Term Rewriting Systems

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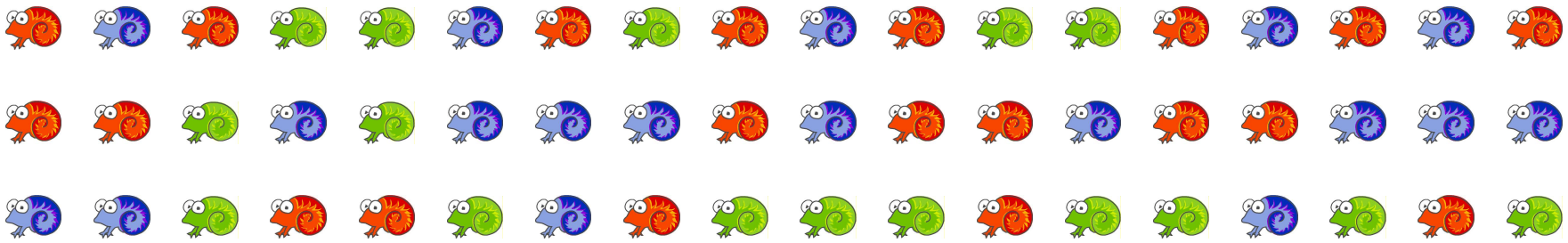
- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

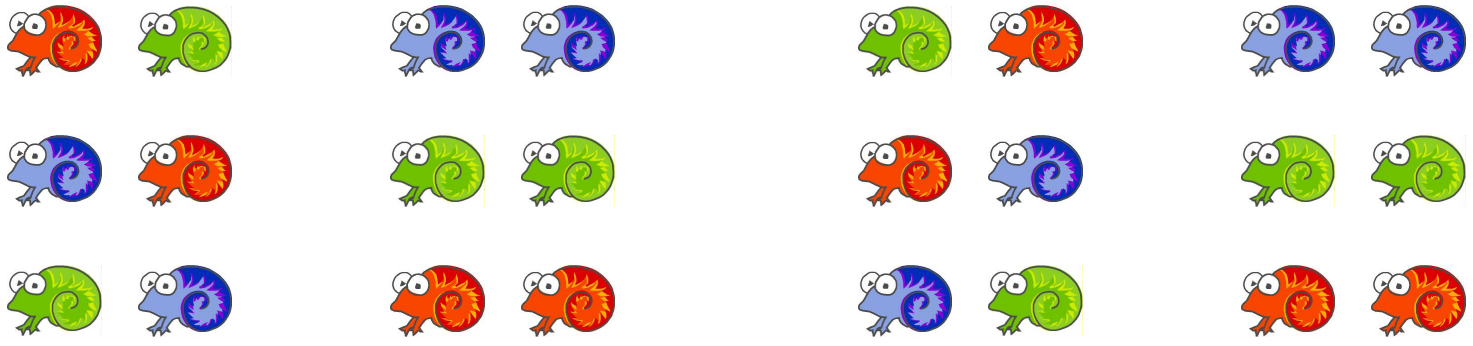
Outline

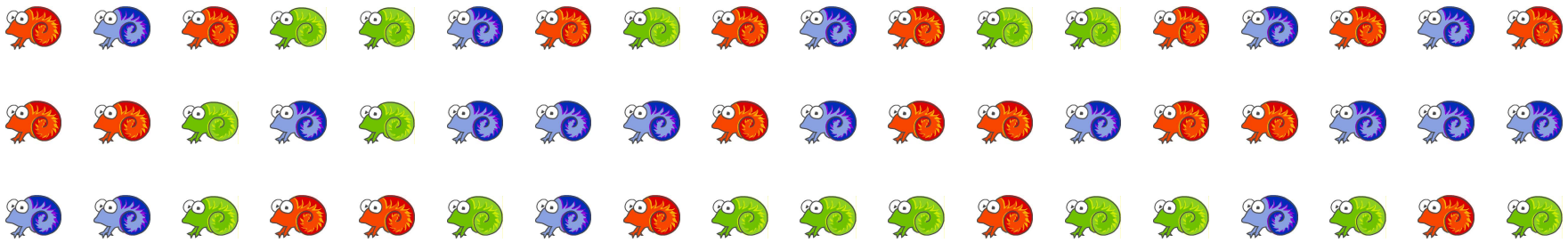
- Overview
- Examples
- Abstract Rewrite Systems
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Examples



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?





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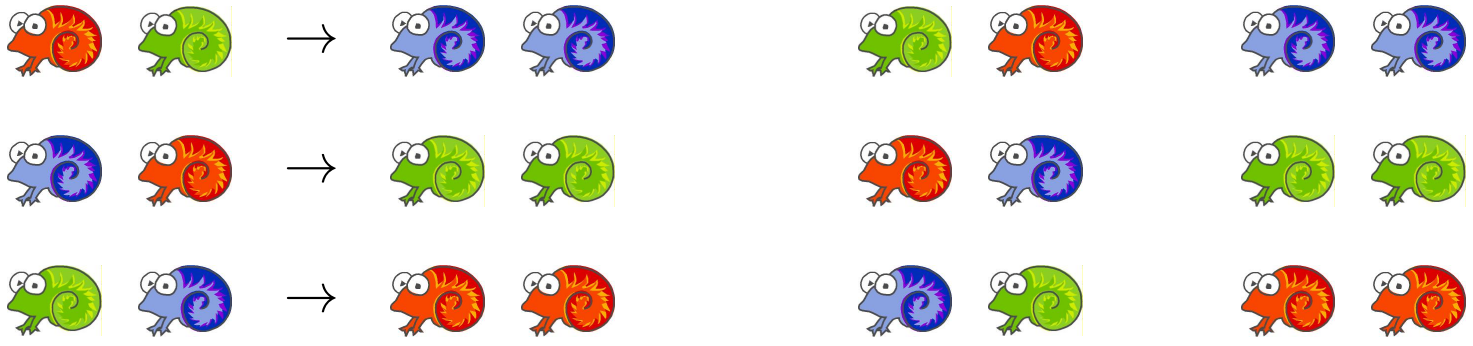

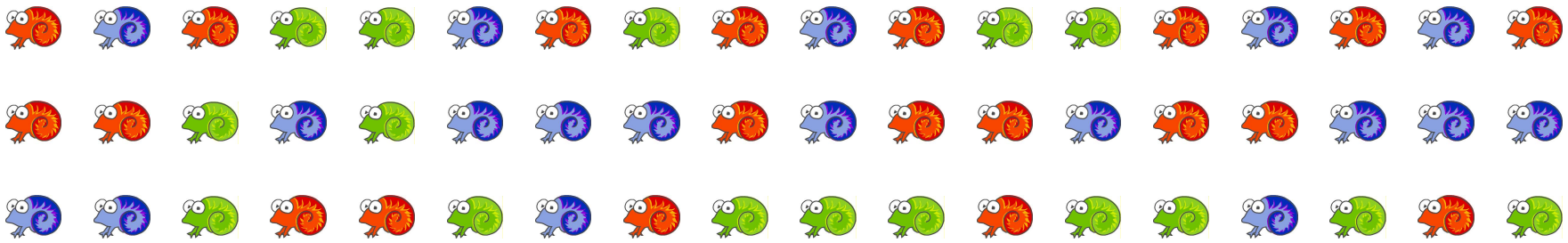

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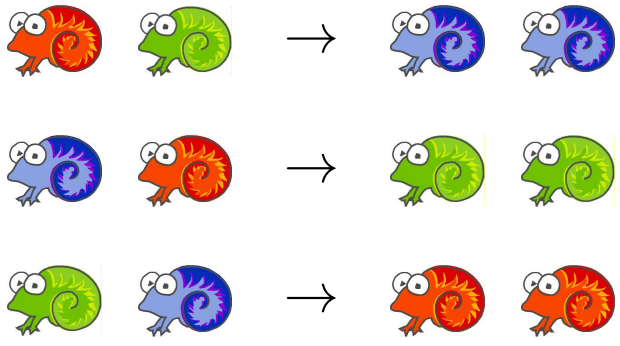

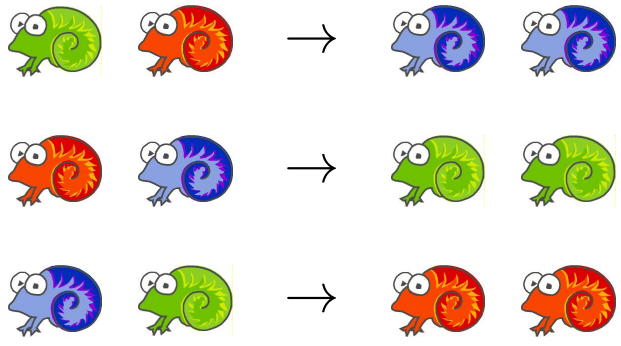

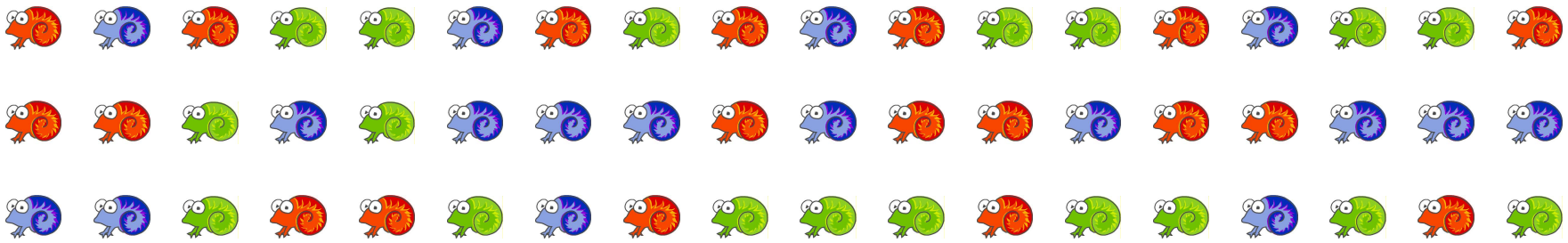

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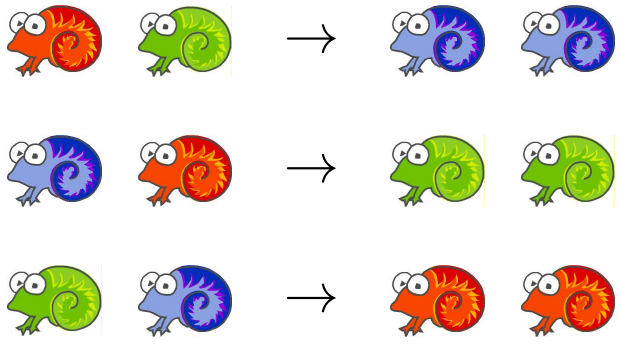

