- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Examples
- Terms
- Term Rewriting

Examples of Term Rewriting Systems

Example (Addition on Natural Numbers in Unary Notation)

signature 0 (constants) s (unary) $+$ (binary, infix)

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rewrite rules $0+y \to y$

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rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$

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Example (Combinatory Logic)

signature S K I (constants)

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signature $\quad S \quad K \quad I$ (constants) $\quad \cdot$ (application, binary, infix)

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$$\mathsf{I} \cdot x \to x$$
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rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

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inventor $\quad$ Moses Schönfinkel (1924)

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rewriting $(2:3)+(7:7)$

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rewriting $(2:3)+(7:7) \;\to^*\; 7:((2+1):0)$

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rewriting $(2:3)+(7:7) \to^* (1:0):0$ normal form

Example (Binary Trees)

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$$\cdots$$
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Outline

- Overview
- Examples
- Terms
  - Operations on Terms
  - Contexts
  - Substitutions
- Term Rewriting

Term Rewriting Systems

Definition

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Definitions (Operations on Terms)

- $\mathcal{V}\mathrm{ar}(\,\cdot\,)$, the variables of a term

$$\mathcal{V}\mathrm{ar}(t) = \begin{cases} \{t\} & \text{if } t \in \mathcal{X} \\ \bigcup_{i=1}^{n} \mathcal{V}\mathrm{ar}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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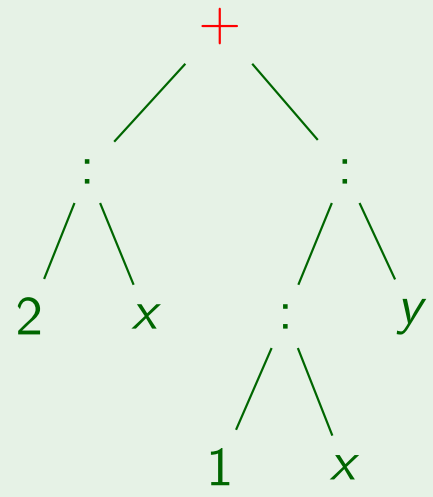
Example

$(2 : x) + ((1 : x) : y)$



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- $|\cdot|$, the size of a term

$$|t| = \begin{cases} 1 & \text{if } t \in \mathcal{X} \\ 1 + \sum_{i=1}^{n} |t_i| & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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$\|(2 : x) + ((1 : x) : y)\| = 6$

Definition (Operations on Terms)

- height($\cdot$), the height (or depth) of a term

$$\text{height}(t) = \begin{cases} 0 & \text{if } t \in \mathcal{X} \text{ or } t \text{ is a constant} \\ 1 + \max\limits_{1 \leqslant i \leqslant n} \text{height}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ with } n \geqslant 1 \end{cases}$$

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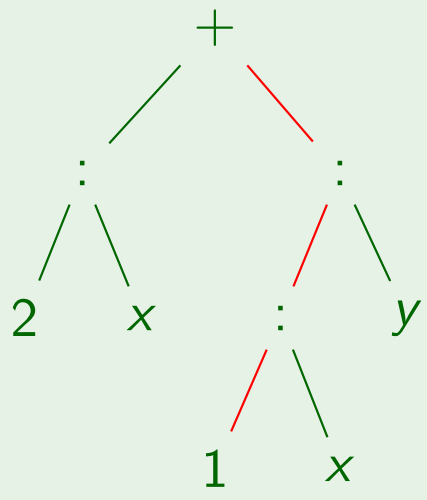
Example

$\text{height}((2 : x) + ((1 : x) : y)) = 3$



Definitions

- $s \trianglelefteq t$ s is subterm of t
  - $s = t$

x $2 : x$ 1 $1 : x$ y $(1 : x) : y$ $(2 : x) + ((1 : x) : y)$

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Definition (Positions)

