Term Rewriting Systems

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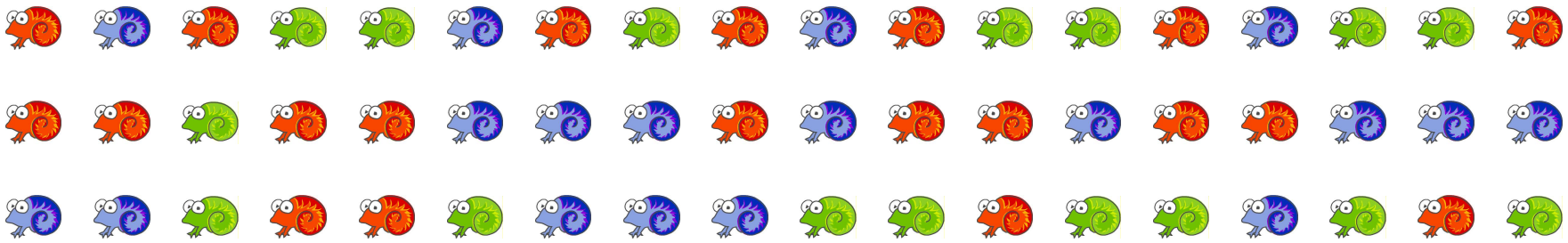
- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

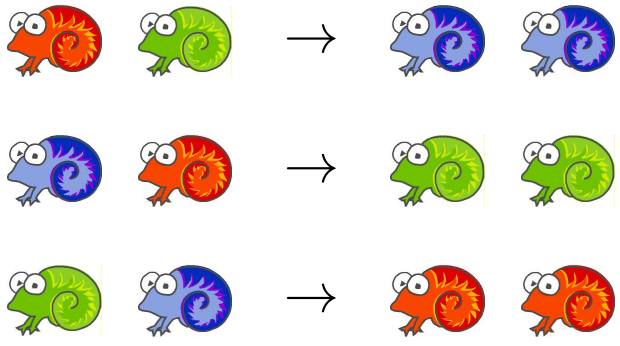
Outline

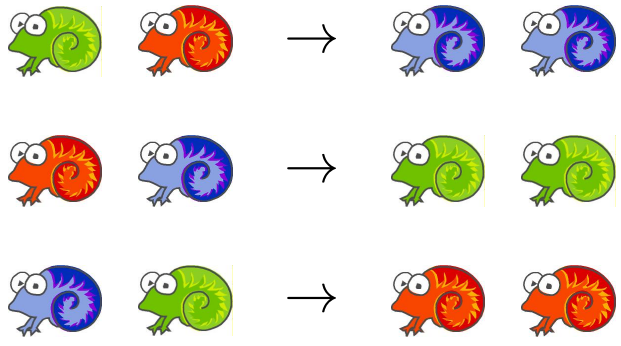
- Overview
- Examples
- Abstract Rewrite Systems
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Examples



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?





A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

$$\text{TAGCTAGCTAGCT}$$

in every fertilized egg into the cola gene

$$\text{CTGACTGACT}$$

Techniques exist to perform the following DNA substitutions

$$\text{TCAT} \leftrightarrow \text{T} \quad \text{GAG} \leftrightarrow \text{AG} \quad \text{CTC} \leftrightarrow \text{TC} \quad \text{AGTA} \leftrightarrow \text{A} \quad \text{TAT} \leftrightarrow \text{CT}$$

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

$$\text{CTGCTACTGACT}$$

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature 0 (constants) s (unary) $+$ (binary, infix)

terms $\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(x)+y$

rewrite rules $0+y \to y$

$\mathsf{s}(x)+y \to \mathsf{s}(x+y)$

rewriting $\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0+\mathsf{s}(\mathsf{s}(0)))$ $y \mapsto \mathsf{s}(\mathsf{s}(0))$

$\to \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

Example (Group Theory)

signature e (constant) $^{-}$ (unary, postfix) $\cdot$ (binary, infix)

equations $\mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^{-} \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^{-} \approx_{\mathcal{E}} y^{-} \cdot x^{-}$

rewrite rules R

$$\begin{array}{rcl@{\qquad}rcl} \mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\ x^{-} \cdot x & \to & \mathsf{e} & x \cdot x^{-} & \to & \mathsf{e} \\ (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\ \mathsf{e}^{-} & \to & \mathsf{e} & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} \\ x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y \end{array}$$

① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

② R admits no infinite computations

① & ② $\implies$ $\mathcal{E}$ has decidable validity problem

Example (Combinatory Logic)

signature $\quad \mathsf{S} \quad \mathsf{K} \quad \mathsf{I}$ (constants) $\quad \cdot$ (application, binary, infix)

terms $\quad \mathsf{S} \quad ((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S} \quad (x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

