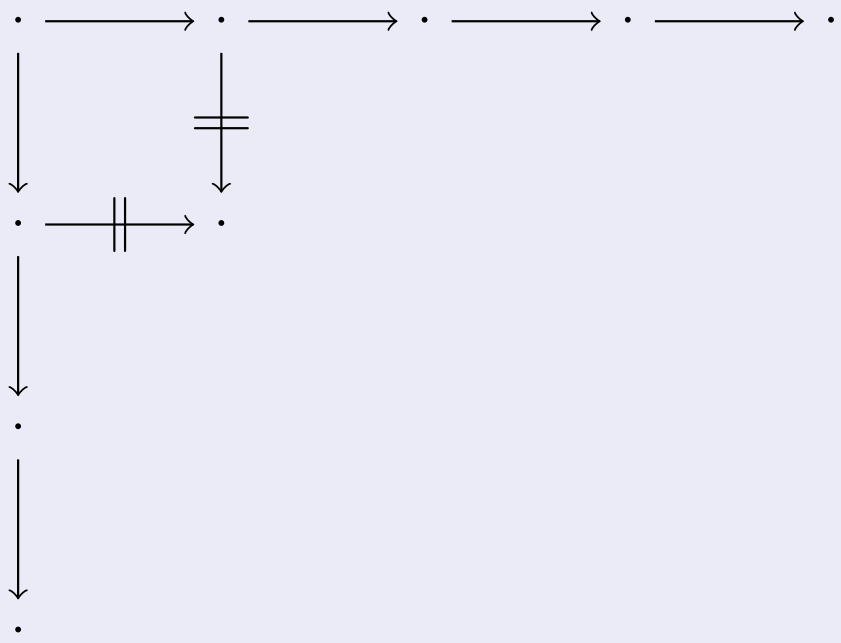

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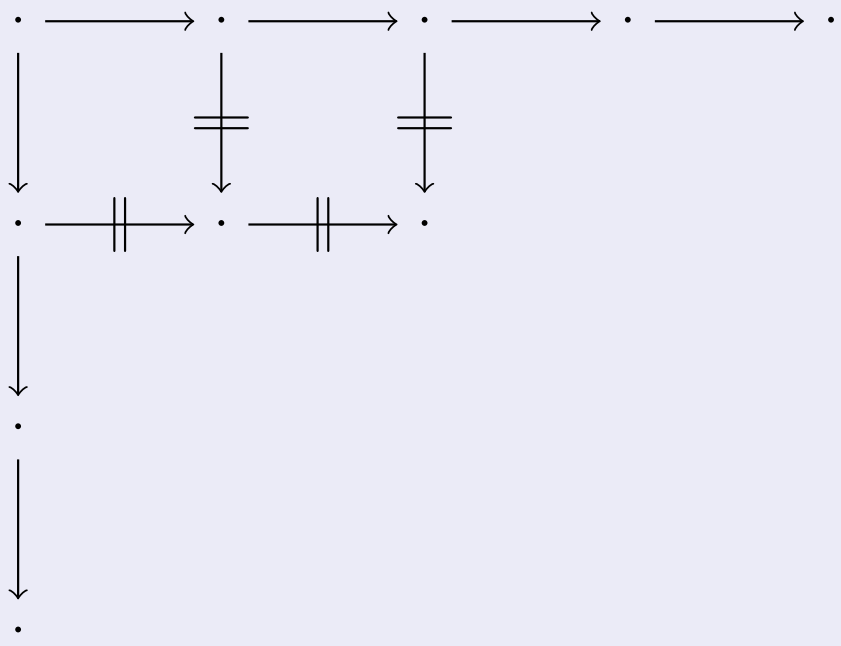

