- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

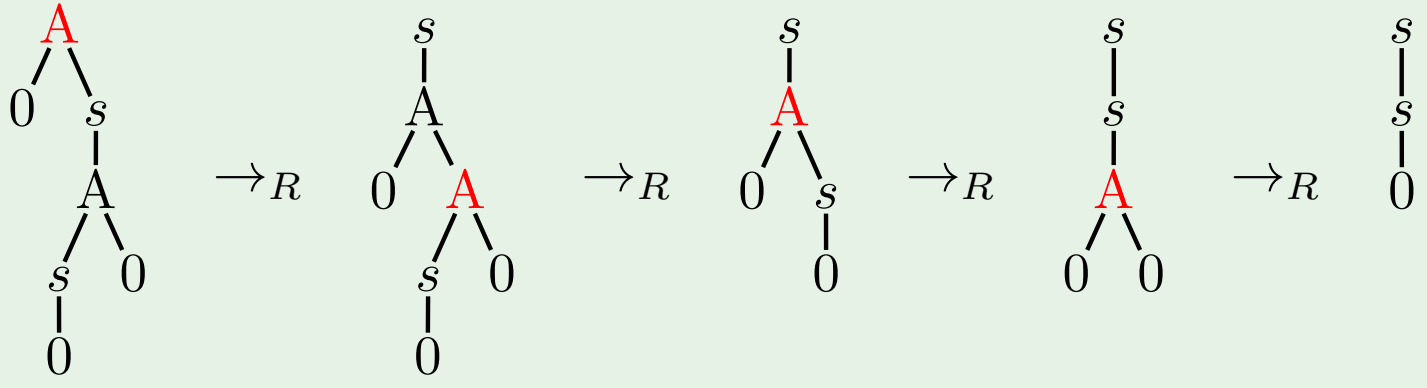
- Overview
- Introduction
- Reduction Orders
- Well-founded Monotone Algebras
- Polynomial Interpretations
- Dependency Pairs
- Stepwise Termination Proofs
- Dependency Graphs

Termination

Termination, Example 1

Example

$$\mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y))$$
$$\mathrm{A}(x, 0) \to x$$



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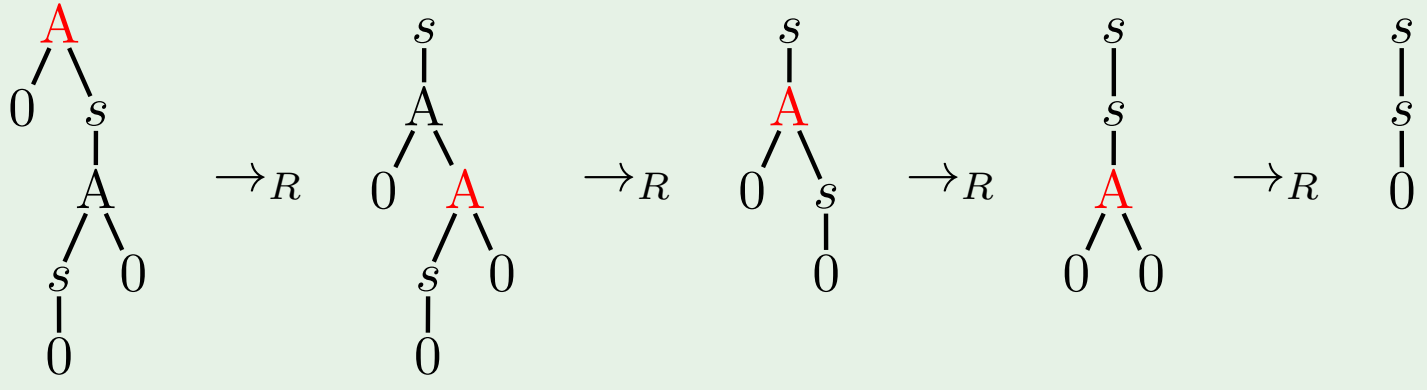

Looks terminating:

- second rule makes terms smaller
- first rule makes ‘s’ move upwards

Termination, Example 2

Example

$$\mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{f}(x))$$

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Looks terminating:

- f's move to the right
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Termination, Example 3

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Termination Methods 1967

Knuth-Bendix order

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Termination Research

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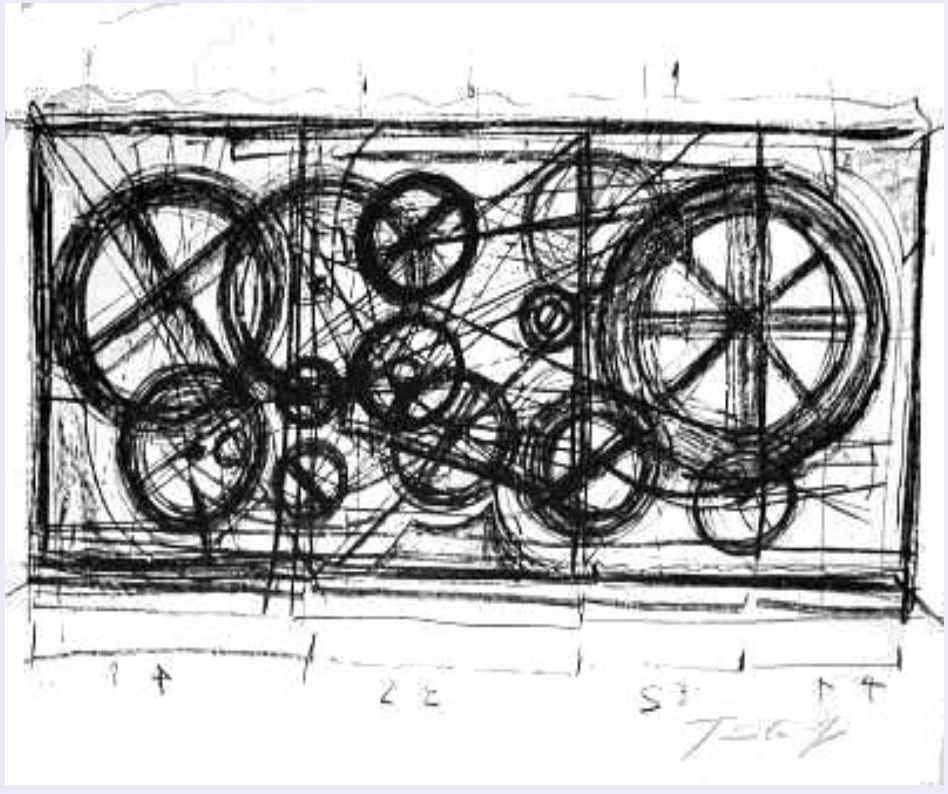

Termination Tools

AProVE, Cariboo, CiME, Jambox, Termptation, Matchbox, MuTerm, NTI, Torpa, TPA, T_TT_2, VMTL, ...