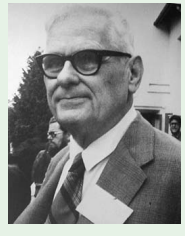
rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

inventor $\quad$ Moses Schönfinkel (1924)

Example (Lambda Calculus)

signature	λ (binds variables) $\cdot$ (application, binary, infix)
terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
α conversion	$\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$
β reduction	$(\lambda x.\, M) \cdot N \to_\beta M[x := N]$
	replace free occurrences of x in M by N
rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$
inventor	Alonzo Church (1936)



Motivation

Term rewriting is used in:

- functional programming (higher order term rewriting)
- model checking (e.g. mCRL)
- compiler construction (graph rewriting)
- computer algebra systems (e.g. Mathematica, Wolfram Alpha)
- proof assistants / automated theorem provers
- deciding equality in equational systems (axiom systems)
- abstract model of computation
- ...

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  - Definitions
  - Properties
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Abstract Rewrite Systems

Motivation

concrete rewrite formalisms

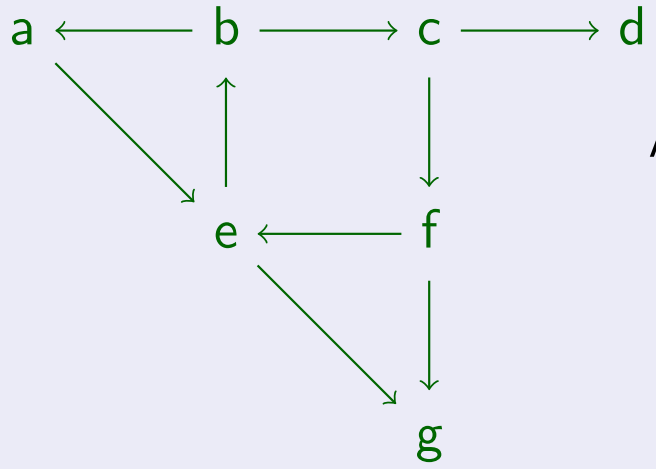
- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- ...

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$



ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow = \left\{ \begin{matrix} (a,e),(b,a),(b,c),(c,d),(c,f) \\ (e,b),(e,g),(f,e),(f,g) \end{matrix} \right\}$

- rewrite sequence

rewrite step

• finite	$a \rightarrow e \rightarrow b \rightarrow c \rightarrow f$	length 4	$a \rightarrow^* f$
• empty	a	length 0	$a \rightarrow^* a$
• infinite	$a \rightarrow e \rightarrow b \rightarrow a \rightarrow e \rightarrow b \rightarrow \cdots$	length ω	

The length of a rewrite sequence is the number of rewrite steps.
We write $x \rightarrow^* y$ if x rewrites to y in 0 or more steps.

Definition (Derived Relations of $\to$)

- $\leftarrow$ or $\to^{-1}$ inverse of $\to$
- $\to^{=}$ reflexive closure of $\to$
- $\to^{+}$ transitive closure of $\to$
- $\to^{*}$ or $\twoheadrightarrow$ transitive and reflexive closure of $\to$
- ${}^{*}\leftarrow$ or $\twoheadleftarrow$ inverse of $\to^{*}$ (transitive and reflexive closure of $\leftarrow$)
- $\leftrightarrow$ symmetric closure of $\to$, that is, $\leftrightarrow = \to \cup \leftarrow$
- $\leftrightarrow^{*}$ conversion (equivalence relation generated by $\to$)
- $\downarrow$ joinability $\downarrow = \to^{*} \cdot {}^{*}\leftarrow$
- $\uparrow$ meetability $\uparrow = {}^{*}\leftarrow \cdot \to^{*}$

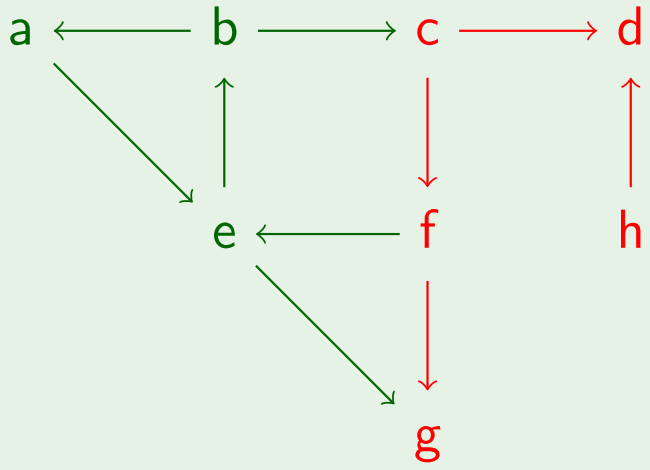
a relation R is

- reflexive if $a\ R\ a$ for all $a \in A$,
- transitive if $a\ R\ c$ whenenver $a\ R\ b$ and $b\ R\ c$,
- symmetric if $a\ R\ b$ whenenver $b\ R\ a$.