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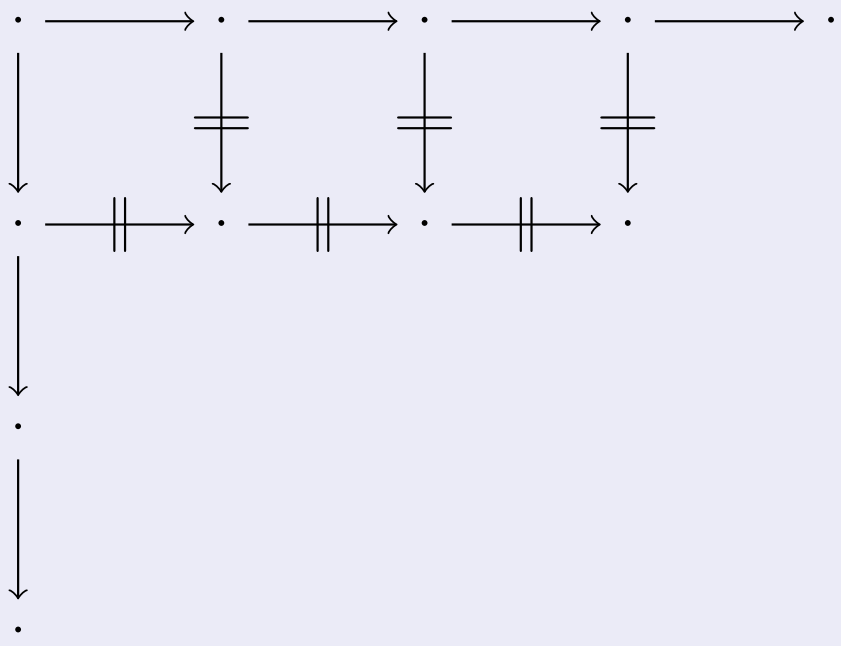

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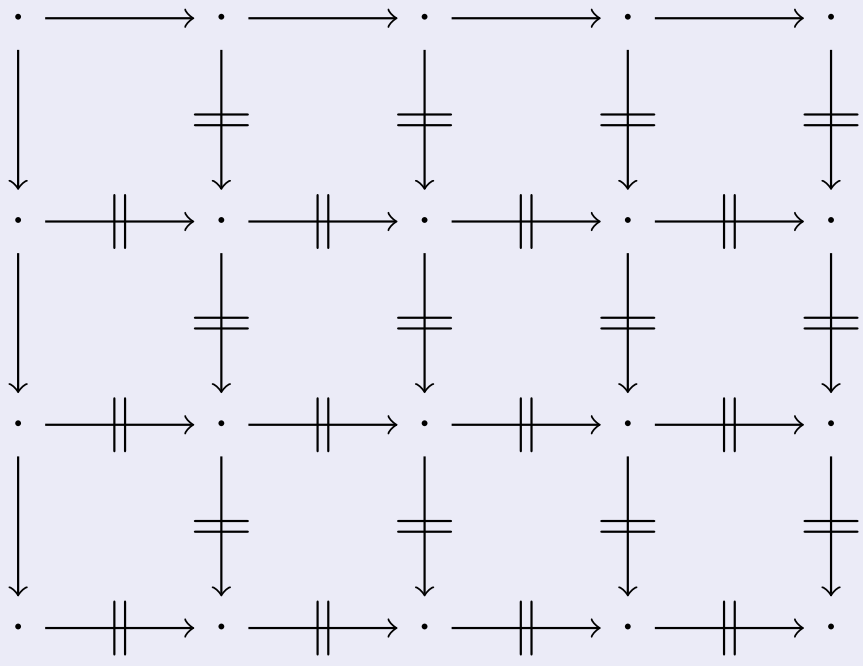

Outline

- Overview
- Local Confluence
- Orthogonality
- Beyond Orthogonality

Definition

A critical pair $s \leftarrow \rtimes \rightarrow t$ is trivial if $s = t$.

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A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

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Examples

$$x \vee \mathsf{T} \rightarrow \mathsf{T}$$
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Every weakly orthogonal TRSs is confluent.

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Proof Sketch.