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- Subterm Criterion
- Iterative Lexicographic Path Order

Reduction Orders

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TRS R is terminating

$\Longleftrightarrow$

$\exists$ *well-founded order* $>$ *on terms such that* $s > t$ *whenever* $s \to_R t$

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- well-founded order $>$

$$s > t \iff \varphi(s) >_{\mathbb{N}} \varphi(t) \text{ with } \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = \mathsf{s}(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

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Remark

(very) inconvenient to check all rewrite steps

Definitions

A reduction order is well-founded order $>$ on terms which is

- closed under contexts $\quad s > t \implies C[s] > C[t]$
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Recall that $\to$ is the smallest relation S such that:

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However, this contradicts well-foundedness of $>$. ■

Well-founded Monotone Algebras

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Example (We use the Σ-algebra $(\mathbb{N}, [\cdot])$ with)

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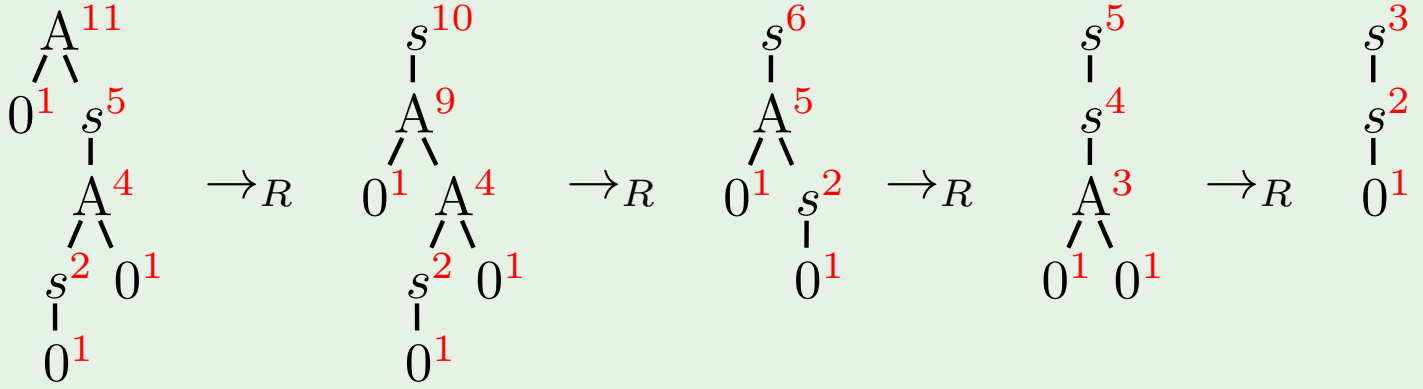

Interpretation of terms

Definition (Interpretation of Terms)

Let $\alpha : \mathcal{X} \to A$ be an interpretation of the variables.
We define the evaluation of terms

$$[\cdot, \alpha] : \mathcal{T}(\Sigma, \mathcal{X}) \to A$$

inductively:

$$[x, \alpha] = \alpha(x) \qquad \text{if } x \in \mathcal{X}$$
$$[f(t_1, \ldots, t_n), \alpha] = [f]([t_1, \alpha], \ldots, [t_n, \alpha])$$

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Relation $>_{\mathcal{A}}$ on terms: $s > t$ if $[s, \alpha] > [t, \alpha]$ for all assignments α.

Lemma

$>_{\mathcal{A}}$ is a reduction order for every well-founded monotone Σ-algebra $(A, [\cdot], >)$

Lemma

Let $\mathcal{A}$ well-founded monotone Σ-algebra. Then $>_{\mathcal{A}}$ is closed under substitutions.

Let σ be a substitution. We show $t\sigma >_{\mathcal{A}} s\sigma$.

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[x\sigma, \alpha] &= [\sigma(x), \alpha] = [x, \beta] \\
[f(t_1, \ldots, t_n)\sigma, \alpha] &= [f]([t_1\sigma, \alpha], \ldots, [t_n\sigma, \alpha]) \\
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Hence $[t\sigma, \alpha] = [t, \beta] > [s, \beta] = [s\sigma, \alpha]$.

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This infinite decreasing $>$ chain contradicts well-foundedness of $>$. ∎

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- $>_{\mathcal{A}}$ is closed under substitutions
- $>_{\mathcal{A}}$ is closed under contexts
- $>_{\mathcal{A}}$ is well-founded

Hence we have proven that:

Lemma

$>_{\mathcal{A}}$ is a reduction order for every well-founded monotone Σ-algebra $(A, [\cdot], >)$

Theorem

TRS R is terminating $\iff$ $R \subseteq >_{\mathcal{A}}$ *for well-founded monotone algebra* $(\mathcal{A}, >)$

$\Leftarrow$ Follows since we have shown that $>_{\mathcal{A}}$ is a reduction order.

$\Rightarrow$ As Σ-algebra we take $(A, [\cdot], >)$ with

- $A = \mathcal{T}(\Sigma, \mathcal{X})$,
- $[f](t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$,
- $> := \rightarrow$.

Then $>$ is well-founded since R is terminating.

Monotonicity of $>$ follows from closure of $\rightarrow$ under contexts:

$$s > t \Rightarrow [f](\ldots, s, \ldots) = f(\ldots, s, \ldots) > f(\ldots, t, \ldots) = [f](\ldots, t, \ldots)$$

$R \subseteq >_{\mathcal{A}}$ since for all $\ell \rightarrow r \in R$ and $\alpha : \mathcal{X} \rightarrow A$:

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Polynomial Interpretations

Outline

- Overview
- Introduction
- Reduction Orders
- Well-founded Monotone Algebras
- Polynomial Interpretations
- Dependency Pairs
- Stepwise Termination Proofs
- Dependency Graphs
- Subterm Criterion
- Iterative Lexicographic Path Order

Polynomial interpretations

Definition

A polynomial interpretations over $\mathbb{N}$ consists of:

- Σ-algebra is $(\mathbb{N}, [\cdot])$,
- $>$ defined as usual on $\mathbb{N}$,
- the interpretations $[f]$ are polynomials.

- monotonicity of the polynoms $[f]$, and
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Example

$$\mathrm{f}(\mathrm{g}(x)) \to \mathrm{f}(\mathrm{f}(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$[\mathrm{f}](x) = ???$ $[\mathrm{g}](x) = ???$

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Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

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&= \alpha(x) + 2 \cdot \alpha(y) + 2 \\
&> \alpha(x) + 2 \cdot \alpha(y) + 1 \\
&= [\mathrm{s}(A(x, y)), \alpha] \\
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Why do we need the conditions?

Example ($>$ not well-founded)

Let $R = \{ \mathrm{f}(x) \to \mathrm{f}(\mathrm{f}(x)) \}$ with the Σ-algebra $(\mathbb{Z}, [\cdot])$ and

$$[\mathrm{f}](x) = x - 1$$

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but

- $[f]$ is monotone, and
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Hence $>$ needs to be well-founded!

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Example ($[\mathrm{f}]$ not monotone)

Let $R = \{\ \mathrm{f}(x) \to \mathrm{g}(\mathrm{f}(x))\ \}$ with the Σ-algebra $(\mathbb{N}, [\cdot])$ and

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Hence the functions $[f]$ need to be monotone!

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Modern Approach

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(d) eliminate universally quantified variables

Question

How to find suitable polynomials ?

Modern Approach

(a) choose abstract polynomial interpretations (linear, quadratic, ...)