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \ {}^*\!\leftarrow y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are convertible

Example

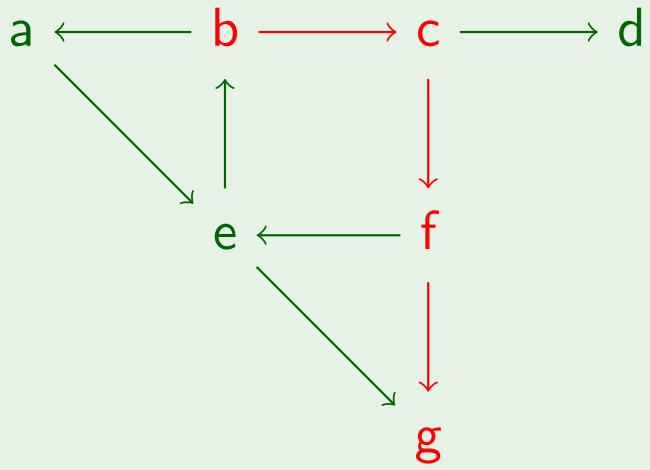


- $a \to^* f$
- $e \downarrow f \quad f \downarrow h \quad$ not $g \downarrow h$
- $g \leftrightarrow^* h$

Definition (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $\mathrm{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \to^{!} y$ if $x \to^{*} y$ for normal form y (x has normal form y)

Example



ARS $\mathcal{A} = \langle A, \to \rangle$

- d is normal form
- $\mathrm{NF}(\mathcal{A}) = \{\mathrm{d}, \mathrm{g}\}$
- $\mathrm{b} \to^{!} \mathrm{g}$

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Definitions

- SN strong normalization or termination
  - no infinite rewrite sequences
- WN weak normalization
  - every element has (rewrites to) at least one normal form
  - $\forall a \; \exists b \quad a \rightarrow^{!} b$

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN

Definitions

- NF normal form property
  - if an element a is convertible with a normal form b, then a rewrites to b
  - $\forall a, b$ if $a \leftrightarrow^* b$ and b is a normal form then $a \rightarrow^* b$
- UN unique normal forms
  - convertible normal forms are equal
  - $\forall a, b$ if $a \leftrightarrow^* b$ and a, b are normal forms then $a = b$
- $\mathrm{UN}^{\rightarrow}$ unique normal forms with respect to reduction
  - no element has more than one normal form
  - $\forall a, b, c$ if $a \rightarrow^! b$ and $a \rightarrow^! c$ then $b = c$
  - ${}^!\leftarrow \cdot \rightarrow^! \subseteq =$

Lemmata

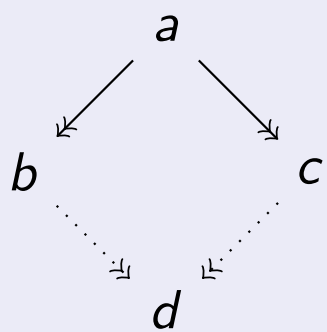
1. $\mathrm{SN} \Longrightarrow \mathrm{WN}$
2. $\mathrm{SN} \not\Longleftarrow \mathrm{WN}$

3. $\mathrm{NF} \Longrightarrow \mathrm{UN} \Longrightarrow \mathrm{UN}^{\rightarrow}$
4. $\mathrm{UN} \not\Longleftarrow \mathrm{UN}^{\rightarrow}$

5. $\mathrm{NF} \not\Longleftarrow \mathrm{UN}$

Definition

- CR confluence or Church-Rosser property
  - $\uparrow \subseteq \downarrow$
  - $\forall a, b, c$



in diagrams: $\twoheadrightarrow$ for $\rightarrow^*$

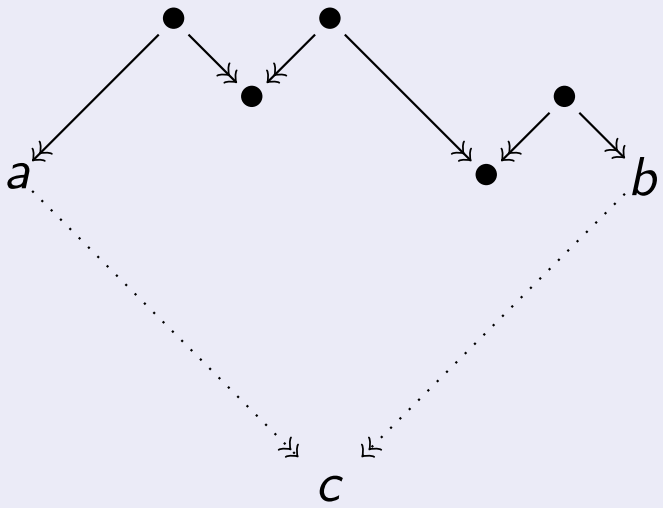
$\exists d$

  - $\forall a, b, c.\quad a \rightarrow^* b \ \wedge\ a \rightarrow^* c \quad \Rightarrow \quad \exists d.\quad b \rightarrow^* d \ \wedge\ c \rightarrow^* d$