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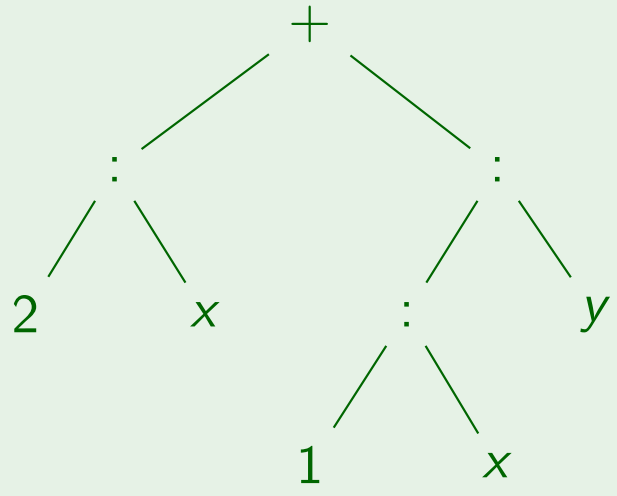

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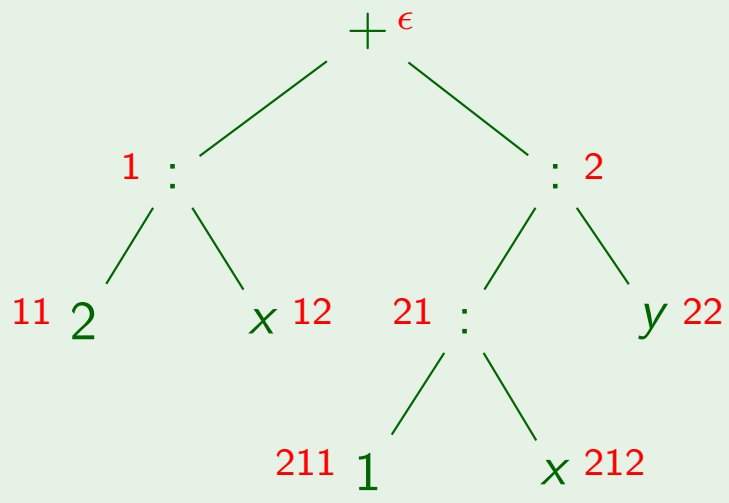

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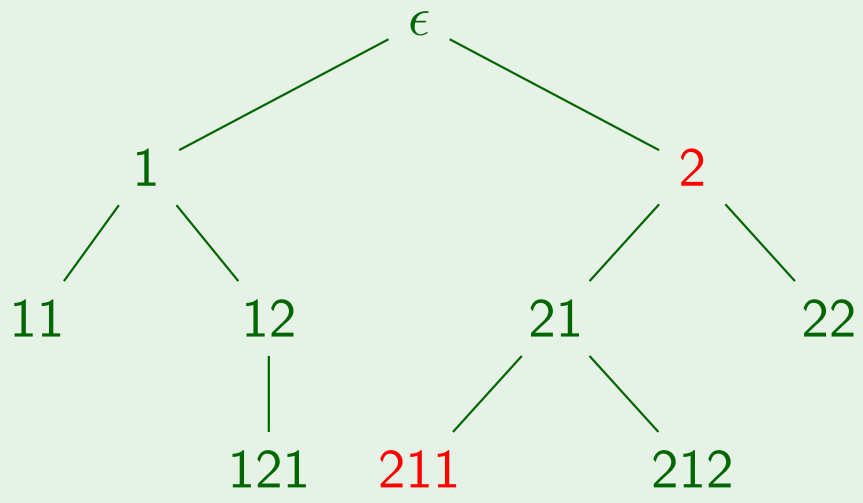