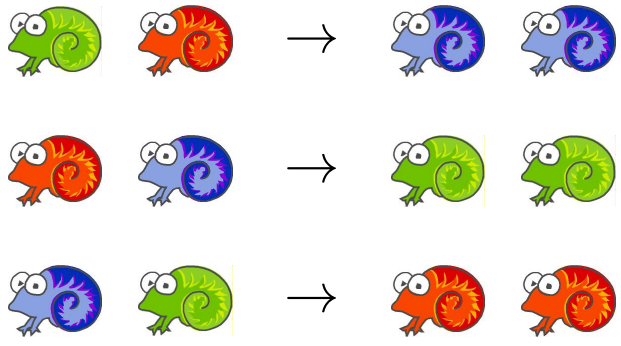

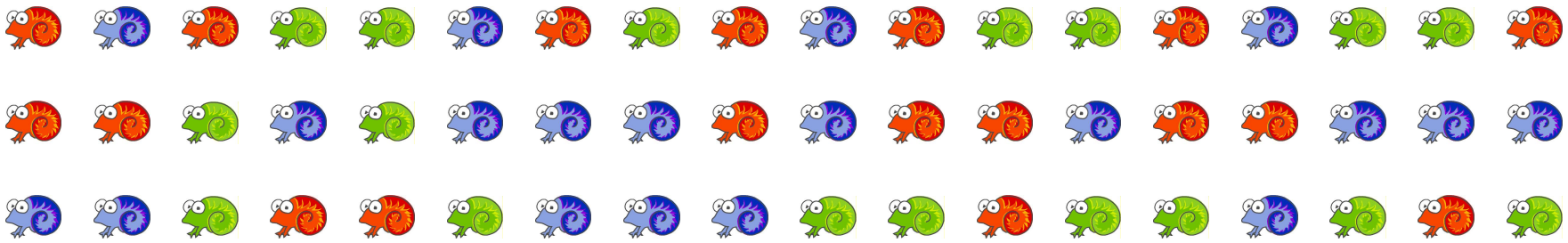

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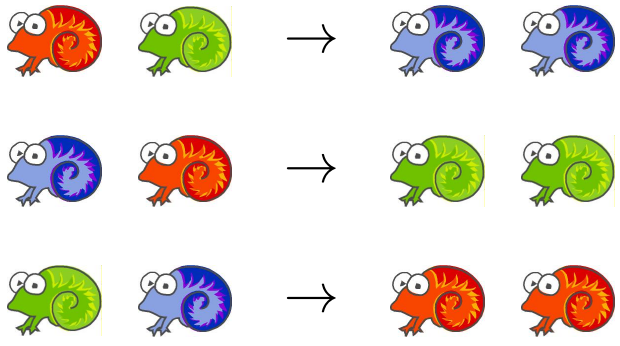

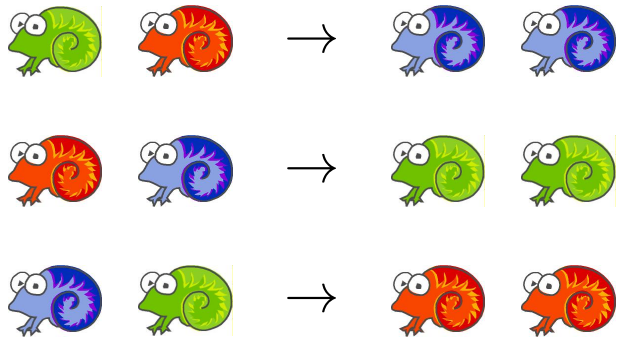

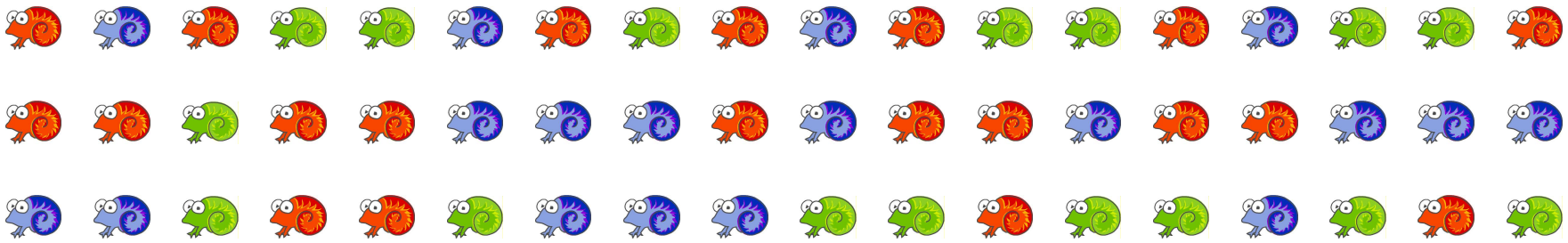

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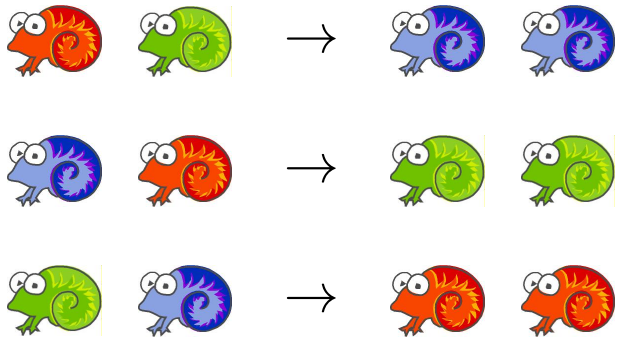

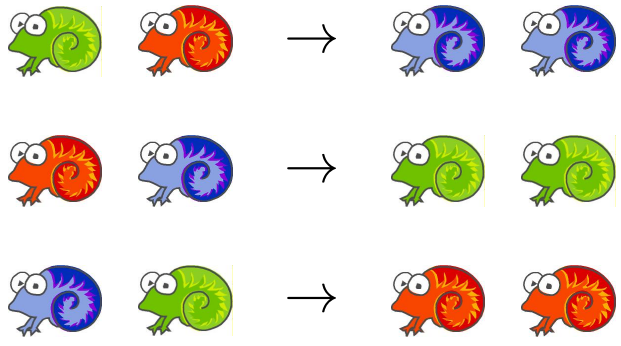

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Techniques exist to perform the following DNA substitutions

$$\text{TCAT} \leftrightarrow \text{T} \quad \text{GAG} \leftrightarrow \text{AG} \quad \text{CTC} \leftrightarrow \text{TC} \quad \text{AGTA} \leftrightarrow \text{A} \quad \text{TAT} \leftrightarrow \text{CT}$$

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Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

$$\text{CTGCTACTGACT}$$

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature 0 (constants) s (unary) $+$ (binary, infix)

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① & ② $\implies$ $\mathcal{E}$ has decidable validity problem

Example (Combinatory Logic)

signature S K I (constants)

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rewriting $\quad ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x$

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rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

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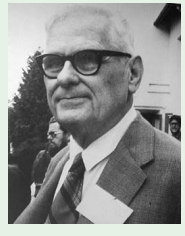
rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

inventor $\quad$ Moses Schönfinkel (1924)

Example (Lambda Calculus)

signature λ (binds variables)



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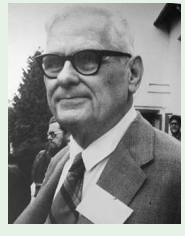

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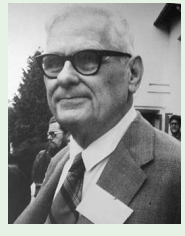
inventor	Alonzo Church (1936)



Motivation

Term rewriting is used in:

- functional programming (higher order term rewriting)
- model checking (e.g. mCRL)
- compiler construction (graph rewriting)
- computer algebra systems (e.g. Mathematica, Wolfram Alpha)
- proof assistants / automated theorem provers
- deciding equality in equational systems (axiom systems)
- abstract model of computation
- ...