(b) transform rewrite rules into polynomial ordering constraints

(c) add monotonicity and well-definedness constraints

(d) eliminate universally quantified variables

(e) translate resulting constraints to SAT or SMT problem

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- rewrite system

$$0 + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

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$$\begin{gathered} (e-1)y + da + f > 0 \\ (e - be)y + dc + f - bf - c > 0 \\ a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0 \end{gathered}$$

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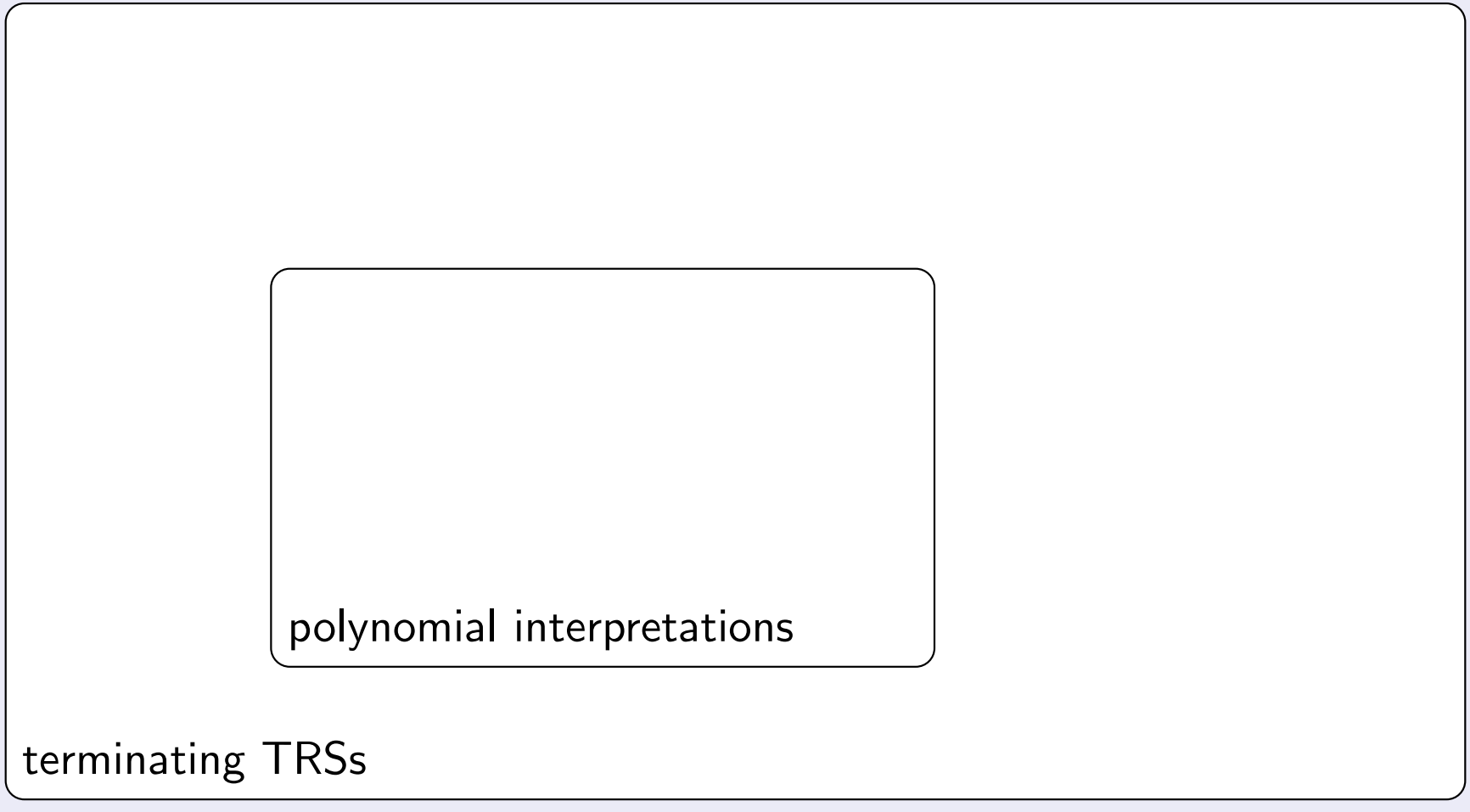

A non-simply terminating TRS: $f(f(x)) \to f(g(f(x)))$

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Thus we need another Σ-algebra, for example $A = \mathbb{N}^2$.

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We choose the Σ-algebra $(\mathbb{N}^2, [\cdot])$ with:

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Hence S is terminating.

Dependency Pairs

Minimal Terms

We call a term is minimal if all its strict subterms are terminating.

We prove: every non-terminating term t contains a minimal subterm.

We use induction on the term structure of t:

- Base case: all strict subterms of t are terminating. Then t is minimal itself.
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Steps below the root do not change minimality:

Then since t is minimal, it follows that all t_k are terminating.

Since $t_k \to^* s_k$, we obtain s_k is terminating for all $1 \leq k \leq n$.

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Notation: we use $\xrightarrow{\text{top}}_R$ to denote root rewrite steps.

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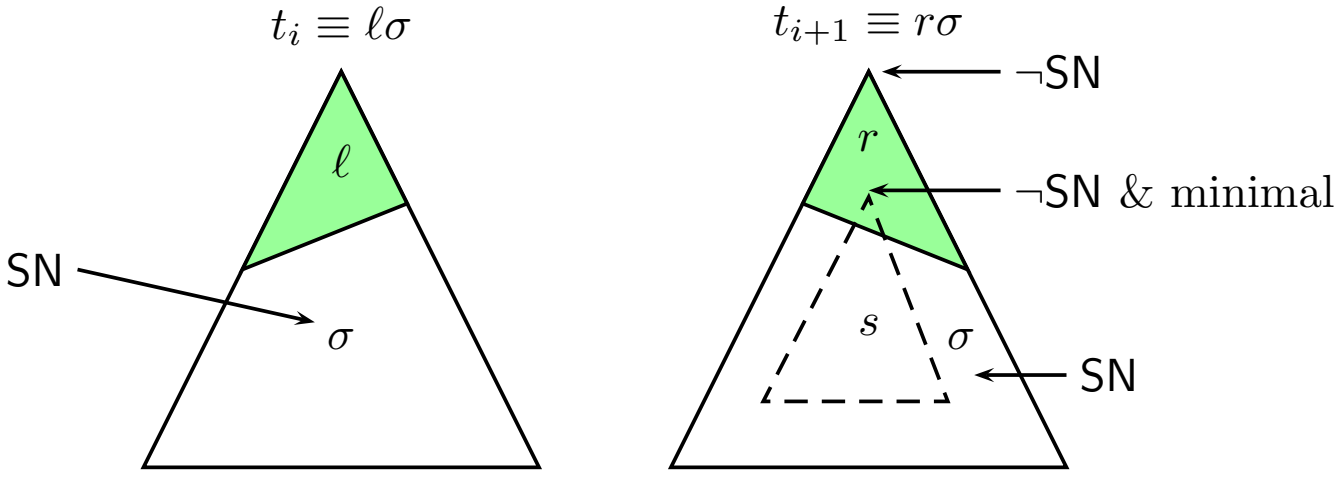

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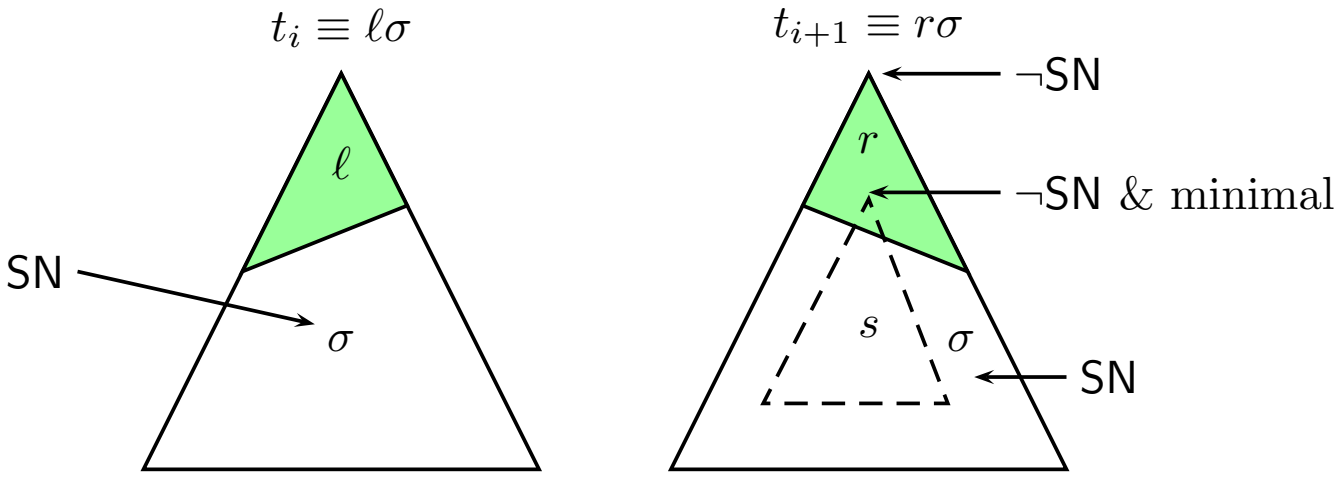