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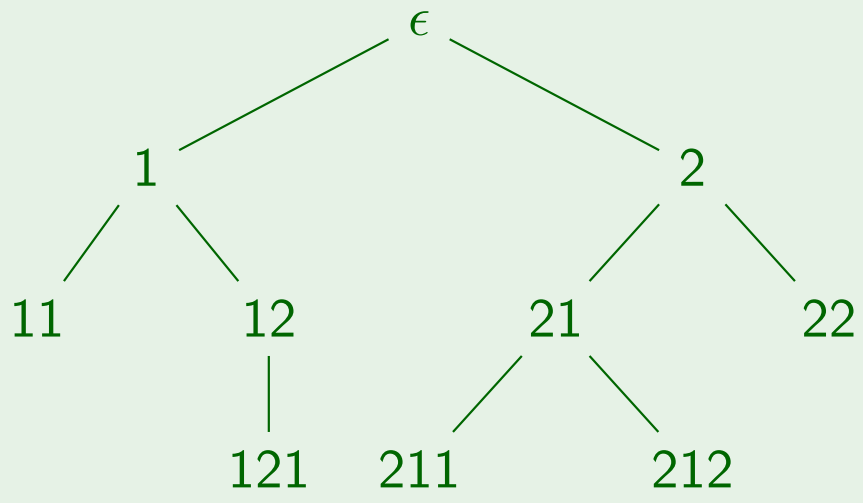

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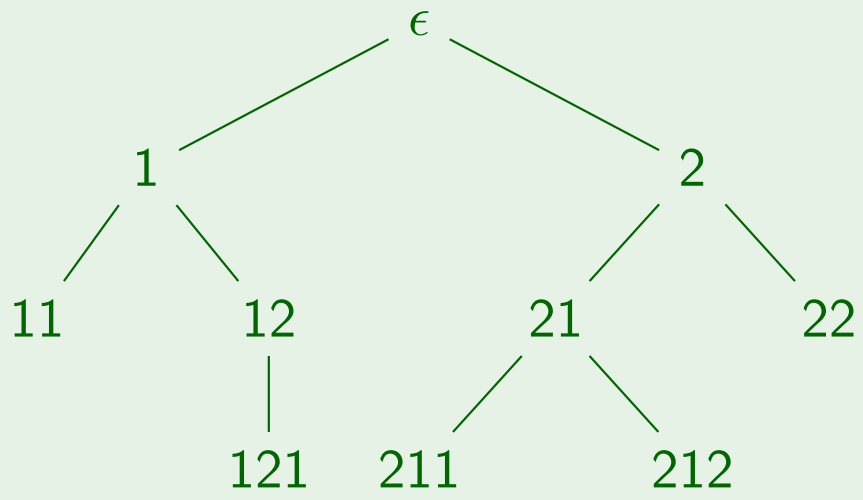

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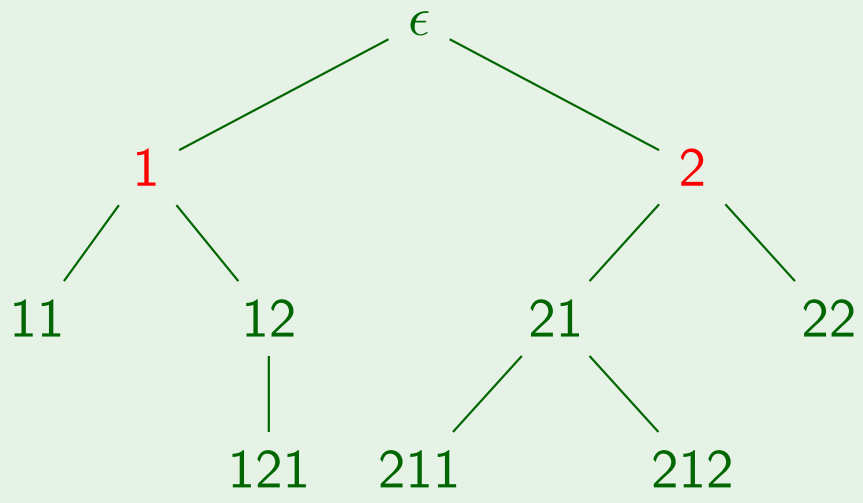