Outline

- Overview
- Examples
- Abstract Rewrite Systems
  - Definitions
  - Properties
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Abstract Rewrite Systems

Motivation

concrete rewrite formalisms

- string rewriting

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- term rewriting

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abstract rewriting

- no structure on objects that are rewritten

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- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Definitions

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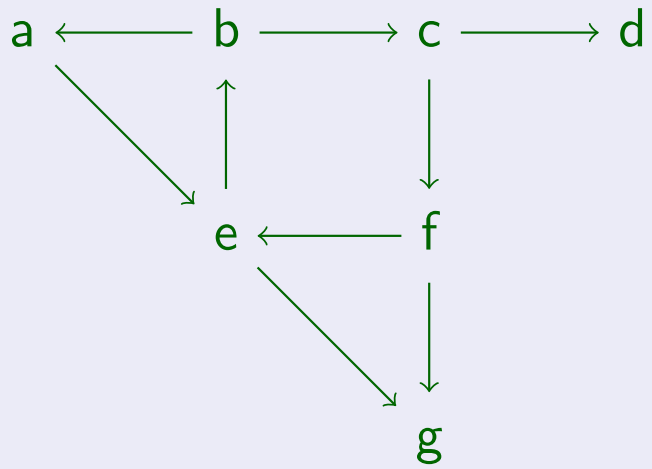

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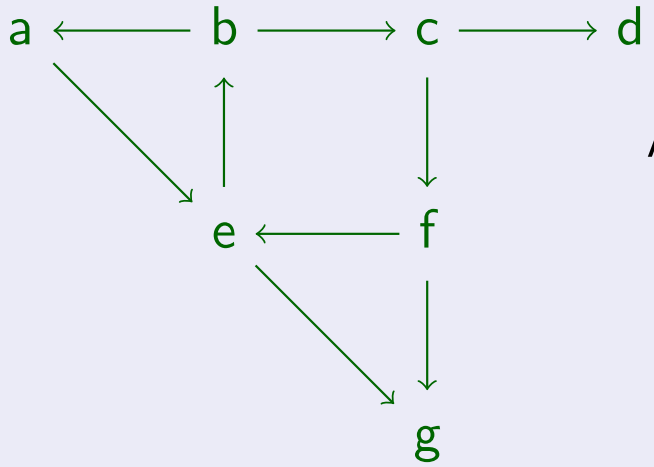

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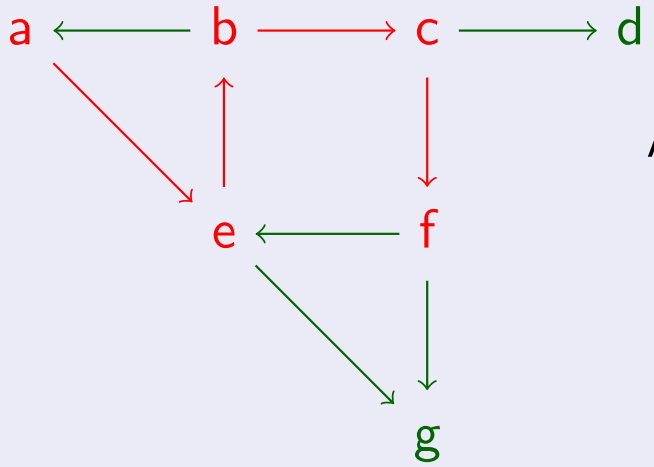

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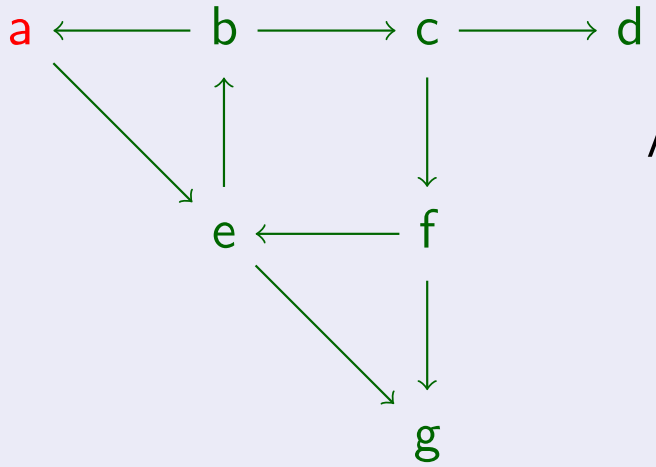

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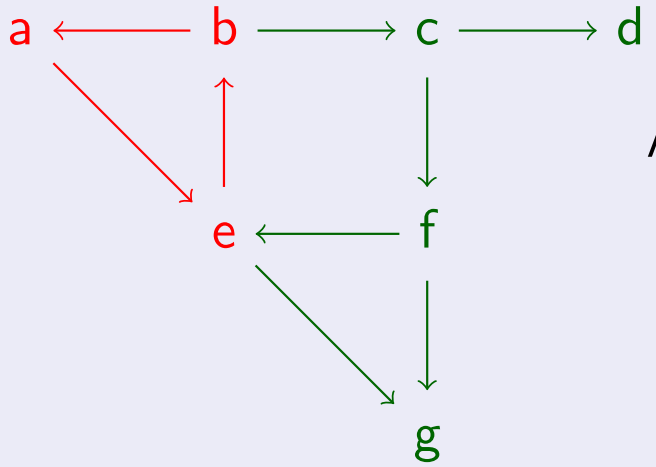

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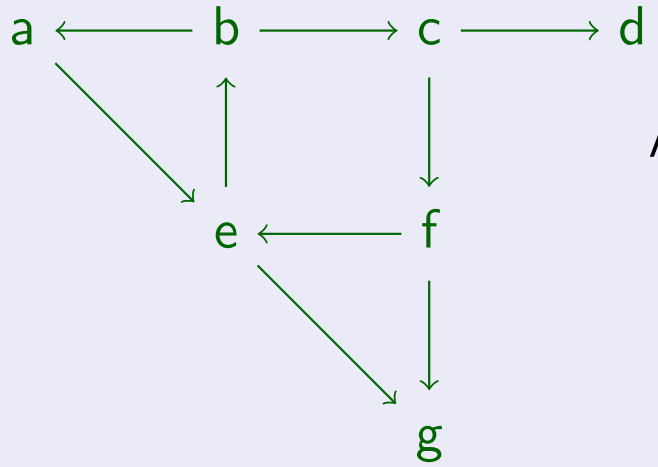

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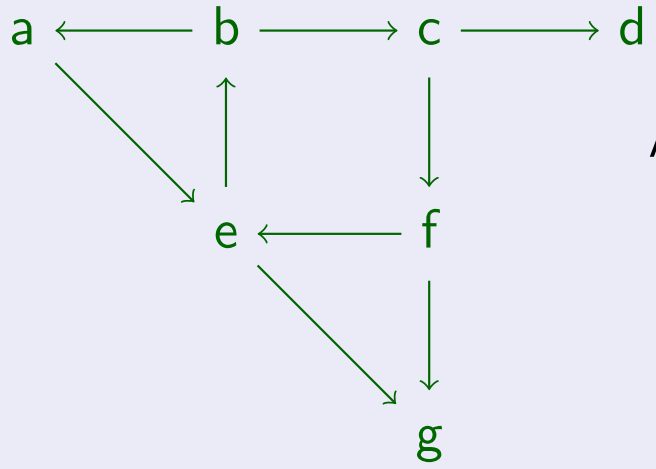

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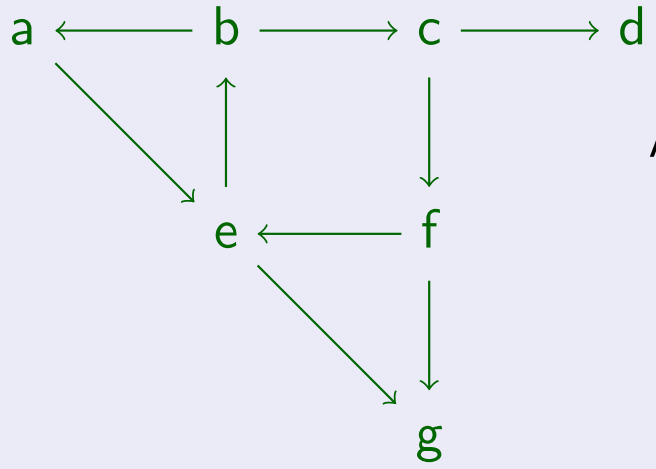

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• finite	$a \rightarrow e \rightarrow b \rightarrow c \rightarrow f$	length 4	$a \rightarrow^* f$
• empty	a	length 0	$a \rightarrow^* a$
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The length of a rewrite sequence is the number of rewrite steps.
We write $x \rightarrow^* y$ if x rewrites to y in 0 or more steps.

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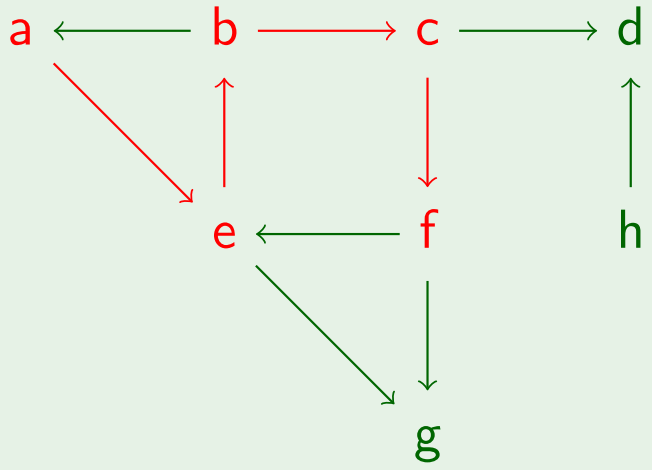

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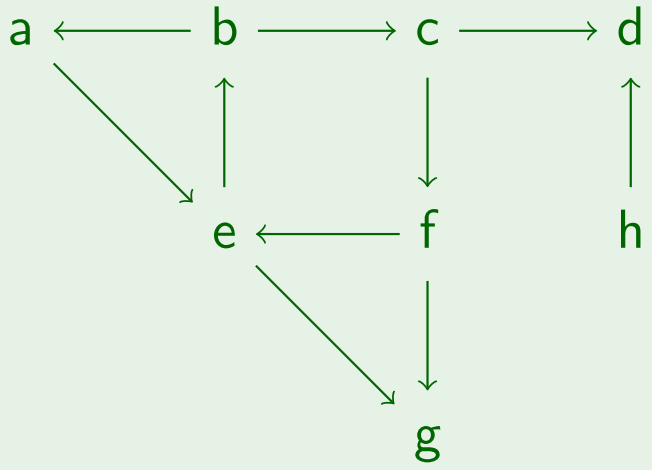