Lemma (An Equivalent Formulation of Confluence)

Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

- $\leftrightarrow^* \subseteq \downarrow$
- $\forall a, b$



$\exists c$

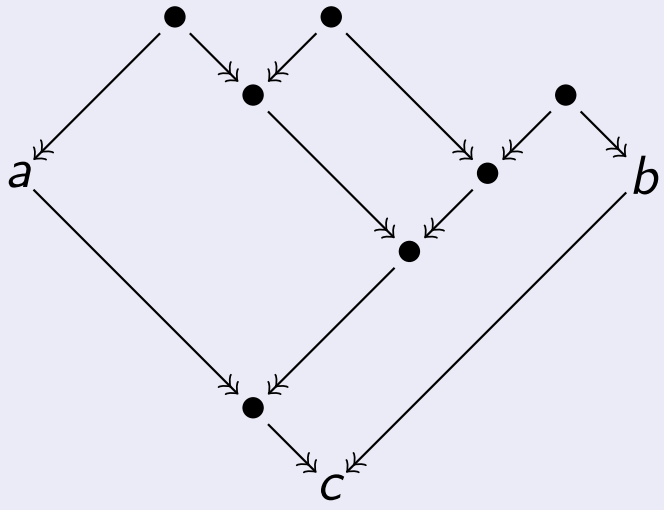
- $\forall a, b. \quad a \leftrightarrow^* b \quad \Rightarrow \quad \exists c. \quad a \rightarrow^* c \ \wedge \ b \rightarrow^* c$

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

First proof: diagram tiling.

- $\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.
- $\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We show $\leftrightarrow^* \subseteq \downarrow$. Therefore let $a, b \in A$ such that $a \leftrightarrow^* b$. Then $a \, (\leftarrow^* \cdot \rightarrow^*)^* \, b$ since $\leftrightarrow \subseteq \leftarrow^* \cdot \rightarrow^*$. Hence:



- we stepwise replace peaks $e \, {}^*\!\leftarrow d \rightarrow^* f$ by valleys $e \rightarrow^* \cdot \, {}^*\!\leftarrow f$
- after each step one peak less
- hence finally no peaks left $\Longrightarrow \quad a \downarrow b$

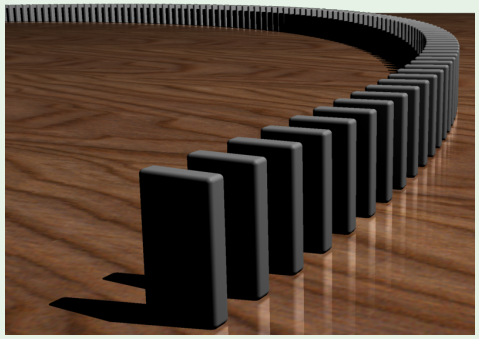
Induction

Lemma (Induction)

To prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$ do:

1. *The base case:*
   show that the statement holds for $n = 0$.
2. *The inductive step:*
   show for all n that if the $P(n)$ holds, then also $P(n+1)$ holds.

Example



Wikipedia

- Base case: proof that the first domino falls
- Induction step: proof that if the n-th domino falls then the $(n+1)$-st domino falls

Then you have proven that all dominoes will fall.

Example

We use induction to prove that:

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

1. Base case $n = 0$: Then

$$1 + 2 + \ldots + 0 = 0 \qquad \text{and} \qquad \frac{0 \cdot (0+1)}{2} = 0$$

   Thus the statement holds for $n = 0$.

2. Induction step:
   Induction hypothesis (IH): Assume the statement hold for n.

   We show it for $n + 1$:

$$1 + 2 + \ldots + n + (n+1) = \frac{n \cdot (n+1)}{2} + (n+1) \quad \text{by IH}$$
$$= \frac{n \cdot (n+1) + 2 \cdot (n+1)}{2} = \frac{(n+1) \cdot ((n+1)+1)}{2}$$

Hence the formula holds for all $n \in \mathbb{N}$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

$\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We show $\leftrightarrow^* \subseteq \downarrow$.

We proof by induction on n that $a\ (\leftarrow^* \cdot \rightarrow^*)^n\ b$ implies $a \rightarrow^* \cdot\ ^*\leftarrow b$.