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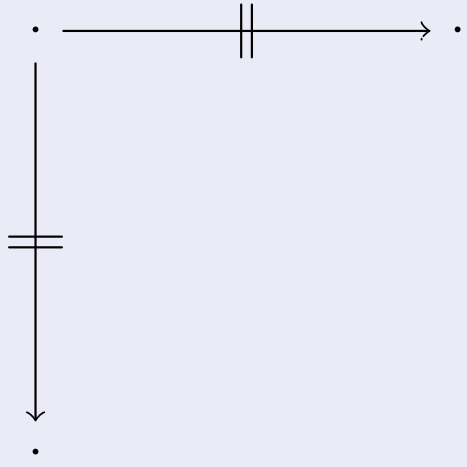

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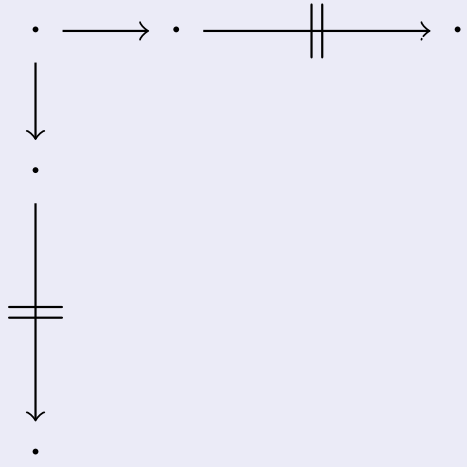

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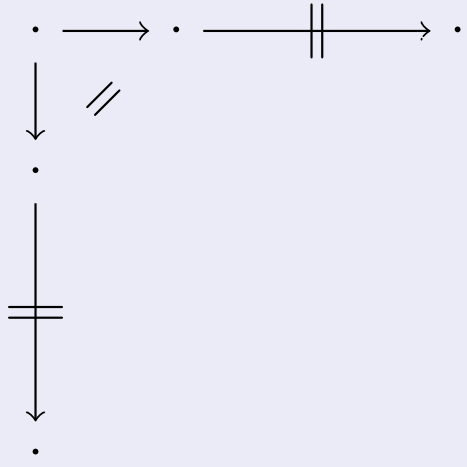

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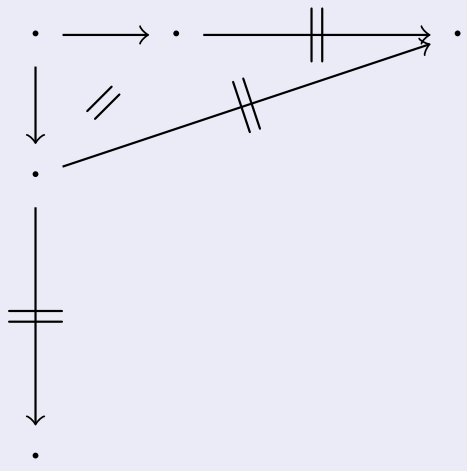

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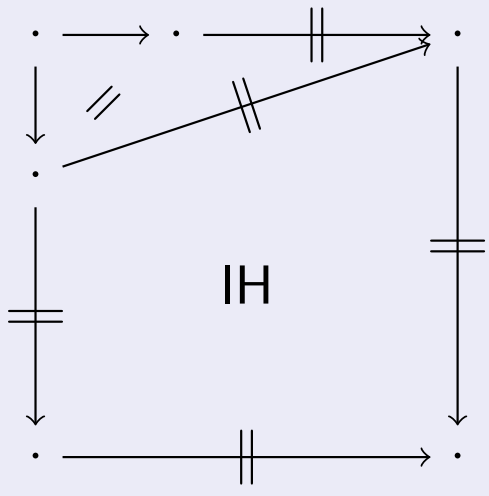

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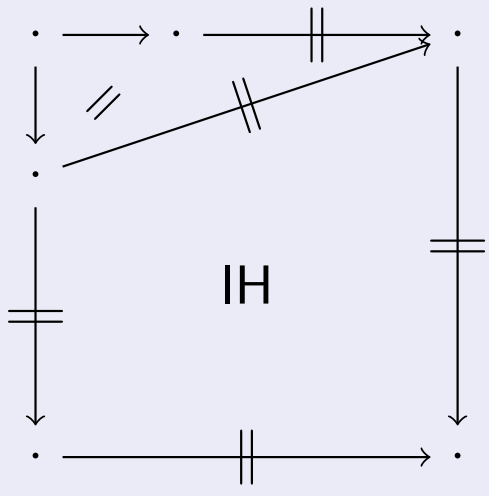