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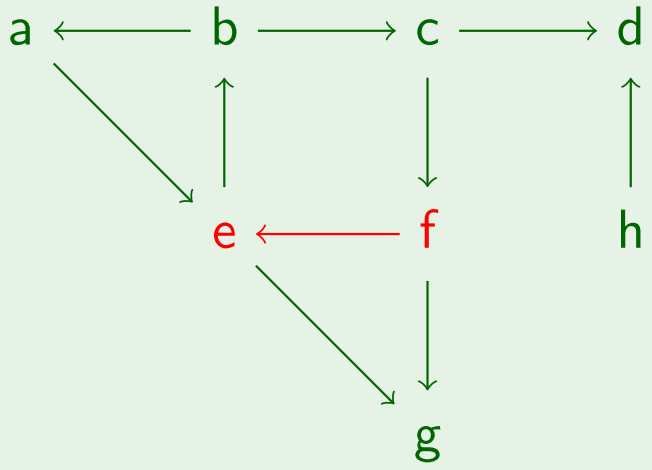

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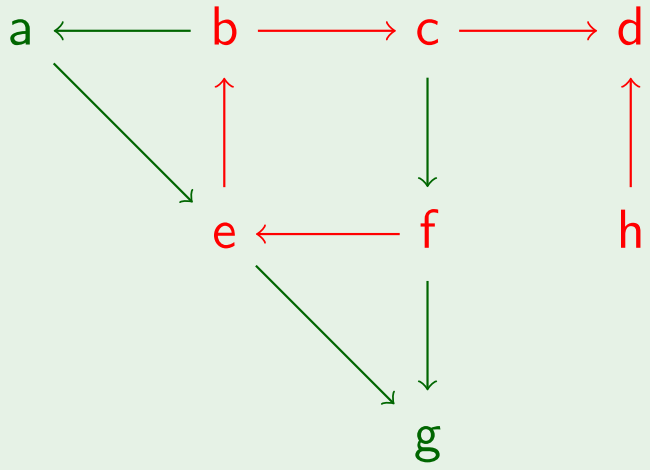

- $a \to^* f$
- $e \downarrow f \quad f \downarrow h$

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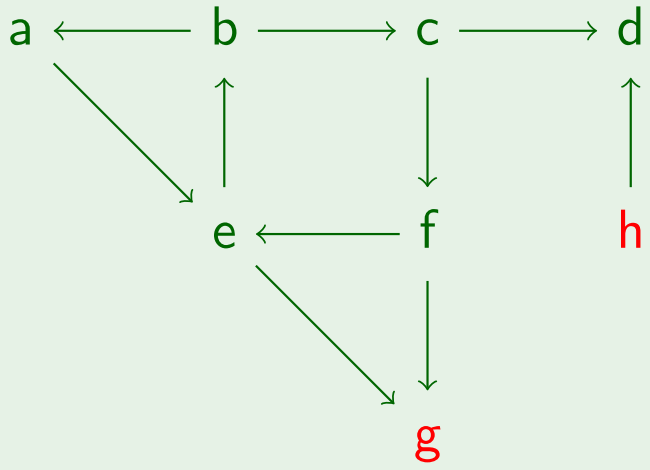

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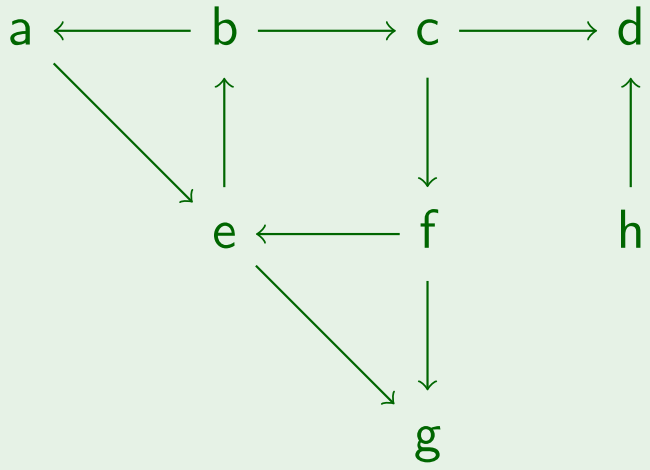

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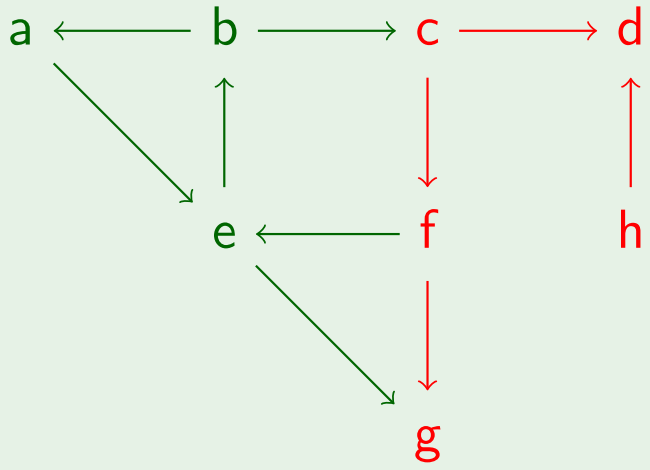

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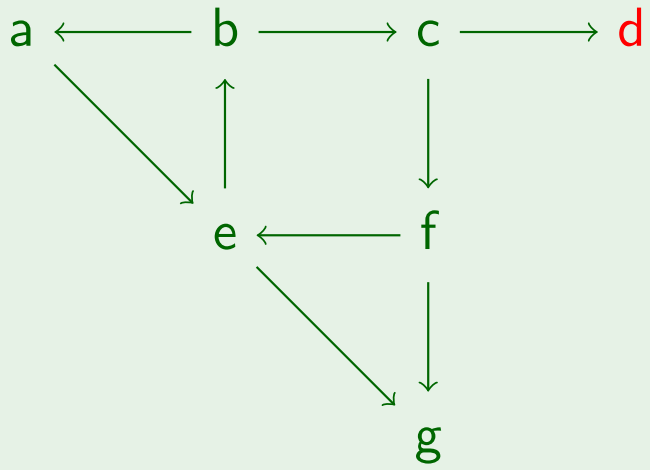

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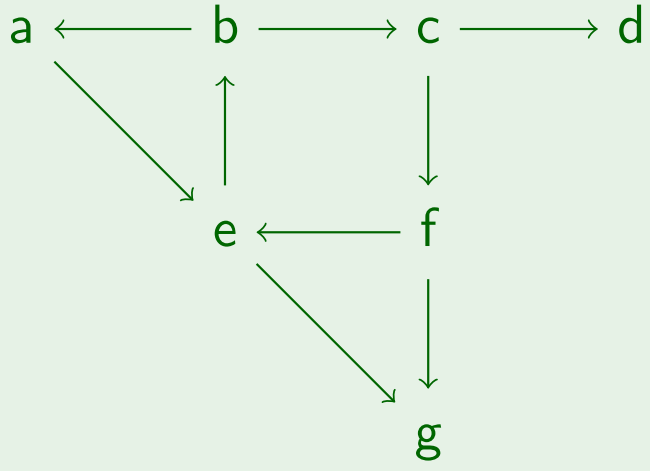

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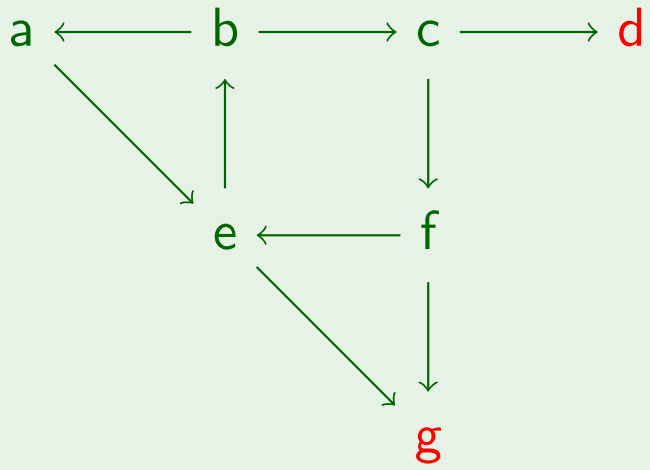

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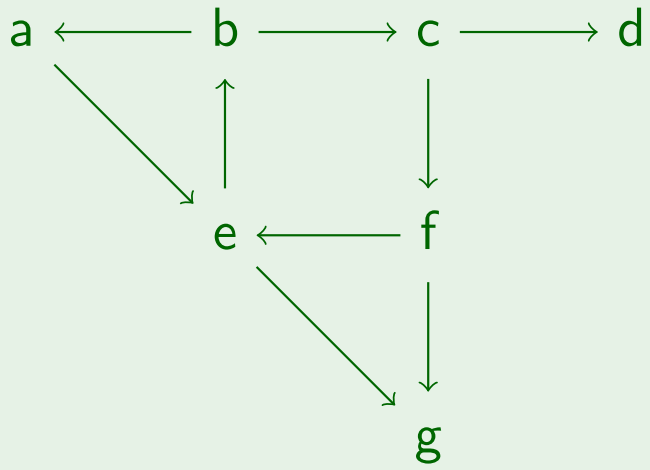