- Base case $n = 0$: $a\ (\leftarrow^* \cdot \rightarrow^*)^0\ b$. Then $a = b$ and hence $a \rightarrow^* \cdot\ ^*\leftarrow b$.
- Induction step $n + 1$: (assume it holds for n, show it for $n + 1$)

  Let $a\ (\leftarrow^* \cdot \rightarrow^*)^{n+1}\ b$. Then $a\ (\leftarrow^* \cdot \rightarrow^*)^n\ d \leftarrow^* e \rightarrow^* b$ for some d, e.
  - Hence $a \rightarrow^* f\ ^*\leftarrow d$ for some f by induction hypothesis.
  - Now $f \leftarrow^* e \rightarrow^* b$ and thus $f \rightarrow^* \cdot\ ^*\leftarrow b$ since by assumption $\uparrow \subseteq \downarrow$.

  We conclude $a \rightarrow^* \cdot\ ^*\leftarrow b$, that is, $a \downarrow b$.

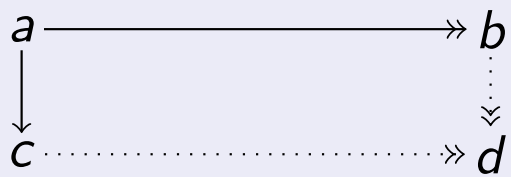
Hence we have shown $\uparrow^* \subseteq \downarrow$. It follows $\leftrightarrow^* \subseteq \downarrow$ since $\leftrightarrow^* \subseteq \uparrow^*$. ■

Lemma

Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$
- $\forall a, b, c$

  $\exists d$



- $\forall a, b, c. \quad a \rightarrow^* b \ \wedge \ a \rightarrow c \quad \Rightarrow \quad \exists d. \quad b \rightarrow^* d \ \wedge \ c \rightarrow^* d$

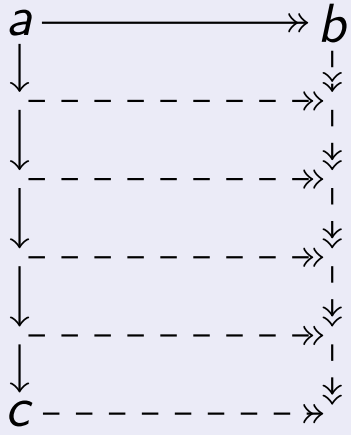
Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



By induction on n we show: ${}^n\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^0\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$:

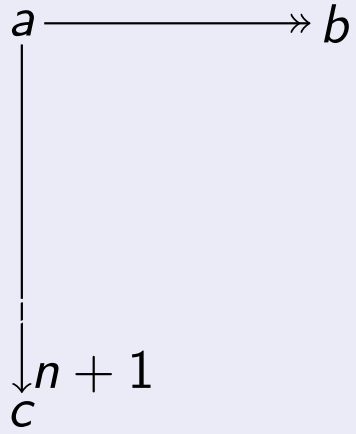
Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



By induction on n we show: ${}^n\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^0\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$: let $c \; {}^{n+1}\leftarrow a \rightarrow^* b$.

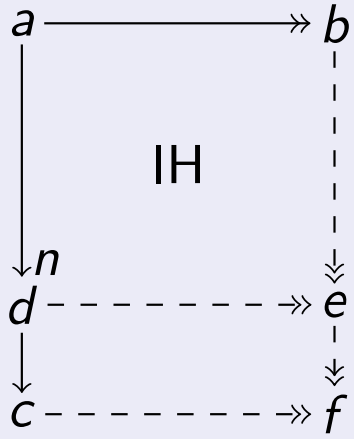
Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

$\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



By induction on n we show: ${}^{n}\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n+1$: let $c \; {}^{n+1}\leftarrow a \rightarrow^* b$.

  Then $c \leftarrow d \; {}^{n}\leftarrow a \rightarrow^* b$ for some d, and:

  - By induction hypothesis $d \rightarrow^* e \; {}^*\leftarrow b$.
  - Then $c \rightarrow^* f \; {}^*\leftarrow e$ since $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

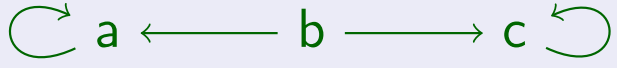
  Hence $c \rightarrow^* \cdot \; {}^*\leftarrow b$.