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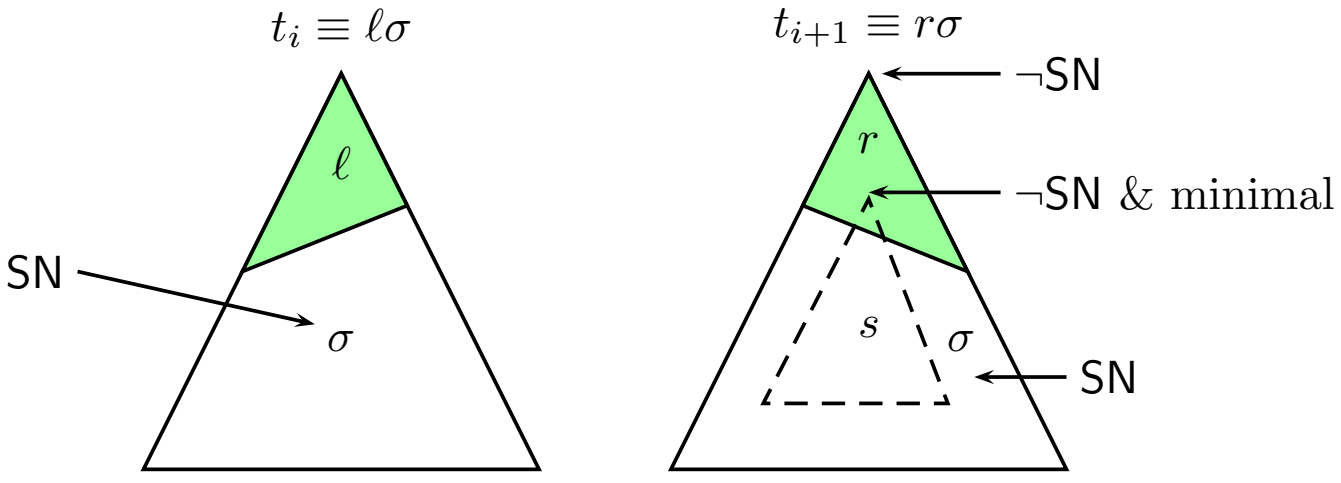

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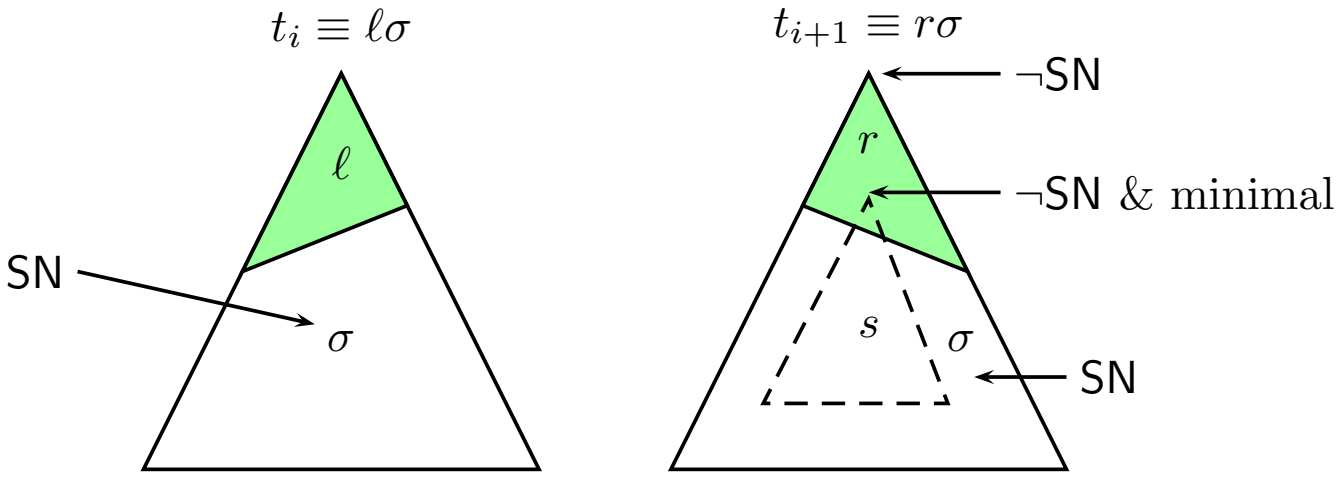

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Hence there exists a non-variable subterm r' of r such that $s = r'\sigma$.

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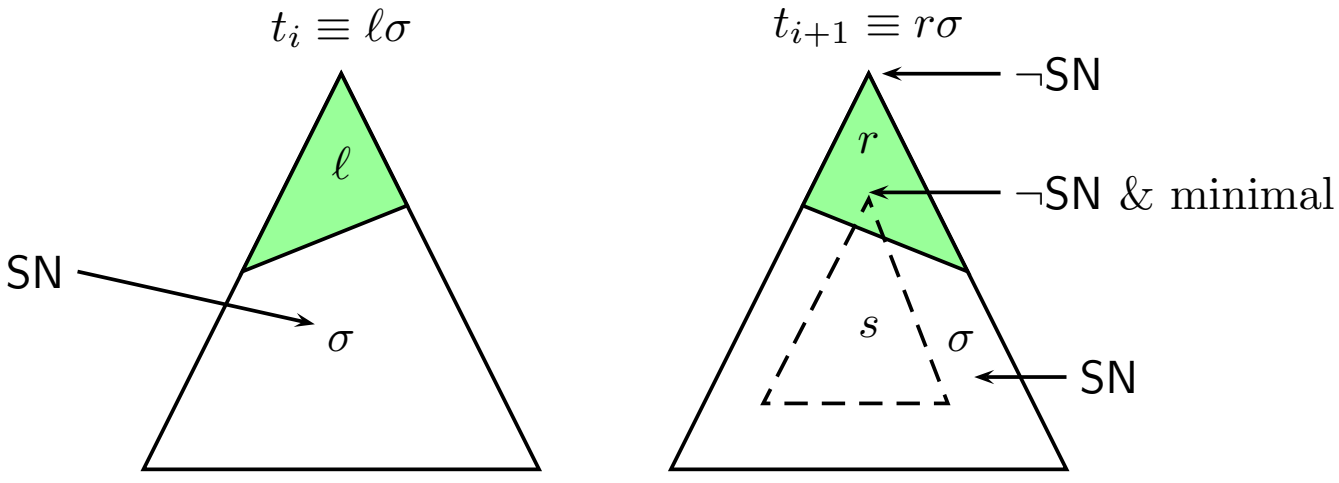

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Idea: add a rule $\ell \to r'$ then $t_i \to s$.

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For every $f \in \Sigma$ let $f_\#$ be a fresh symbol with the same arity as f.