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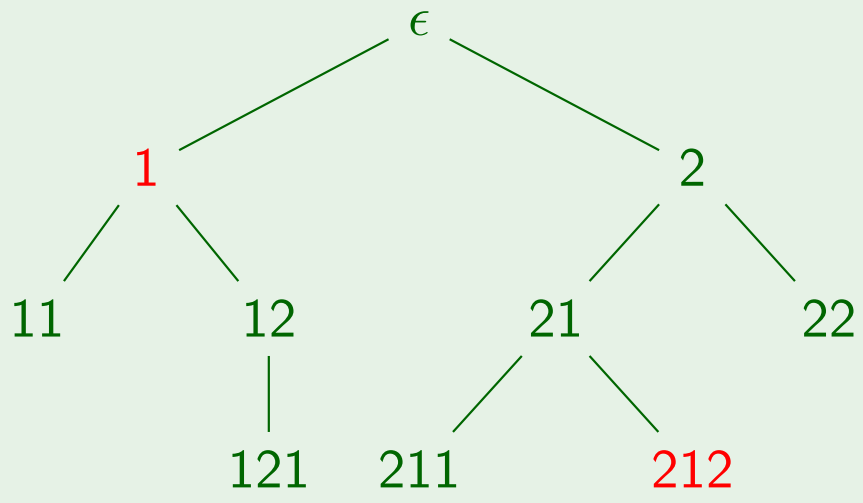

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- $t|_p$ subterm of t at position p

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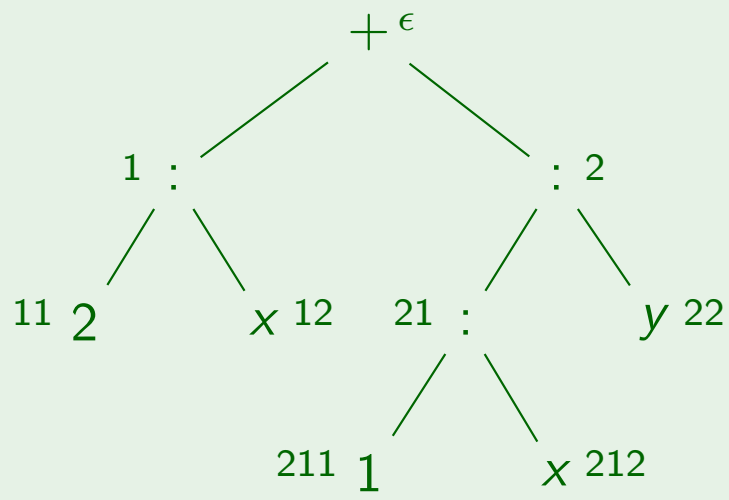

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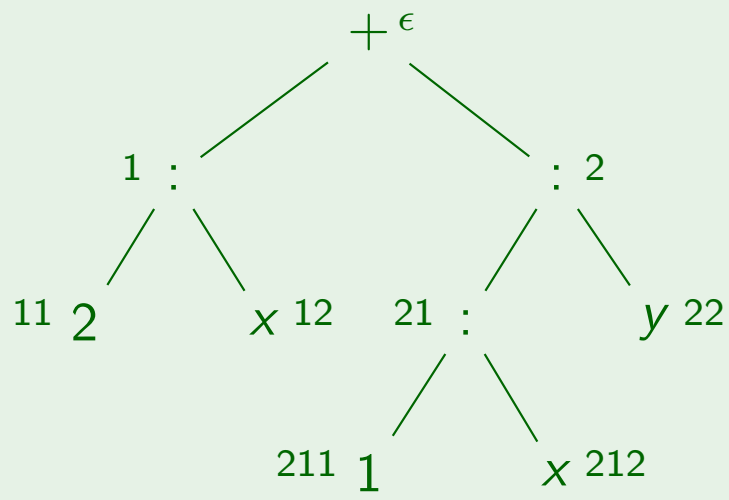