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Theorem (Huet 1980)

left-linearity & $\leftarrow \rtimes \rightarrow \subseteq \Vdash$ $\Longrightarrow$ *CR*

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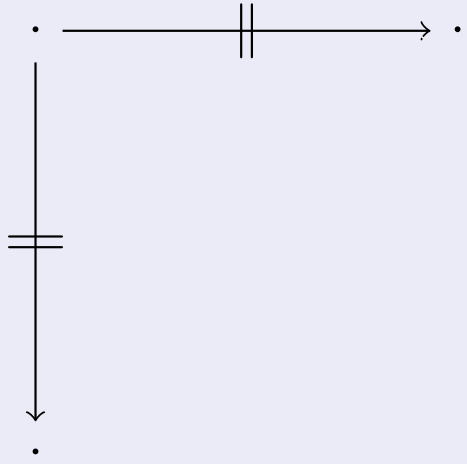

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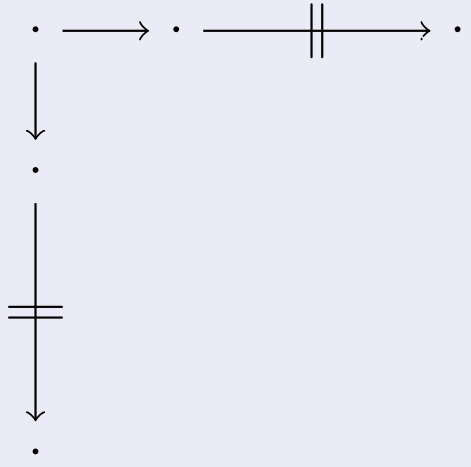

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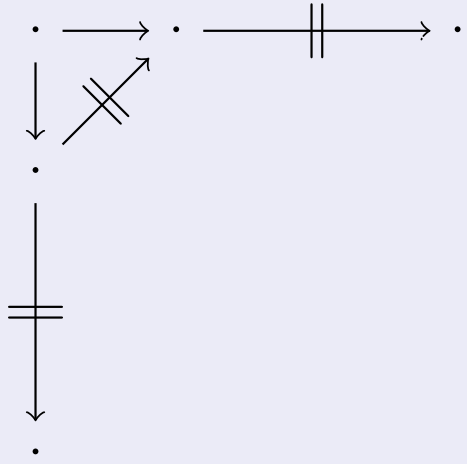

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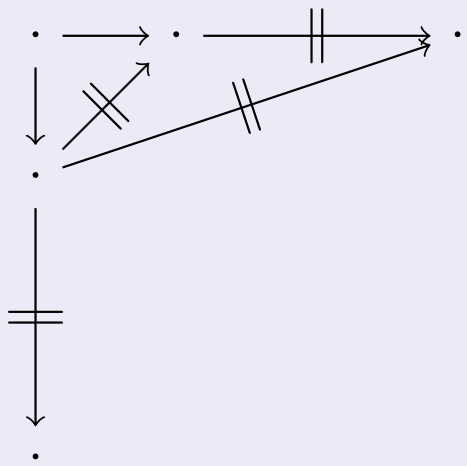

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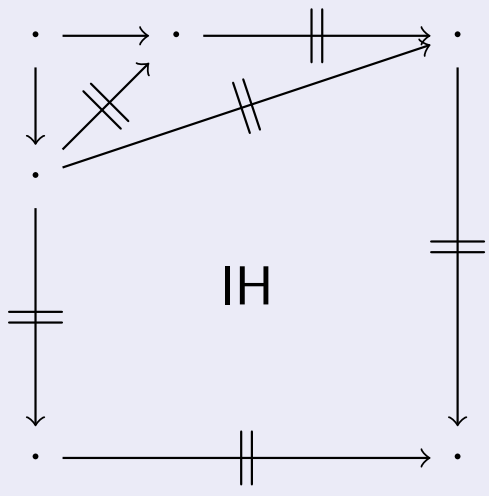