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Definition (Dependency Pairs)

$$\mathsf{DP}(R) = \{\ell_\# \to r'_\# \mid \ell \to r \in R,\ r' \trianglelefteq r \text{ with } r' \notin \mathcal{X}\}$$

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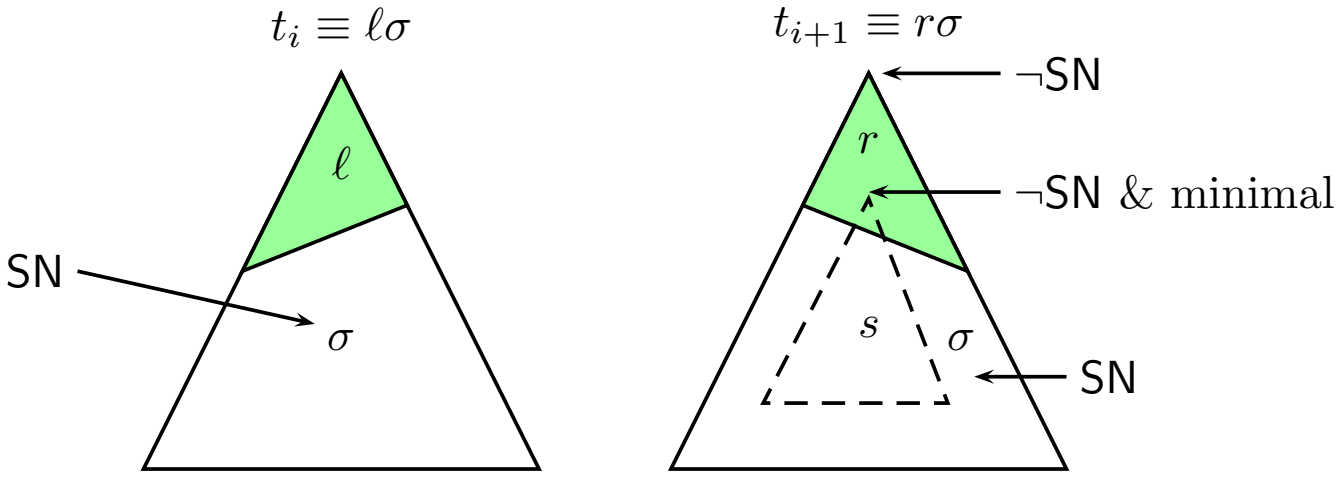
Let t_0 be a minimal and $t_0 \to_R t_1 \to_R t_2 \to_R \ldots$ an infinite rewrite sequence.

We consider the first root rewrite step $t_i \to t_{i+1}$:



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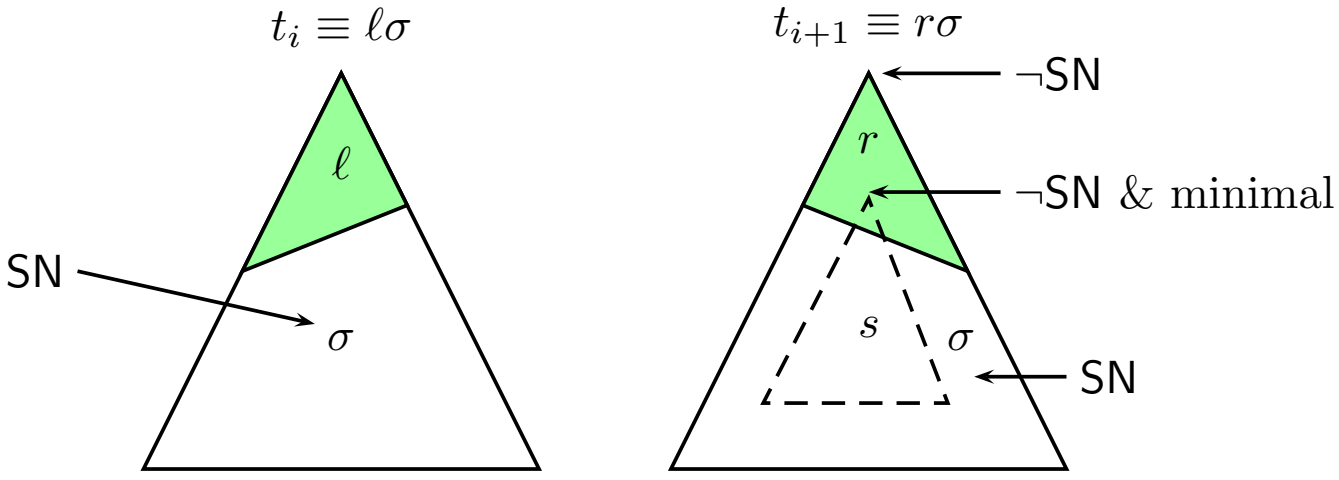

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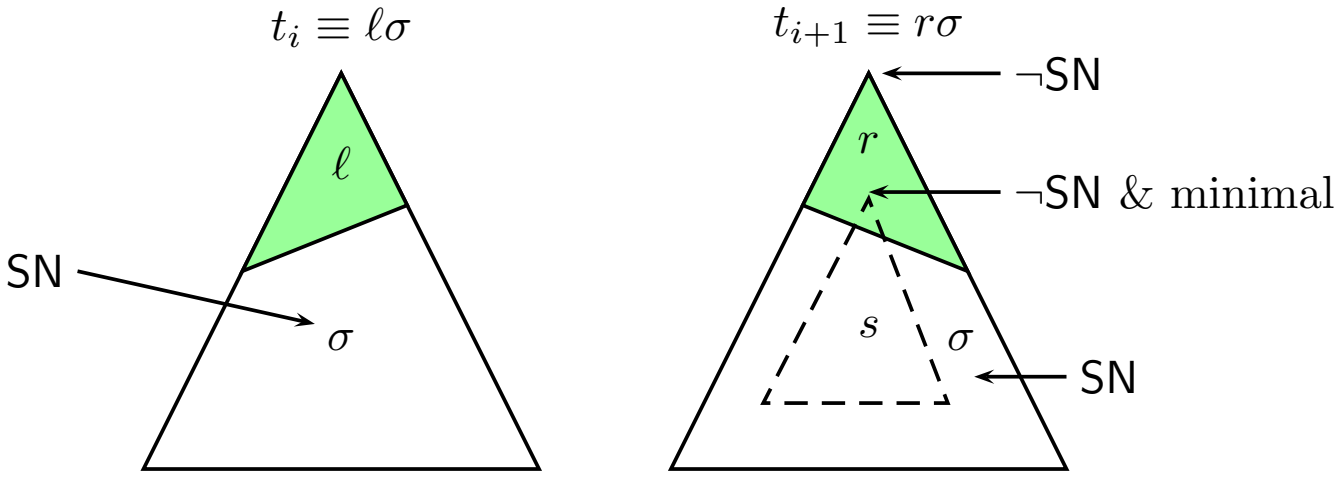

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Repeating the construction with s yields:

$$t_{0,\#} \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdots$$

an infinite rewrite sequence containing infinitely many $\mathsf{DP}(R)$ steps.