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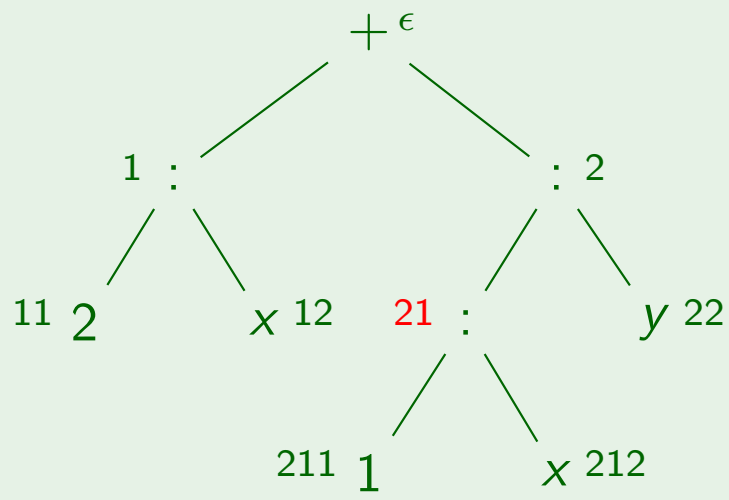

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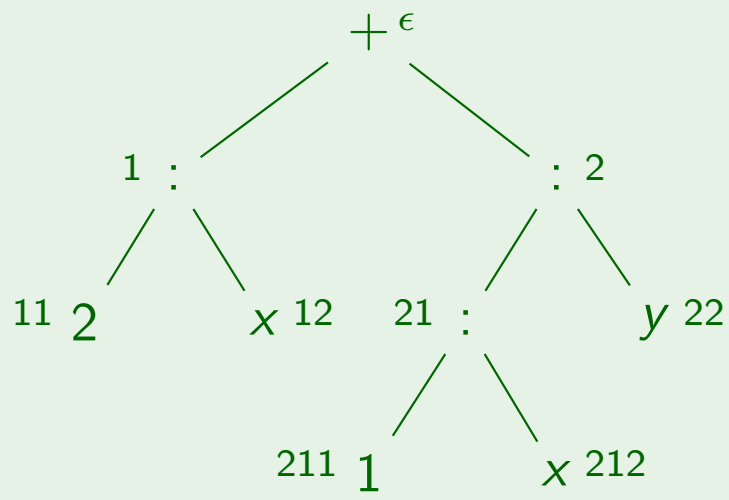

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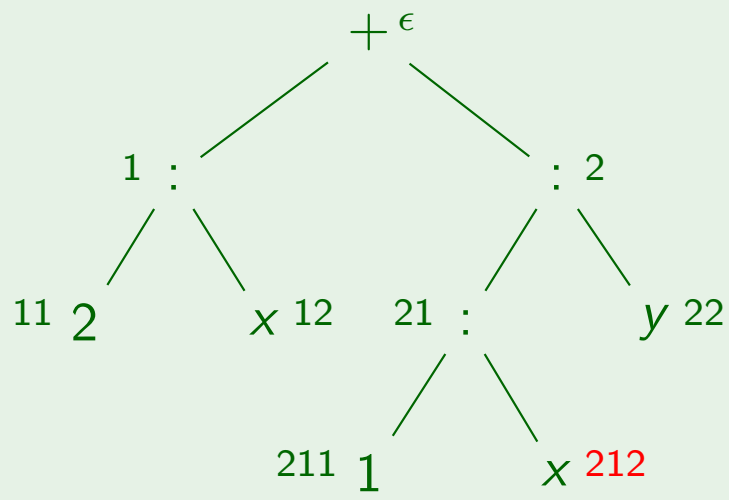