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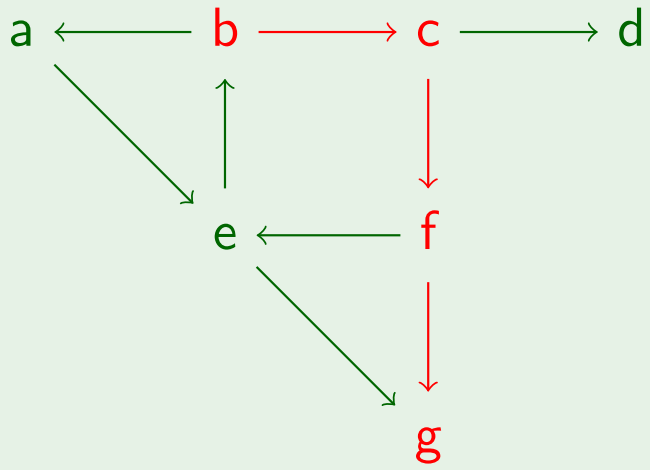
- $\mathrm{NF}(\mathcal{A})$ denotes set of normal forms of ARS $\mathcal{A}$
- $x \to^{!} y$ if $x \to^{*} y$ for normal form y (x has normal form y)

Example



ARS $\mathcal{A} = \langle A, \to \rangle$

- d is normal form
- $\mathrm{NF}(\mathcal{A}) = \{\mathrm{d}, \mathrm{g}\}$
- $\mathrm{b} \to^{!} \mathrm{g}$

Outline

- Overview
- Examples
- Abstract Rewrite Systems
  - Definitions
  - Properties
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

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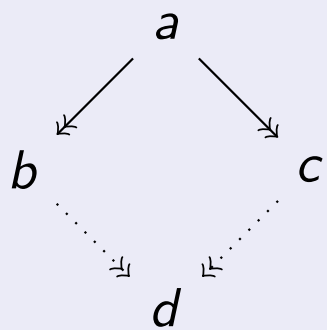

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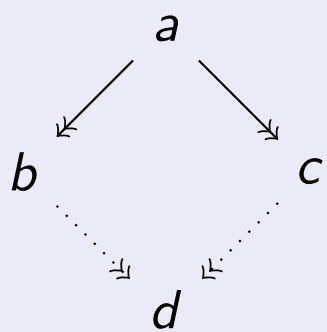

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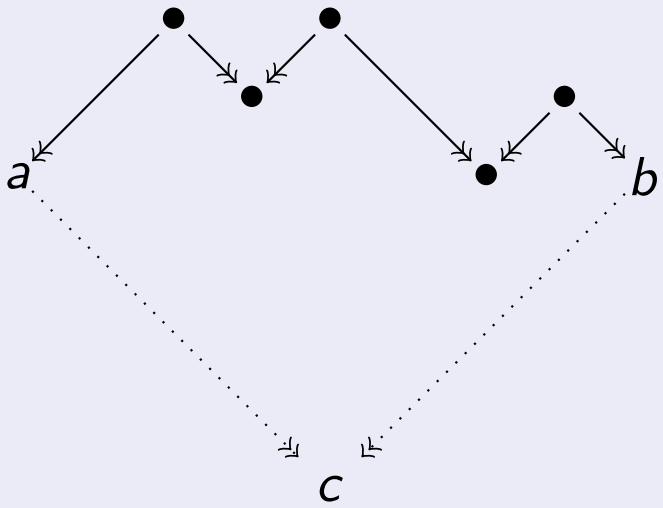

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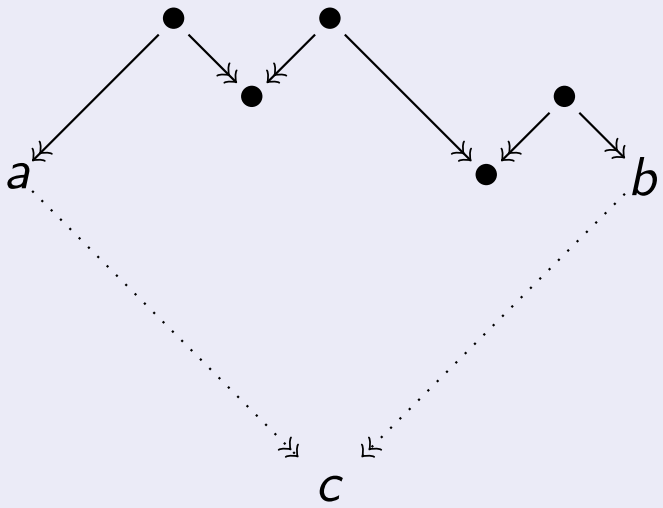

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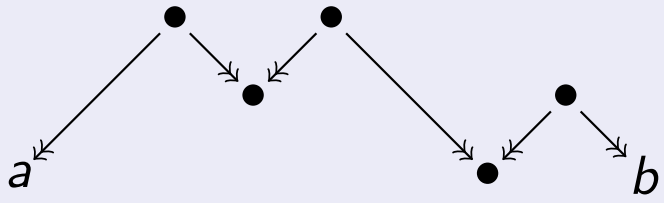

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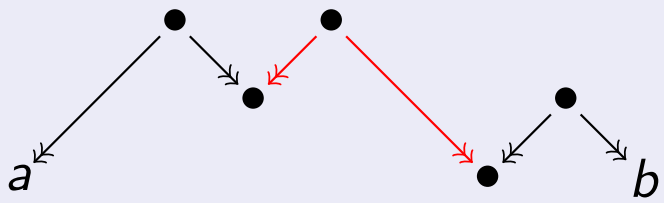

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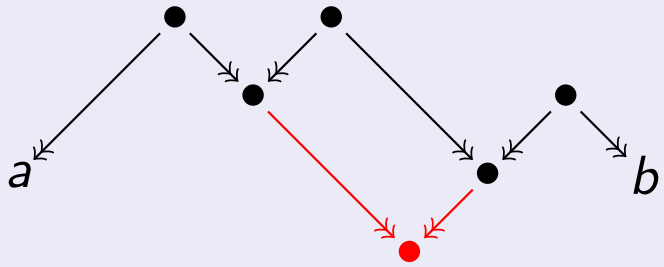

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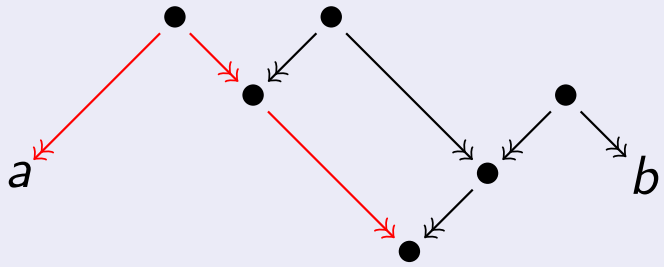

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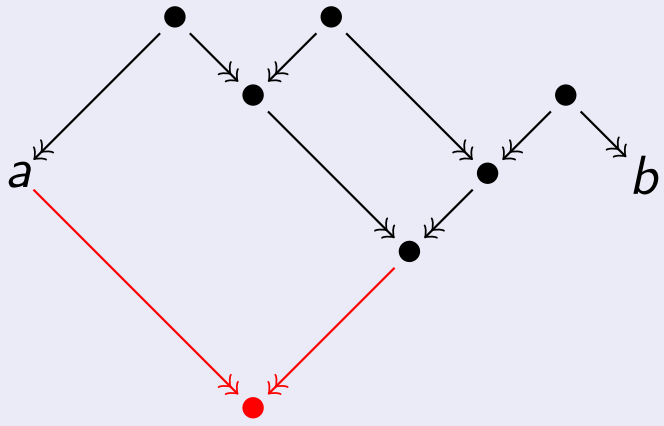

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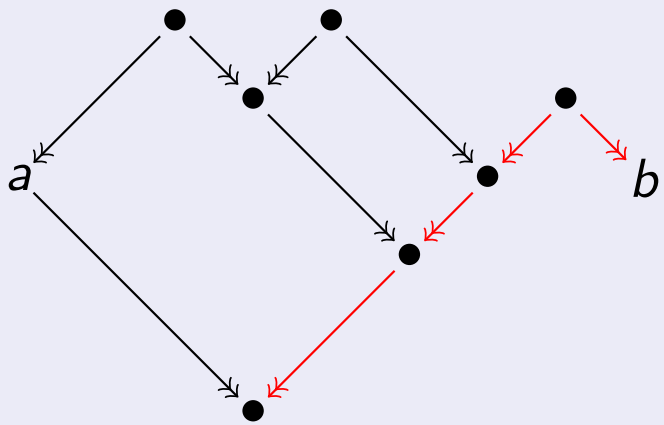

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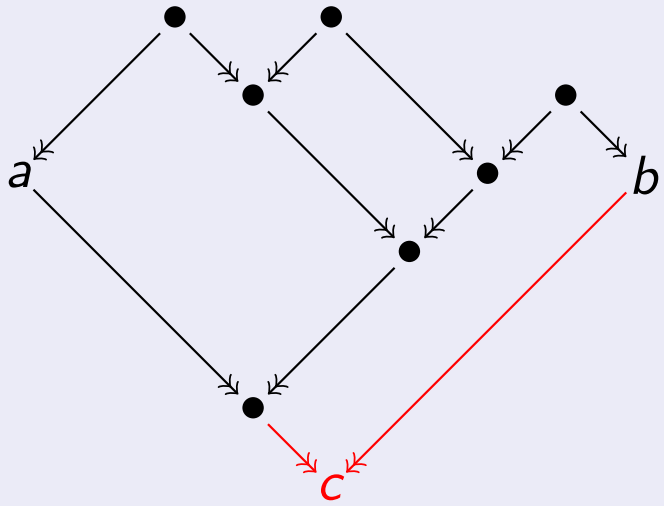

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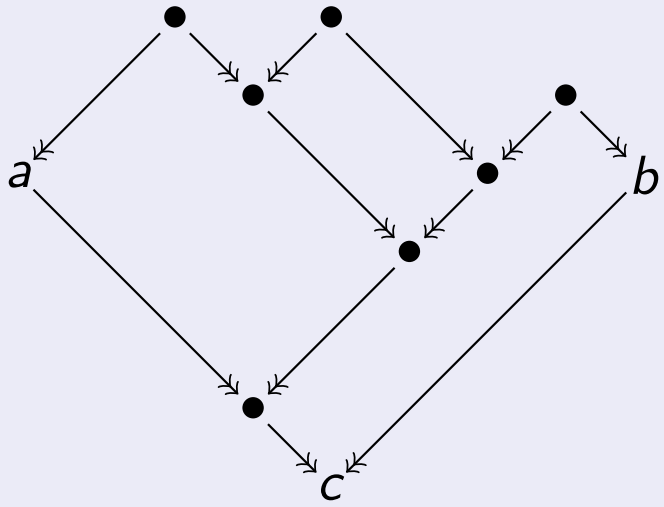