Lemmata

1. $\mathrm{SN} \Longrightarrow \mathrm{WN}$
2. $\mathrm{SN} \not\Longleftarrow \mathrm{WN}$

3. $\mathrm{CR} \Longrightarrow \mathrm{NF} \Longrightarrow \mathrm{UN} \Longrightarrow \mathrm{UN}^{\rightarrow}$
4. $\mathrm{UN} \not\Longleftarrow \mathrm{UN}^{\rightarrow}$

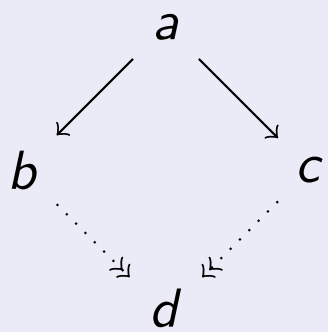
5. $\mathrm{NF} \not\Longleftarrow \mathrm{UN}$

6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

7. $\mathrm{CR} \Longleftrightarrow \leftrightarrow^* \subseteq \downarrow \Longleftrightarrow \leftrightarrow^* = \downarrow \Longleftrightarrow \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathrm{WN}\ \&\ \mathrm{UN}^{\rightarrow} \Longrightarrow \mathrm{CR}$

Definitions

- WCR local confluence or weak Church-Rosser property
  - $\leftarrow \cdot \rightarrow \subseteq \downarrow$
  - $\forall a, b, c$



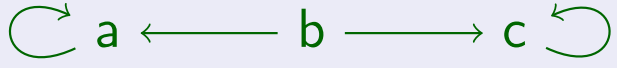
$\exists d$

Lemmata

1. $SN \Longrightarrow WN$
2. $SN \not\Longleftarrow WN$

3. $CR \Longrightarrow NF \Longrightarrow UN \Longrightarrow UN^{\rightarrow}$
4. $UN \not\Longleftarrow UN^{\rightarrow}$

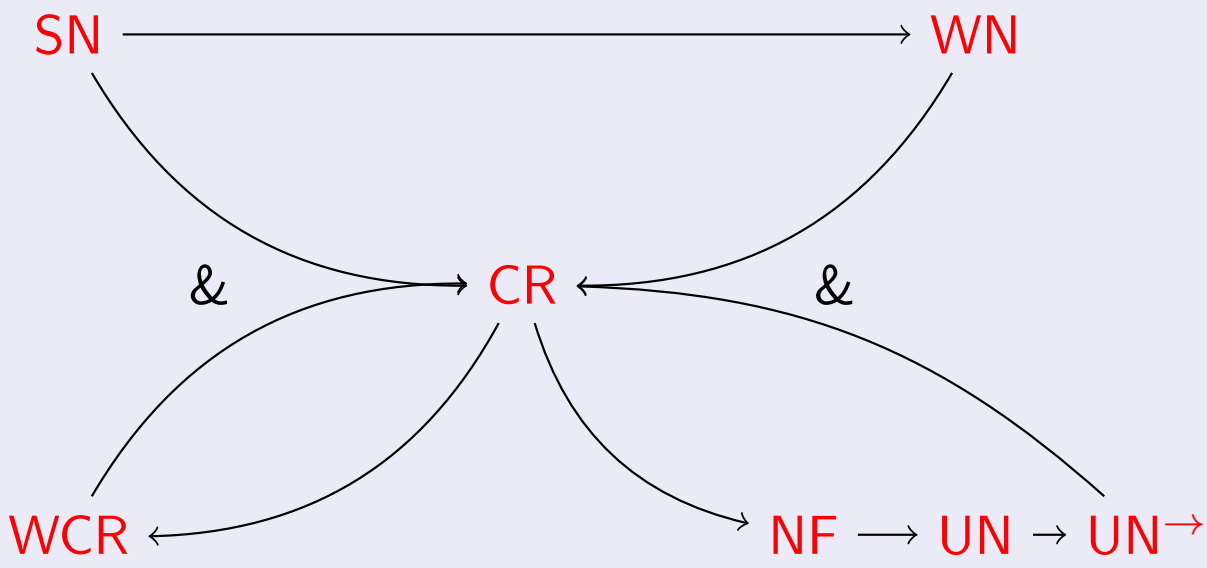
5. $NF \not\Longleftarrow UN$

6. $CR \not\Longleftarrow NF$ 

7. $CR \Longleftrightarrow {\leftrightarrow}^* \subseteq {\downarrow} \Longleftrightarrow {\leftrightarrow}^* = {\downarrow} \Longleftrightarrow {\leftarrow} \cdot {\rightarrow}^* \subseteq {\downarrow}$
8. $WN \ \& \ UN^{\rightarrow} \Longrightarrow CR$
9. $CR \Longrightarrow WCR$
10. $CR \not\Longleftarrow WCR$

11. $SN \ \& \ WCR \Longrightarrow CR$ Newman's Lemma

Summary

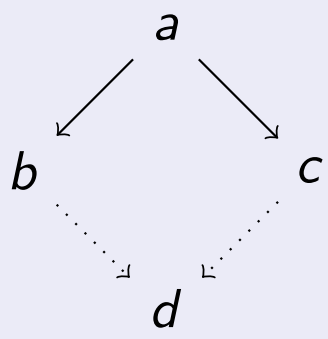


Definitions

- semi-completeness
  - CR & WN
  - every element has unique normal form
- completeness
  - CR & SN

Definition

- diamond property $\diamond$
  - $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
  - $\forall a, b, c$



$\exists d$

Lemma

An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow$ has the diamond property.