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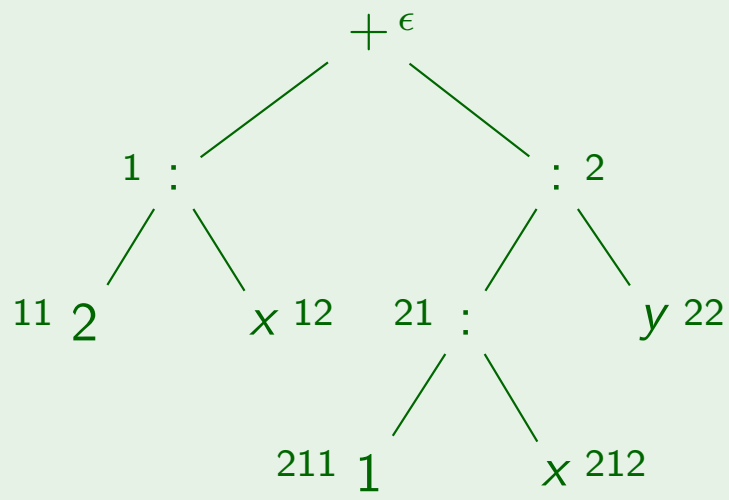

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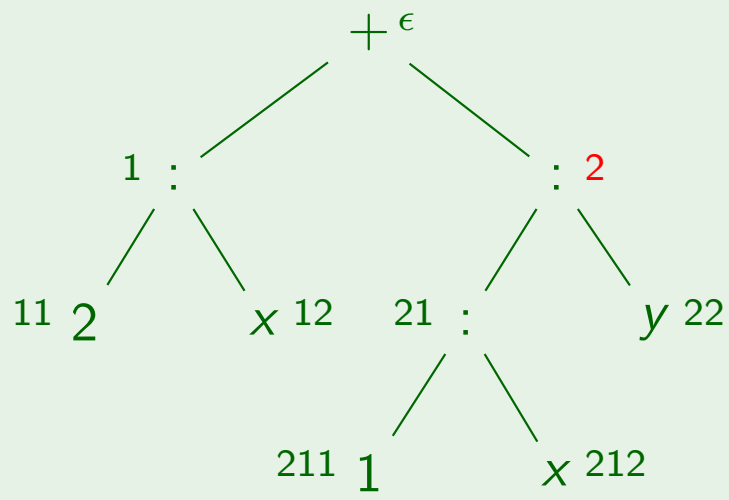

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Example

TRS (Σ, R) with signature Σ

$\mathsf{0}$ (constant) s (unary) add (binary)

$\#(\mathsf{0}) = 0$, $\#(\mathsf{s}) = 1$, $\#(\mathsf{add}) = 2$, and rewrite rules R

$$\mathsf{add}(\mathsf{0}, y) \to y$$
$$\mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

Definition (Term Rewriting)

Let $\mathcal{R} = (\Sigma, R)$ be a TRS. The rewrite relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\Sigma, \mathcal{X})$ is defined as:

$$C[\ell\sigma] \to_{\mathcal{R}} C[r\sigma]$$

for every:

- rule $\ell \to r \in R$,
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Example

The rule $\mathsf{add}(0, y) \to y$ with $y \mapsto \mathsf{s}(0)$ and $C = \mathsf{add}(0, \Box)$ gives rise to the step:

$$\mathsf{add}(0, \mathsf{add}(0, \mathsf{s}(0))) \to \mathsf{add}(0, \mathsf{s}(0))$$

at position 2.

Lemma

The rewrite relation $\rightarrow_{\mathcal{R}}$ is the smallest relation on $\mathcal{T}(\Sigma, \mathcal{X})$ that:

- *contains the rules R,*
- *is closed under context, and*
- *is closed under substitutions.*

That is, $\rightarrow_{\mathcal{R}}$ is the closure of R under contexts and substitutions.

Properties of Term Rewriting Systems

We consider TRSs $\mathcal{R} = (\Sigma, R)$ as

$$\text{ARSs } \langle \mathcal{T}(\Sigma, \mathcal{X}),\ \to_{\mathcal{R}} \rangle$$

TRSs inherit the properties: CR, WCR, SN, WN, NF, UN, $\text{UN}^{\to}$, ...