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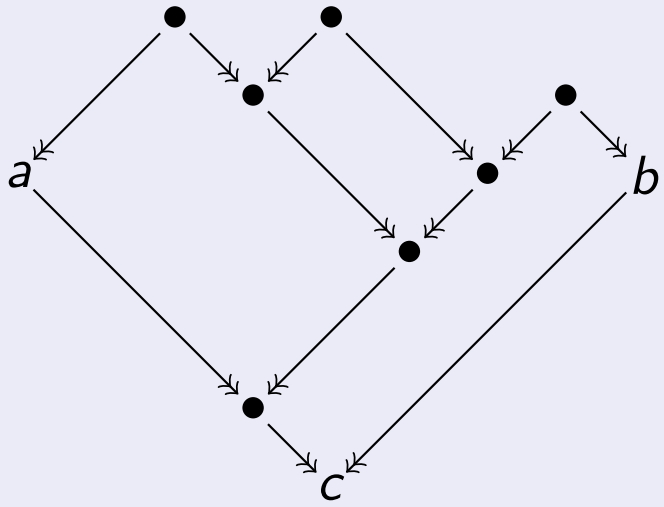

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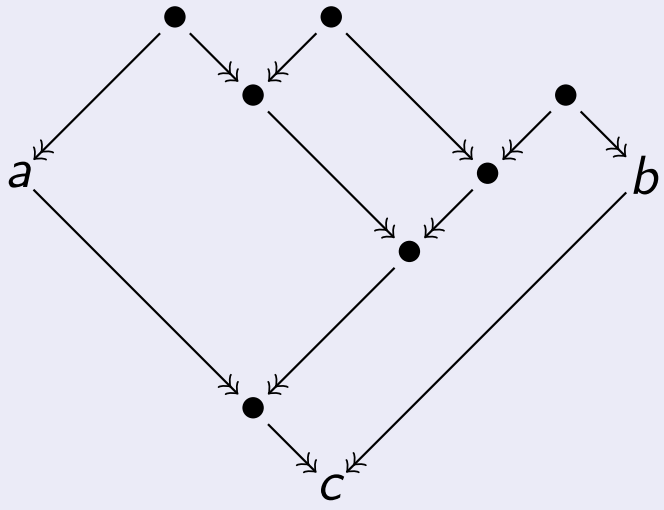

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Lemma (Induction)

To prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$ do:

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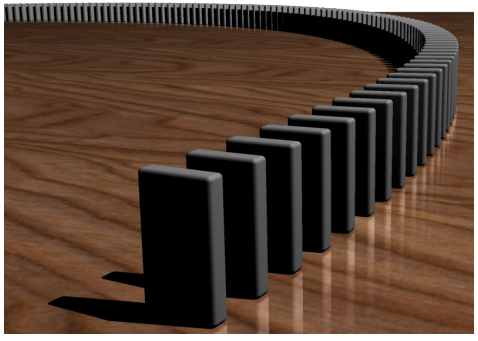

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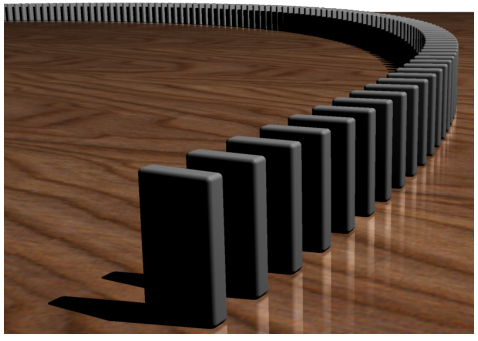

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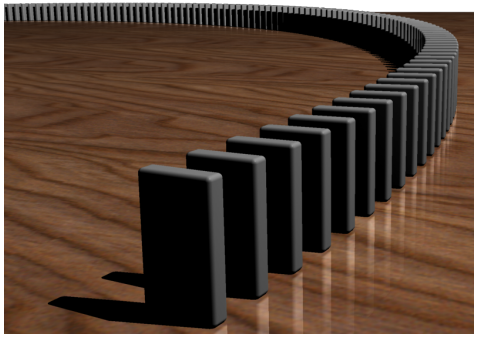

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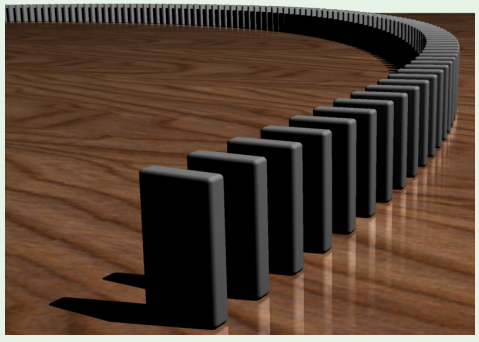

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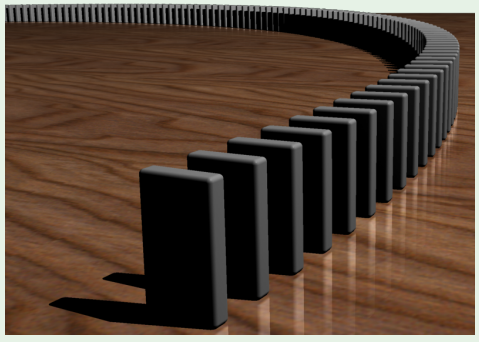

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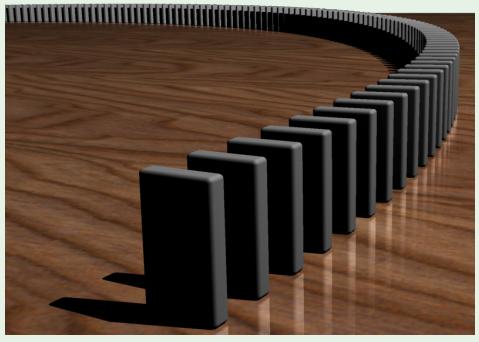

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- Base case: proof that the first domino falls
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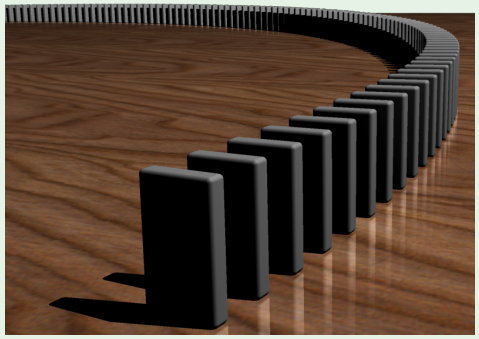

Wikipedia

- Base case: proof that the first domino falls
- Induction step: proof that if the n-th domino falls then the $(n+1)$-st domino falls

Then you have proven that all dominoes will fall.

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We use induction to prove that:

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

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Hence the formula holds for all $n \in \mathbb{N}$.

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$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

- $\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.
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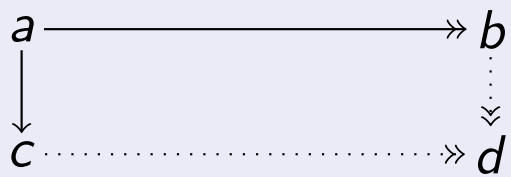

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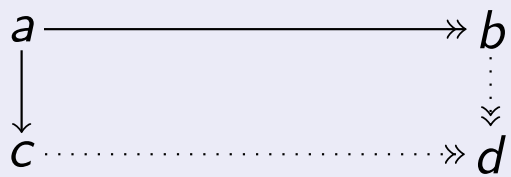

- $\forall a, b, c. \quad a \rightarrow^* b \ \wedge \ a \rightarrow c \quad \Rightarrow \quad \exists d. \quad b \rightarrow^* d \ \wedge \ c \rightarrow^* d$

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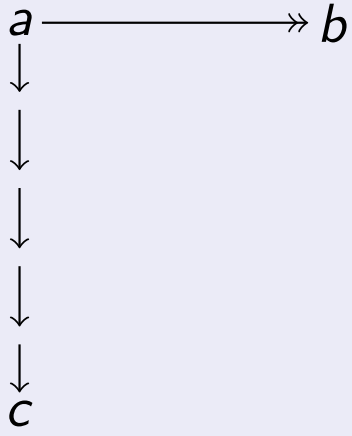

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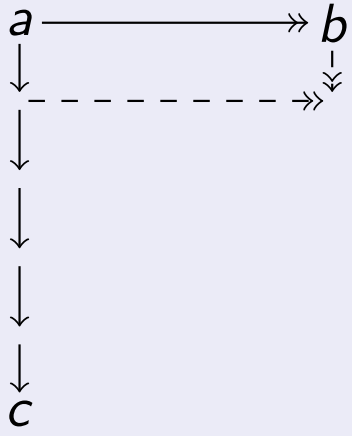

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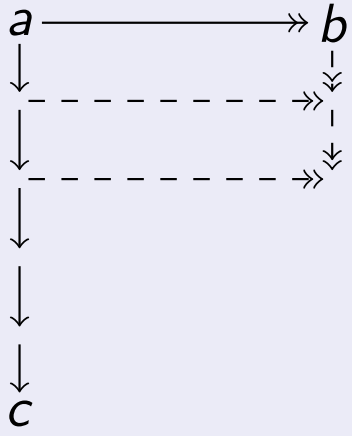