Dependency Pairs

Lemma

Let R be a non-terminating TRS. Then there exists a rewrite sequence:

$$t_{0,\#} \to_R^* \cdot \xrightarrow{top}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{top}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{top}_{\mathsf{DP}(R)} \cdots$$

such that:

- *the sequence contains infinitely many* $\mathsf{DP}(R)$ *steps,*
- *all R steps are below the root, and*
- *all* $\mathsf{DP}(R)$ *steps are at the root.*

Dependency Pairs, Examples

Example

$$R = \{\ \mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{f}(x))))\ \}$$

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$$\mathrm{f}_\#(\mathrm{g}(\mathrm{g}(x))) \xrightarrow{\mathrm{top}}_{\mathrm{DP}(R)} \mathrm{f}_\#(\mathrm{f}(\mathrm{g}(x))) \to_R \mathrm{f}_\#(\mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{f}(x))))) \to \dots$$

Dependency Pairs, Examples

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$$R = \{\ \mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y)),$$
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Dependency Pairs, Main Theorem

Definition (Relative Termination)

A relation $\to_1$ is called terminating relative to $\to_2$, denoted $\mathrm{SN}(\to_1 \,/\, \to_2)$, if every $\to_1 \cup \to_2$ rewrite sequence contains only finitely many $\to_1$ steps.

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The main theorem from dependency pairs is:

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That is, a TRS R is terminating if and only if $\xrightarrow{\text{top}}_{\mathsf{DP}(R)}$ terminates relative to $\to_R$.

Dependency Pairs, Termination Proofs

Definition

A well-founded weakly monotone Σ-algebra $(A, [\cdot], >, \succeq)$ consists of:

- a Σ-algebra $(A, [\cdot])$ with relations $>$, $\succeq$ on A
- $>$ is well-founded,
- $\mathrm{SN}(>/\succeq)$ (compatibility), (e.g. $> \cdot \succeq \subseteq >$ suffices)
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$\mathsf{SN}(\mathsf{DP}(R)_{top}/R)$ *if there exists a weakly monotone* Σ*-algebra s.t.*

- $\mathsf{DP}(R) \subseteq >$ *that is,* $[\ell, \alpha] > [r, \alpha] \quad \forall \alpha,\ \ell \to r \in \mathsf{DP}(R)$
- $R \subseteq \succeq$ *that is,* $[\ell, \alpha] \succeq [r, \alpha] \quad \forall \alpha,\ \ell \to r \in R$

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Advantages: no monotonicity for $>$, and $\succeq$ not well-founded.

Frequently used are polynomial interpretations over $\mathbb{N}$:

- $>$ as usual on $\mathbb{N}$ and $\succeq := \geq$
- the interpretations $[f]$ are polynomials

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We will see some examples...

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  Yes, since whenever $a \geq b$, then
  $[\mathrm{f}](a) = a + 1 \geq b + 1 = [\mathrm{f}](b)$,
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Hence we have proven termination.

Stepwise Termination Proofs

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Stepwise termination proofs with monotone Σ-algebras:

Theorem

If there exists a well-founded monotone Σ-algebra $(A, [\cdot], >)$ s.t.

- $R \subseteq \geq$, *and*
- $R' \subseteq >$, *and*

where $\geq := > \cup =$. Then

$$\mathsf{SN}(R) \implies \mathsf{SN}(R \cup R')$$

This theorem allows us to stepwise remove rules until none are left.

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Remark

Instead of $\geq$ we can more generally use a monotone relation $\succeq$ with $\mathsf{SN}(>/\succeq)$.

Example

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We use the following interpretation:

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We use the following interpretation:

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We get the following interpretation of rules:

$$[\mathrm{f}(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 > \alpha(x) + 1 = [\mathrm{g}(x), \alpha]$$
$$[\mathrm{f}(\mathrm{g}(x)), \alpha] = \alpha(x) + 2 \geq \alpha(x) + 2 = [\mathrm{g}(\mathrm{f}(x)), \alpha]$$

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We use the following interpretation:

$$[\mathrm{f}](x) = x + 1 \qquad [\mathrm{g}](x) = x + 1$$

We get the following interpretation of rules:

$$[\mathrm{f}(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 > \alpha(x) + 1 = [\mathrm{g}(x), \alpha]$$
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The first rule is strictly decreasing, hence we can remove it.

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The first rule is strictly decreasing, hence we can remove it.

Thus for termination it sufficies to show $\mathsf{SN}(\mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{f}(x)))$.

We have already shown this a few slides ago.
Hence we have proven termination.

Stepwise Termination Proofs

Stepwise termination proofs with dependency pairs:

Theorem

If there exists a weakly monotone Σ-algebra s.t.

- $T_1 \subseteq >$
- $T_2 \cup R \subseteq \succeq$

Then

$$\mathsf{SN}(T_{2,top}/R) \implies \mathsf{SN}((T_1 \cup T_2)_{top}/R)$$

That is, we may remove the strictly decreasing top-rules.

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Typically, $T_1, T_2 \subseteq \mathsf{DP}(R)$. Works also for other top-termination problems.

We are not allowed to remove strictly decreasing rules in R!
For removing from R we need monotonic interpretations!

Example

$$\begin{aligned}
\mathrm{minus}(x, 0) &\to x \\
\mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}(x, y) \\
\mathrm{quot}(0, \mathrm{s}(y)) &\to 0 \\
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We use the interpretation:

$$[\mathrm{minus}_{\#}](x, y) = 1 \quad [\mathrm{quot}_{\#}](x, y) = 1 \quad [\mathrm{minus}](x, y) = x \quad [\mathrm{s}](x) = x$$

$$[\mathrm{f}](\vec{x}) = 0 \text{ for all other symbols } \mathrm{f}$$

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The following interpretation removes the remaining DP rules:

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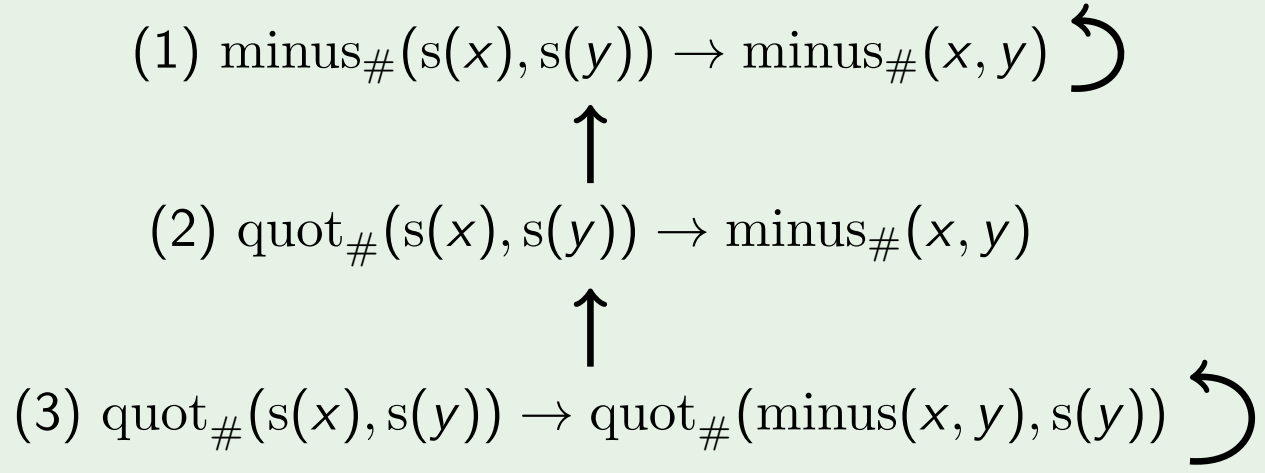
Dependency graph: analysis which DP-rules may follow each other