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Example

TRS R modeling Sieve of Eratostheness for generating list of prime numbers

$$\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y)\\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x))\\
\mathsf{head}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w)\\
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- how to compute normal forms in R ? strategy (lecture 9)

Example (Combinatory Logic)

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

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$$\mathsf{I} \cdot x \to x \qquad \mathsf{I}\, x \to x$$
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- applicative notation: surpress $\cdot$

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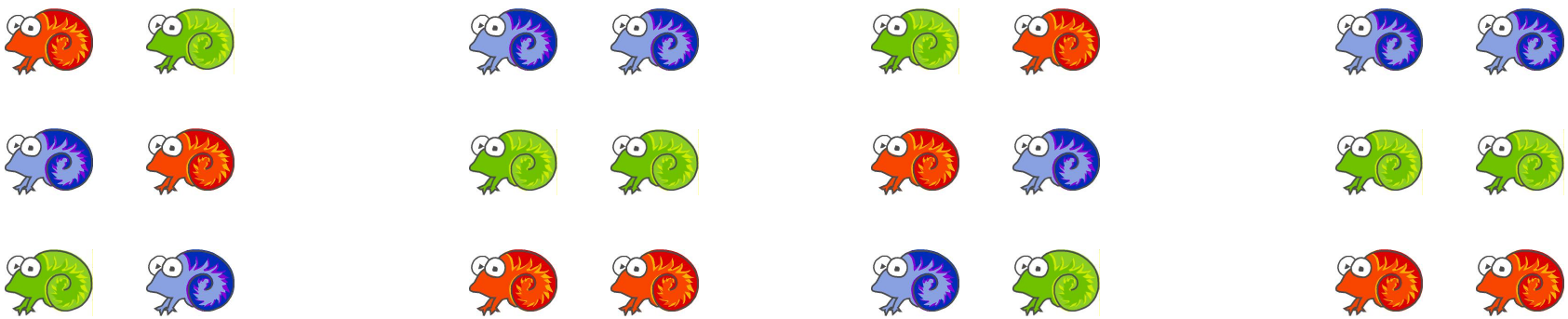
- CL is consistent

$$\mathsf{S} \not\to^* \mathsf{K}$$

Definition

TRS R over signature Σ is string rewrite system (SRS) if Σ consists of unary function symbols



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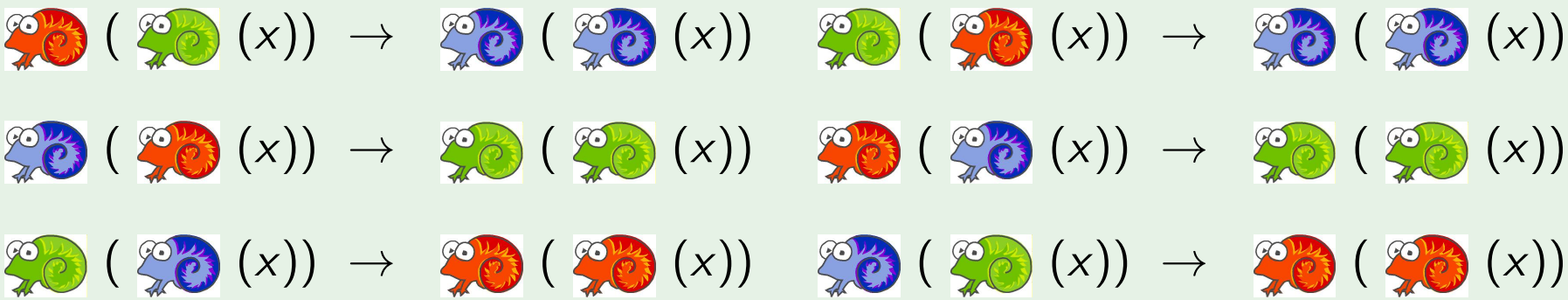

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