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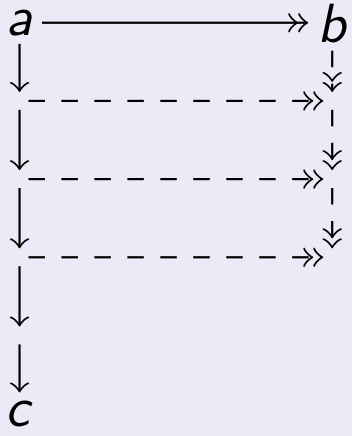

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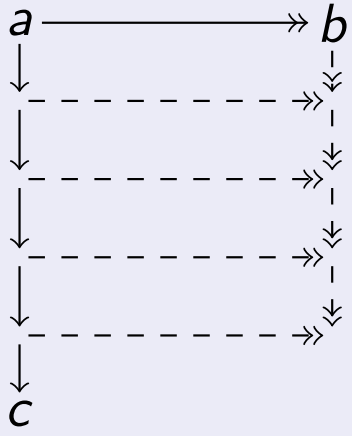

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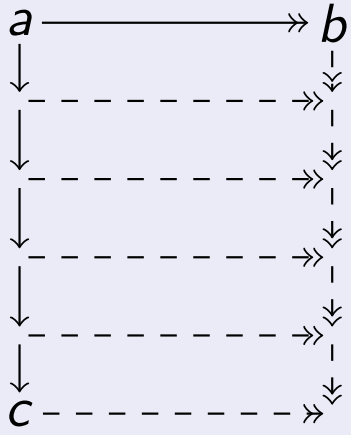

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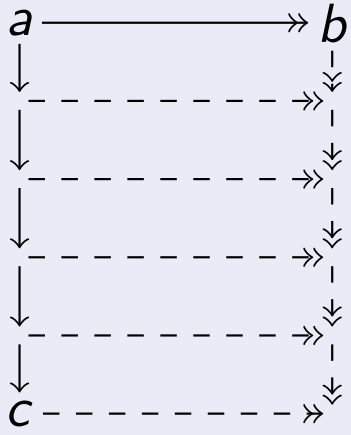
By induction on n we show: ${}^n\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.



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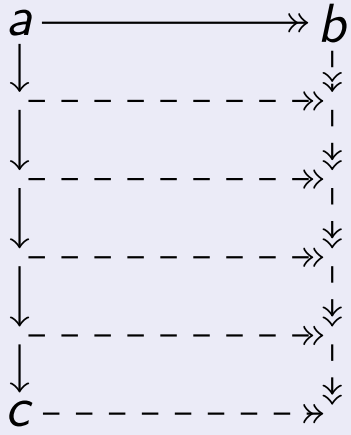

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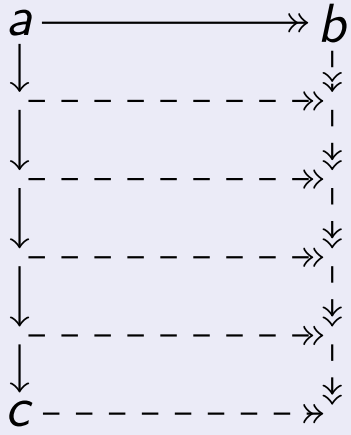

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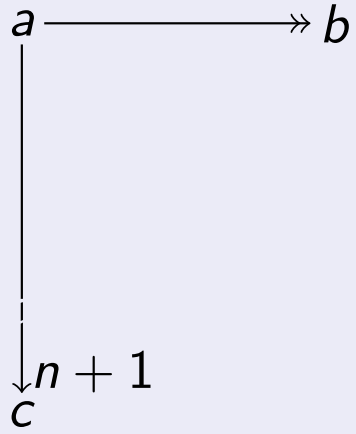

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$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

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$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

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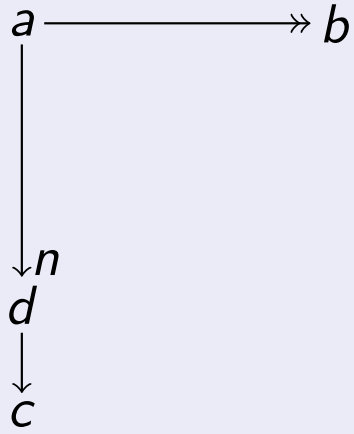

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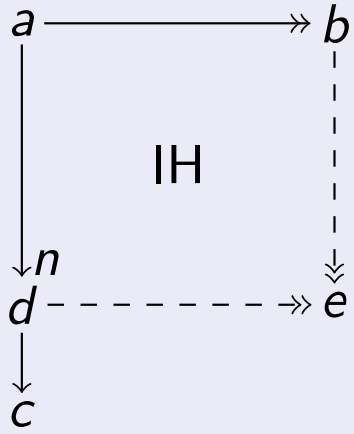

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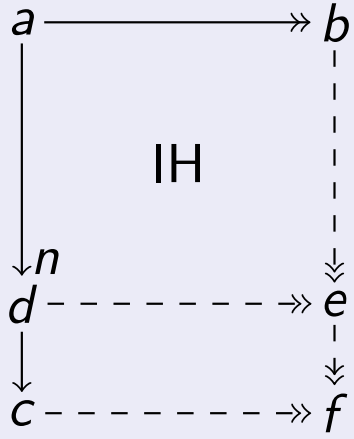

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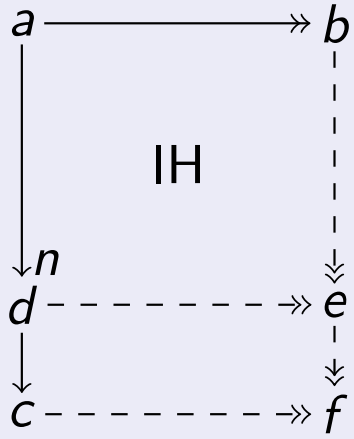

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Lemmata

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3. $\mathrm{CR} \Longrightarrow \mathrm{NF} \Longrightarrow \mathrm{UN} \Longrightarrow \mathrm{UN}^{\rightarrow}$
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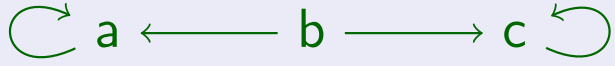
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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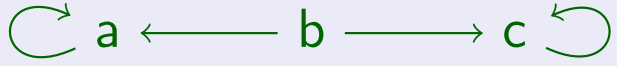
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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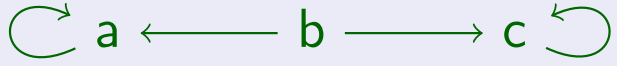
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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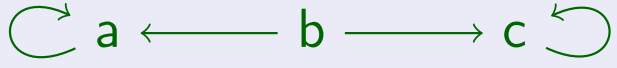
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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Definitions

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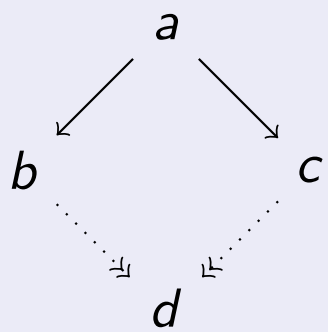

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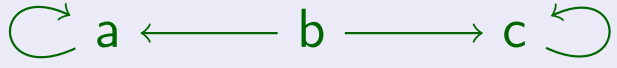
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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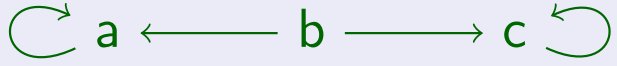
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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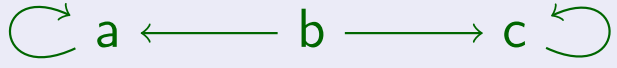
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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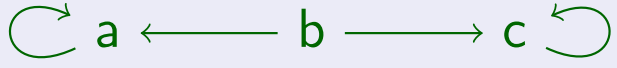
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ 

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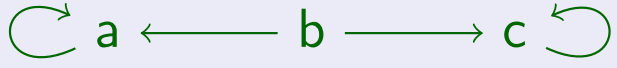
6. $CR \not\Longleftarrow NF$ 

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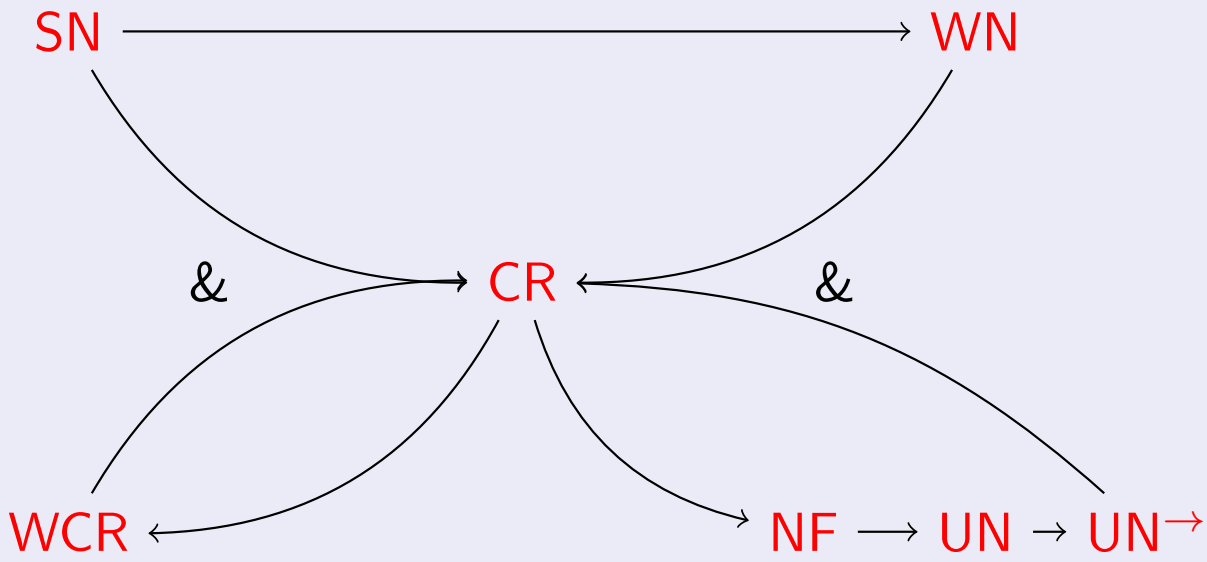
11. $SN \ \& \ WCR \Longrightarrow CR$ Newman's Lemma