Proof.

Exercise. ∎

Lemma

ARS $\mathcal{A} = \langle A, \to_1 \rangle$ *is confluent if*

- $\to_1 \subseteq \to_2 \subseteq \to_1^*$

for a confluent relation $\to_2$ *on* A.

Proof.

Assume $\to_2$ is confluent, that is, ${}^*_2\!\leftarrow \cdot \to_2^* \subseteq \to_2^* \cdot {}^*_2\!\leftarrow$.

- From $\to_1 \subseteq \to_2$ follows $\to_1^* \subseteq \to_2^*$.
- Moreover $\to_2^* \subseteq \to_1^*$ since $\to_1^*$ is transitive and contains $\to_2$.

Hence $\to_1^* = \to_2^*$.

$\Longrightarrow \quad {}^*_1\!\leftarrow \cdot \to_1^* \subseteq \to_1^* \cdot {}^*_1\!\leftarrow$

Outline

- Overview
- Examples
- Abstract Rewrite Systems
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Well-Founded Induction

given

- strongly normalizing ARS $\mathcal{A} = \langle A, \to \rangle$
- property P over the elements of $\mathcal{A}$

to conclude

- $\forall a \in A\colon \mathrm{P}(a)$

it is sufficient to prove

- if $\underbrace{\mathrm{P}(b) \text{ for every } b \text{ with } a \to b}_{\text{induction hypothesis}}$ then $\mathrm{P}(a)$

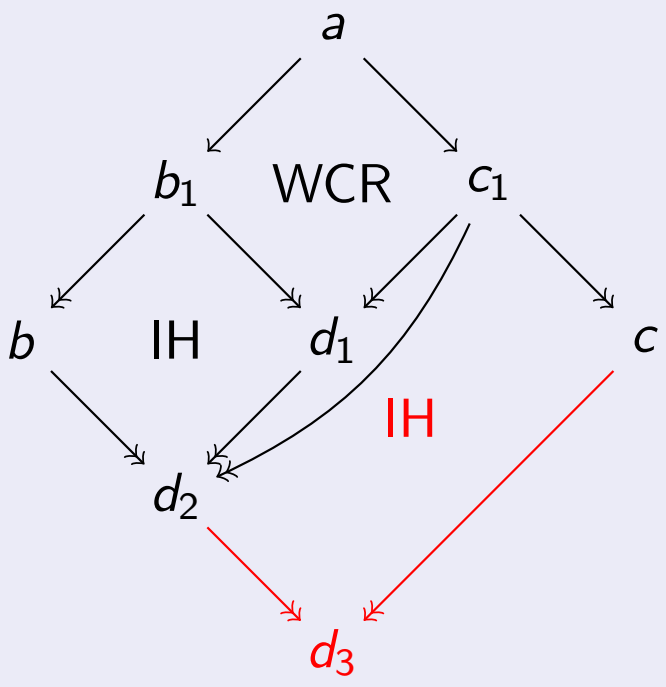
for arbitrary element a

$$\Big(\forall a\colon \big(\forall b\colon a \to b \implies \mathrm{P}(b)\big) \implies \mathrm{P}(a)\Big) \implies \forall a\colon \mathrm{P}(a)$$

Newman's Lemma

SN($\mathcal{A}$) & WCR($\mathcal{A}$) $\Longrightarrow$ CR($\mathcal{A}$)

Proof.



induction hypothesis CR(c_1)

$\forall a'$: if $a \to a'$ then CR(a')

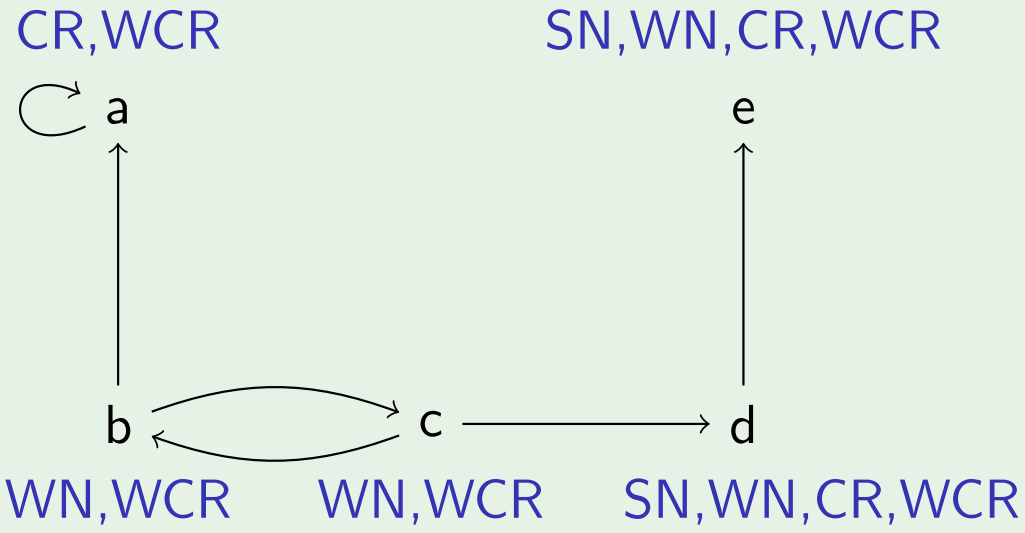
Definitions (Properties of Elements)