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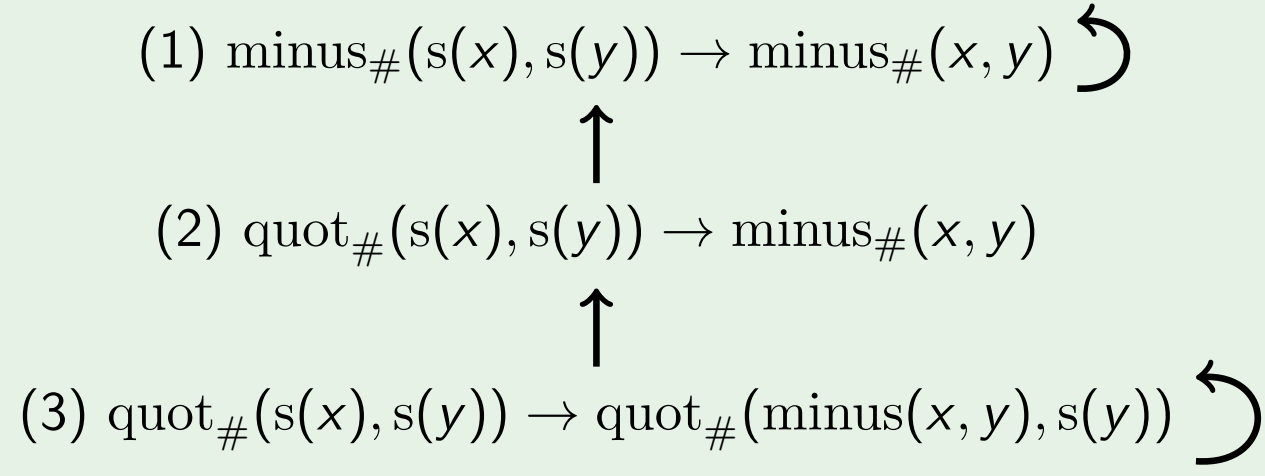

Idea: consider only strongly connected components $\mathsf{SN}(\{1\}_{top}/R)$, $\mathsf{SN}(\{3\}_{top}/R)$.

Subterm Criterion

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Theorem (Subterm Criterion)

Let R be a TRS, $T_1, T_2 \subseteq \mathsf{DP}(R)$, and $\pi : \Sigma_{\#} \to \mathbb{N}$ such that:

- $s_{\pi(f_\#)} \rhd t_{\pi(g_\#)}$ *for every rule* $f_\#(s_1, \ldots, s_n) \to g_\#(t_1, \ldots, t_m) \in T_1$
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Then:

$$\mathsf{SN}(T_{2,top}/R) \implies \mathsf{SN}((T_1 \cup T_2)_{top}/R)$$

We can only finitely often make a terminating term smaller ($\rhd$). ■

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After the dependency pairs transformation, we consider only minimal terms.

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We can remove the first two DP-rules.

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We can remove the remaining DP-rule.

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Hence we have proven termination.

Iterative Lexicographic Path Order

Iterative Lexicographic Path Order (ILPO)

ILPO. . . Historical overview

Kamin and Lévy [1980] (lexicographic path order, LPO):

- Kruskal's Tree Theorem was used in the original proofs
- Buchholz [1995] simplified the proof: Kruskal not needed

Bergsta and Klop [1985]:

- Iterative version of RPO: 'star method'
- Operational definition of reduction order
  via an auxiliary term rewriting system (with stars)

Klop, van Oostrom and de Vrijer [2005]:

- Extension of the star method to LPO
- Iterative lexicographic path order (ILPO)

ILPO... The star TRS $\mathcal{L}\mathrm{ex}_{\succ}$

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}_{\succ}$

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- Reduction rules (four types, for arbitrary f, $g \in \Sigma$):

ILPO... The star TRS $\mathcal{L}\mathrm{ex}_{\succ}$

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}_{\succ}$

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$$f^*(\vec{x}, g(\vec{y}), \vec{z}) \to_{\mathsf{lex}} f(\vec{x}, g^*(\vec{y}), l, \ldots, l) \qquad \text{where } l = f^*(\vec{x}, g(\vec{y}), \vec{z})$$

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Definition (ILPO)

$\succ_{ilpo}$ is the restriction of $\to^+_{\mathcal{L}\mathrm{ex}_\succ}$ to terms over Σ, i.e.

$$t \succ_{ilpo} s \iff t \to^+_{\mathcal{L}\mathrm{ex}_\succ} s \ \wedge \ t, s \in T(\Sigma \uplus V)$$

ILPO

Claim

$\succ_{ilpo}$ is a reduction order, that is:

- $\succ_{ilpo}$ is well-founded, and
- closed under substitution and contexts

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Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

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Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

Proof. $\succ_{ilpo}$ is a reduction order with $\mathcal{R} \subseteq \succ_{ilpo}$.

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\to x \\
\mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
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\end{aligned}
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Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

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\mathtt{A}(x, \mathtt{0}) &\to x \\
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Example, Addition and multiplication

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\mathtt{A}(x, \mathtt{0}) &\to x \\
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\mathtt{A}(x, \mathtt{0}) &\to x \\
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For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0})$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0}) \to_{\mathsf{select}} x$$

Example, Addition and multiplication

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\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\to x \\
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$$\mathtt{A}(x, \mathtt{S}(y))$$

Example, Addition and multiplication

$$\begin{aligned}
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\mathtt{A}(x, \mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0}) \to_{\mathsf{select}} x \\
\mathtt{A}(x, \mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y)))
\end{aligned}
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&\to_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y)))
\end{aligned}
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Example, Addition and multiplication

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\mathtt{A}(x, \mathtt{0}) &\to x \\
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\mathtt{A}(x, \mathtt{0}) &\to x \\
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\mathtt{M}(x, \mathtt{0})
\end{aligned}
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\mathtt{M}(x, \mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x, \mathtt{0})
\end{aligned}
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\mathtt{A}(x, \mathtt{0}) &\to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0}) \to_{\mathsf{select}} x \\
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\mathtt{M}(x, \mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x, \mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y)))
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\mathtt{M}(x, \mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x, \mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \\
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\end{aligned}
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Example, Addition and multiplication

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\mathtt{A}(x, \mathtt{0}) &\to x \\
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\end{aligned}
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Example, Addition and multiplication

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\mathtt{A}(x, \mathtt{0}) &\to x \\
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\mathtt{M}(x, \mathtt{0}) &\to_{\mathsf{put}} \mathtt{M}^*(x, \mathtt{0}) \to_{\mathsf{select}} \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y))) \\
&\to_{\mathsf{select}} \mathtt{A}(x, \mathtt{M}^*(x, \mathtt{S}(y))) \to_{\mathsf{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^*(y))) \\
&\to_{\mathsf{select}} \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}\mathrm{ex}^{\omega}$

But $\mathcal{L}\mathrm{ex}$ is in general not terminating, e.g. if $\mathtt{A} > \mathtt{S}$, then

$$\begin{aligned}\mathtt{A}(x,y) &\to_{\mathsf{put}} \mathtt{A}^*(x,y)\\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,y))\\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y)))\\ &\dots\end{aligned}$$

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Starred symbol $\mathtt{A}^*$ is ‘used’ infinitely often.

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Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

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Starred symbol $\mathtt{A}^*$ is 'used' infinitely often.

This is essential in any infinite reduction!

Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

$\Rightarrow$ Yields a terminating TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$.

Auxiliary TRS with numerical control symbols

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$

- Signature: $\Sigma \uplus \Sigma^{\omega}$, where $\Sigma^{\omega} = \{f^n \mid f \in \Sigma,\ n \in \mathbb{N}\}$

  f^n is fresh and has same arity as f

- Reduction rules:

$$
\begin{aligned}
f(\vec{x}) &\to_{\mathsf{put}} f^n(\vec{x}) \\
f^n(\vec{x}) &\to_{\mathsf{select}} x_i \\
f^{n+1}(\vec{x}) &\to_{\mathsf{copy}} g(f^n(\vec{x}), \ldots, f^n(\vec{x})) && \text{if } f \succ g \\
f^{n+1}(\vec{x}, g(\vec{y}), \vec{z}) &\to_{\mathsf{lex}} f(\vec{x}, g^n(\vec{y}), l, \ldots, l) && \text{where } l = f^n(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
$$

Back and forth between $\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^{\omega}$

From $\to_{\mathcal{L}\mathrm{ex}^{\omega}}$ to $\to_{\mathcal{L}\mathrm{ex}}$:

- every reduction can be transformed by replacing f^n by f^*.