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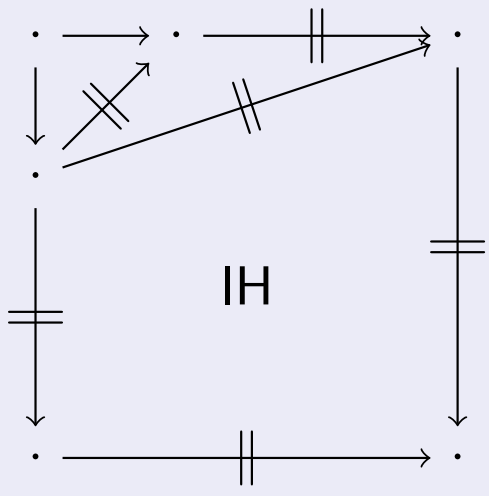

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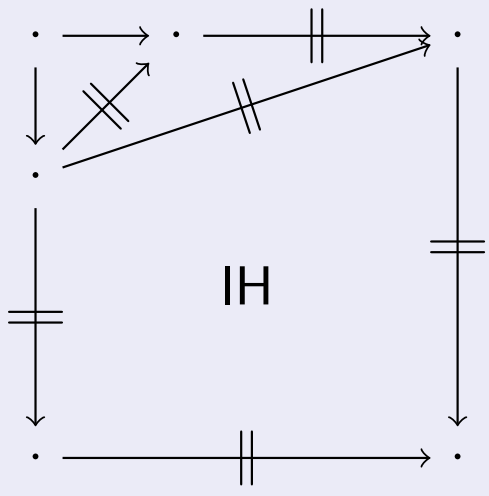

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Open Problem

left-linearity & $\leftarrow \rtimes \rightarrow \subseteq \twoheadleftarrow \implies$ *CR ?*

Example

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x, \mathsf{a}, \mathsf{b})) &\rightarrow x \\ \mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c}, \mathsf{d})), x, y) &\rightarrow \mathsf{h}(\mathsf{k}(x), \mathsf{k}(y)) \\ \mathsf{k}(\mathsf{a}) &\rightarrow \mathsf{c} \\ \mathsf{k}(\mathsf{b}) &\rightarrow \mathsf{d} \end{aligned}$$

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The only critical pair is:

$$\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}), \mathsf{k}(\mathsf{b}))) \quad \leftarrow \quad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c}, \mathsf{d})), \mathsf{a}, \mathsf{b})) \quad \to \quad \mathsf{f}(\mathsf{h}(\mathsf{c}, \mathsf{d}))$$

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Since $\mathcal{R}$ is left-linear and $\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}),\mathsf{k}(\mathsf{b}))) \nrightarrow\!\!\!\!\!\!\rightarrow \mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d}))$, the system is confluent.

Theorem (Huet 1980)

linearity & $\leftarrow \rtimes \rightarrow \subseteq (\rightarrow^{=} \cdot {}^{*}\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\leftarrow) \implies CR$

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Notation

- $s \leftarrow \bowtie \rightarrow t$ for critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2 \rangle$
- $\leftarrow \cdot \rtimes \rightarrow \; = \; \leftarrow \rtimes \rightarrow \setminus \leftarrow \bowtie \rightarrow$

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Theorem (Toyama 1988)

left-linearity & $\leftarrow \cdot \rtimes \rightarrow \subseteq \twoheadrightarrow$ & $\leftarrow \bowtie \rightarrow \subseteq \twoheadrightarrow \cdot {}^{*}\leftarrow \implies$ *CR*

Theorem (Huet 1980)

$$\textit{linearity} \quad \& \quad \leftarrow \rtimes \rightarrow \subseteq (\rightarrow^{=} \cdot {}^{*}\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\leftarrow) \quad \Longrightarrow \quad \textit{CR}$$

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Theorem (van Oostrom 1996)

$$\textit{left-linearity} \quad \& \quad \leftarrow \cdot\rtimes \rightarrow \subseteq \circ\!\!\rightarrow \quad \& \quad \leftarrow \bowtie \rightarrow \subseteq \circ\!\!\rightarrow \cdot {}^{*}\leftarrow \quad \Longrightarrow \quad \textit{CR}$$

Confluence Tool

- ACP