Let $\langle A, \to \rangle$ be an ARS. An element $a \in A$ is called:

- SN strongly normalizing or terminating
  if a admits no infinite rewrite sequence $a = a_1 \to a_2 \to \ldots$
- WN weakly normalizing
  if $\exists b.\ a \to^! b$
- CR confluent or Church Rosser
  if $\forall b, c.\ (c\ {}^*\!\leftarrow a \to^* b \ \Rightarrow\ \exists d.\ c \to^* d\ {}^*\!\leftarrow b)$
- WCR weakly confluent or weakly Church Rosser
  if $\forall b, c.\ (c \leftarrow a \to b \ \Rightarrow\ \exists d.\ c \to^* d\ {}^*\!\leftarrow b)$

An ARS has the property if all its elements have the respective property.

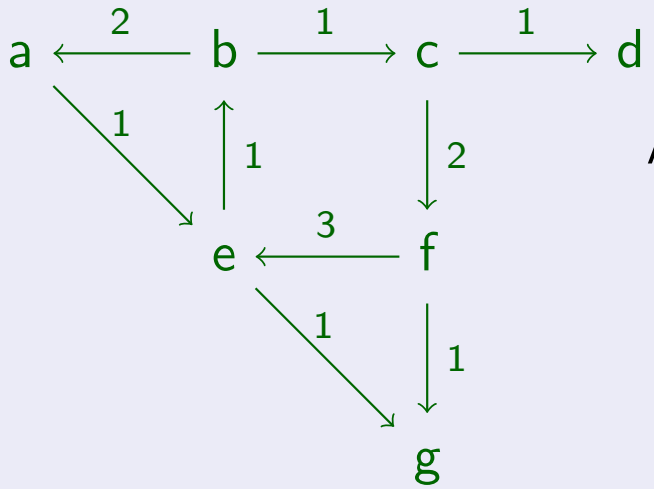
Example



ARSs with Multiple Relations

Definitions

- abstract rewrite system (ARS) is set A with binary relations $\rightarrow_i$ for $i \in \mathcal{I}$



ARS $\mathcal{A} = \langle A, \rightarrow_1, \rightarrow_2, \rightarrow_3 \rangle$

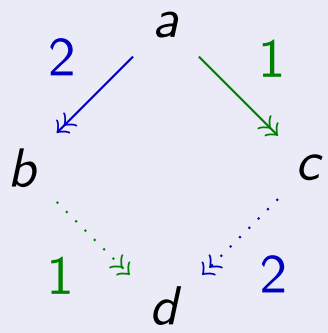
- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow_1 = \left\{ \begin{matrix} (a,e),(b,c),(c,d), \\ (e,b),(e,g),(f,g) \end{matrix} \right\}$
- $\rightarrow_2 = \{(b,a),(c,f)\}$
- $\rightarrow_3 = \{(f,e)\}$

- $\rightarrow = \rightarrow_1 \cup \rightarrow_2 \cup \rightarrow_3$
- $\rightarrow_{12} = \rightarrow_1 \cup \rightarrow_2, \quad \rightarrow_{13} = \rightarrow_1 \cup \rightarrow_3, \quad \ldots$

Definition

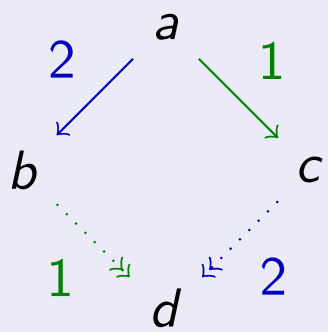
Let $\mathcal{A} = \langle A, \to_1, \to_2 \rangle$ be an ARS.

- $\to_1$ commutes with $\to_2$
  - ${}^*_2\!\leftarrow \cdot \to_1^* \subseteq \to_1^* \cdot {}^*_2\!\leftarrow$
  - $\forall a, b, c$



$\exists d$

- $\to_1$ commutes weakly with $\to_2$
  - ${}_2\!\leftarrow \cdot \to_1 \subseteq \to_1^* \cdot {}^*_2\!\leftarrow$
  - $\forall a, b, c$



$\exists d$