Back and forth between $\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^\omega$

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From $\to_{\mathcal{L}\mathrm{ex}}$ to $\to_{\mathcal{L}\mathrm{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

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From $\to_{\mathcal{L}\mathrm{ex}^\omega}$ to $\to_{\mathcal{L}\mathrm{ex}}$:

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From $\to_{\mathcal{L}\mathrm{ex}}$ to $\to_{\mathcal{L}\mathrm{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

For example:

$$\begin{aligned}\mathtt{M}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y)))\\ &\to_{\mathsf{select}} \mathtt{A}(x,\mathtt{M}^*(x,\mathtt{S}(y))) \to_{\mathsf{lex}} \mathtt{A}(x,\mathtt{M}(x,\mathtt{S}^*(y)))\\ &\to_{\mathsf{select}} \mathtt{A}(x,\mathtt{M}(x,y))\end{aligned}$$

becomes:

$$\begin{aligned}\mathtt{M}(x,\mathtt{S}(y)) &\to_{\mathsf{put}} \mathtt{M}^2(x,\mathtt{S}(y)) \to_{\mathsf{copy}} \mathtt{A}(\mathtt{M}^1(x,\mathtt{S}(y)),\mathtt{M}^1(x,\mathtt{S}(y)))\\ &\to_{\mathsf{select}} \mathtt{A}(x,\mathtt{M}^1(x,\mathtt{S}(y))) \to_{\mathsf{lex}} \mathtt{A}(x,\mathtt{M}(x,\mathtt{S}^0(y)))\\ &\to_{\mathsf{select}} \mathtt{A}(x,\mathtt{M}(x,y))\end{aligned}$$

$\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^{\omega}$

Theorem

$\to^{+}_{\mathcal{L}\mathrm{ex}_{\succ}}$ *and* $\to^{+}_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}}$ *coincide on* $T(\Sigma \uplus V)$

$\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^{\omega}$

Theorem $\to^{+}_{\mathcal{L}\mathrm{ex}_{\succ}}$ *and* $\to^{+}_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}}$ *coincide on* $T(\Sigma \uplus V)$

Note that the infinite reduction

$$\begin{aligned}\mathtt{A}(x,y) &\to_{\mathsf{put}} \mathtt{A}^{*}(x,y)\\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^{*}(x,y))\\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^{*}(x,y)))\\ &\dots\end{aligned}$$

cannot be lifted:

$$\begin{aligned}\mathtt{A}(x,y) &\to_{\mathsf{put}} \mathtt{A}^{?}(x,y)\\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^{?-1}(x,y))\\ &\to_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^{?-2}(x,y)))\\ &\dots\end{aligned}$$

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f,\ \vec{t},\ \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n, > \rangle$.

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- Here $(\to_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

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In general a term is SN if all one-step reducts are SN.

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In general a term is SN if all one-step reducts are SN.

$\Rightarrow$ We check all one step reducts of $f^{\ell}(t_1, \ldots, t_n)$.

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Case 1. Internal step $f^{\ell}(\ldots, t_i, \ldots) \to f^{\ell}(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

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Case 1. Internal step $f^{\ell}(\ldots, t_i, \ldots) \to f^{\ell}(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Case 2. $f(t_1, \ldots, t_n) \to_{\mathsf{put}} f^n(t_1, \ldots, t_n)$.
We have a decrease in the third component.

Termination of $\mathcal{L}ex^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}ex^{\omega}_{\succ}})^n, > \rangle$.

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We have a decrease in the third component.

Case 3. $f^n(t_1, \ldots, t_n) \to_{\mathsf{select}} t_i$.
By assumption t_i is SN.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n, > \rangle$.

Case 4. $f^{n+1}(\vec{t}) \to_{\mathsf{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.

By IH the arguments $f^n(\vec{t})$ of g are SN since $n+1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Termination of $\mathcal{L}ex^{\omega}$ à la Buchholz

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$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

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Case 4. $f^{n+1}(\vec{t}) \to_{\text{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.

By IH the arguments $f^n(\vec{t})$ of g are SN since $n+1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Case 5. $f^{n+1}(\vec{t}, g(\vec{s}), \vec{r}) \to_{\text{lex}} f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$, $l = f^n(\vec{t}, g(\vec{y}), \vec{r})$.

By assumption the arguments $\vec{t}$ and $g(\vec{s})$ are SN.
By IH we get l is SN.
Since $g(\vec{s}) \to_{\text{put}} g^n(\vec{s})$ we get

- $g^n(\vec{s})$ is SN, and
- the triple decreases in the second component.

Thus by IH $f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$ is SN.

Termination of $\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^{\omega}$

Hence we have proven:

Theorem

$\rightarrow_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}}$ *is terminating*

Termination of $\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^{\omega}$

Hence we have proven:

Theorem

$\to_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}}$ *is terminating*

Corollary

$\to^{+}_{\mathcal{L}\mathrm{ex}}$ *is terminating on* $T(\Sigma \uplus V)$

Proof. $\to^{+}_{\mathcal{L}\mathrm{ex}}$ and $\to^{+}_{\mathcal{L}\mathrm{ex}^{\omega}}$ coincide on $T(\Sigma \uplus V)$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

Example, Ackermann Function

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We get the following derivations:

$$\mathrm{Ack}(0, y)$$

Example, Ackermann Function

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Find an order $\succ$ on Σ which proves termination.

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We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y})$$

Example, Ackermann Function

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Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y}))$$

Example, Ackermann Function

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$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

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We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$

Example, Ackermann Function

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$$\mathrm{Ack}(\mathrm{s}(x), 0)$$

Example, Ackermann Function

Example

$$\begin{aligned}\mathrm{Ack}(0, y) &\to \mathrm{s}(y)\\ \mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0))\\ \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))\end{aligned}$$

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Example, Ackermann Function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
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$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$

Example, Ackermann Function

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Example, Ackermann Function

Example

$$\begin{aligned}\mathrm{Ack}(0, y) &\to \mathrm{s}(y)\\ \mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0))\\ \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))\end{aligned}$$

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Example, Ackermann Function

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Example, Ackermann Function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))$$

Example, Ackermann Function

Example

$$\begin{aligned}\mathrm{Ack}(0, y) &\to \mathrm{s}(y)\\ \mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0))\\ \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))\end{aligned}$$

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Example, Ackermann Function

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Hence we have proven termination.

Recursive definition of LPO

Let $\succ$ be a strict order on signature Σ

Define $\succ_{lpo}$ on $T(\Sigma, V)$ by: $s \succ_{lpo} t$ iff

(LPO1) $t \in \mathcal{V}ar(s)$ and $s \neq t$, or

(LPO2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (LPO2a) $\exists 1 \leq i \leq m$, with $s_i = t$ or $s_i \succ_{lpo} t$, or
- (LPO2b) $f \succ g$ and $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, or
- (LPO2c) $f = g$, and

  $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, and
  there exists $1 \leq i \leq m$, s.t.
  $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

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Theorem

$\succ_{ilpo}$ *is equivalent with* $\succ_{lpo}$