Summary



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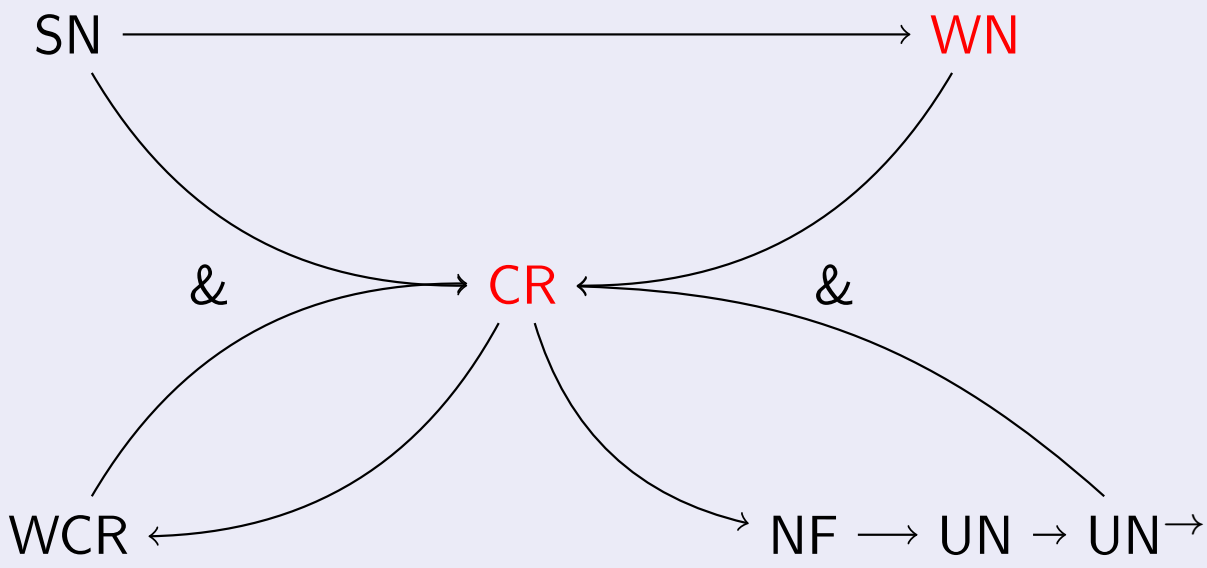

Definitions

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  - CR & WN

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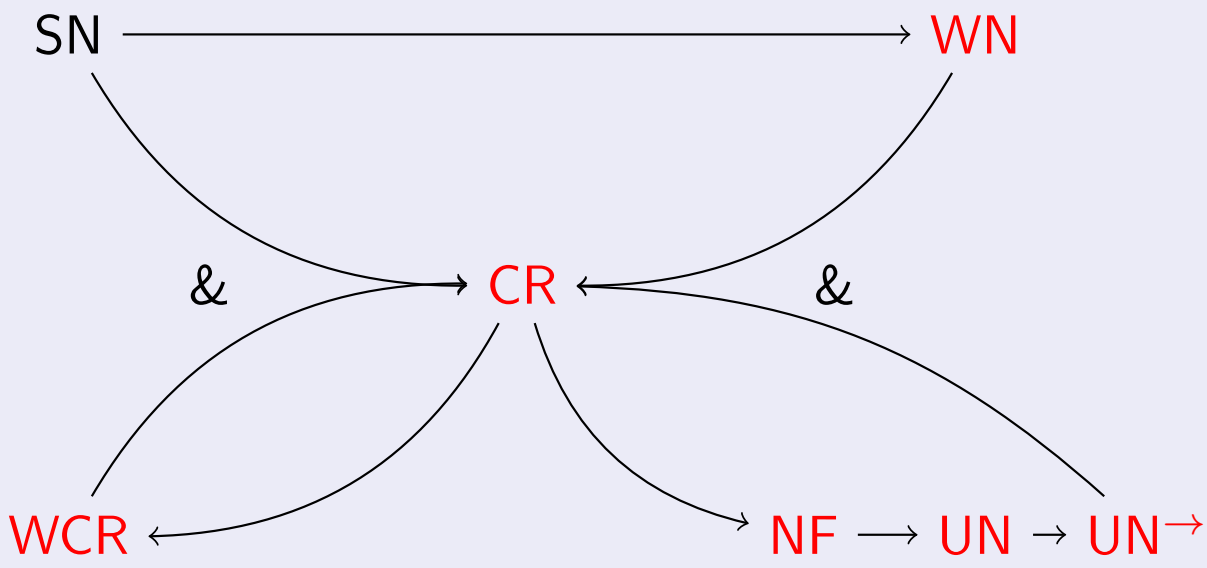

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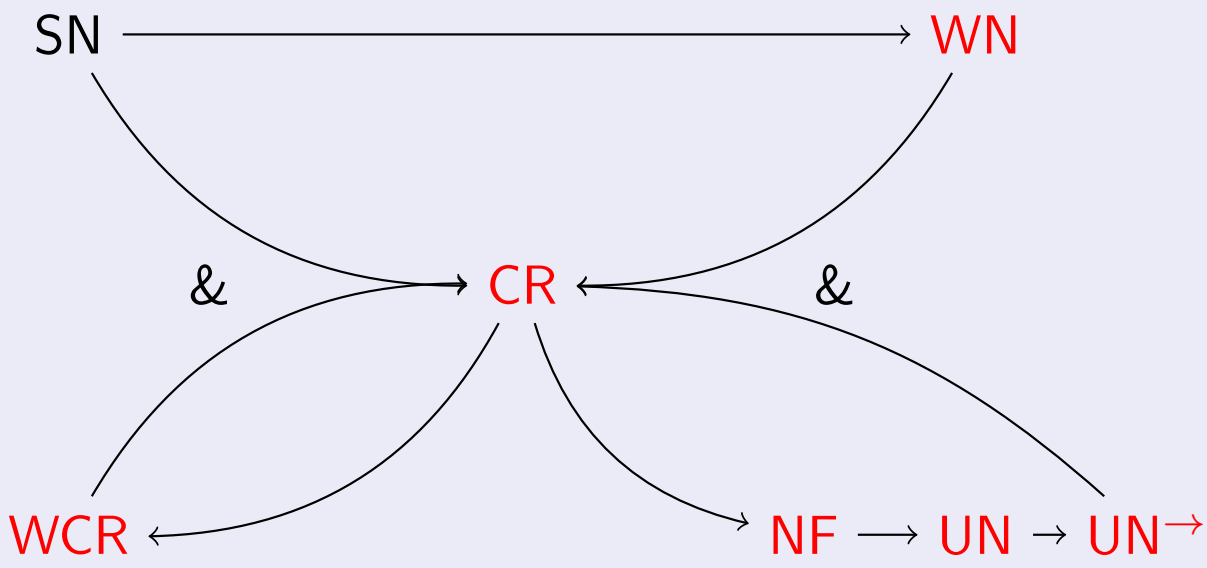

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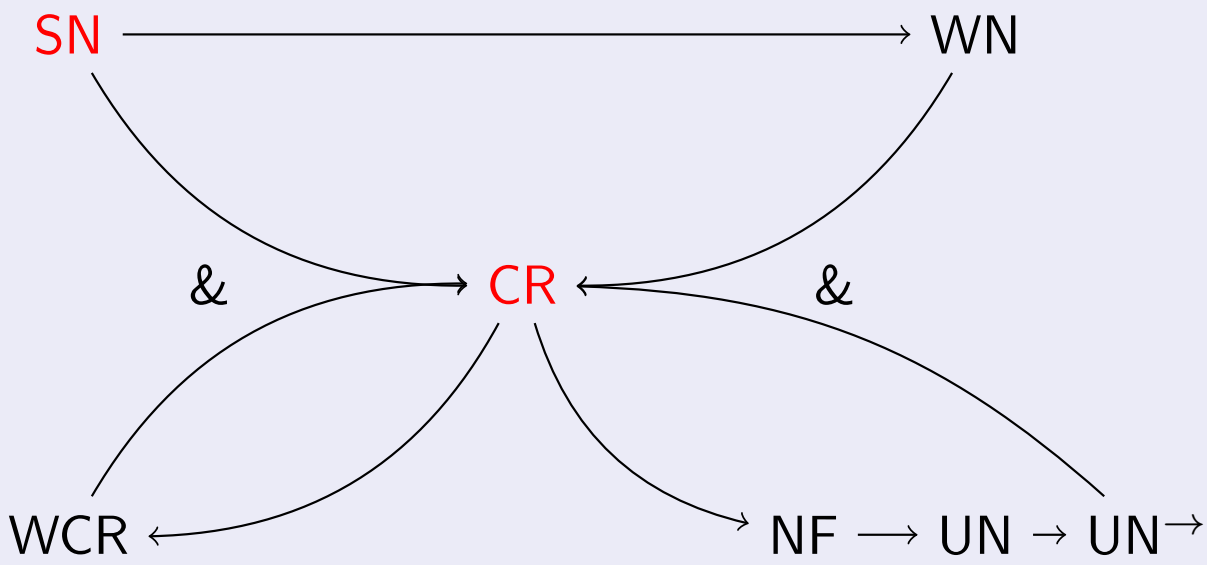

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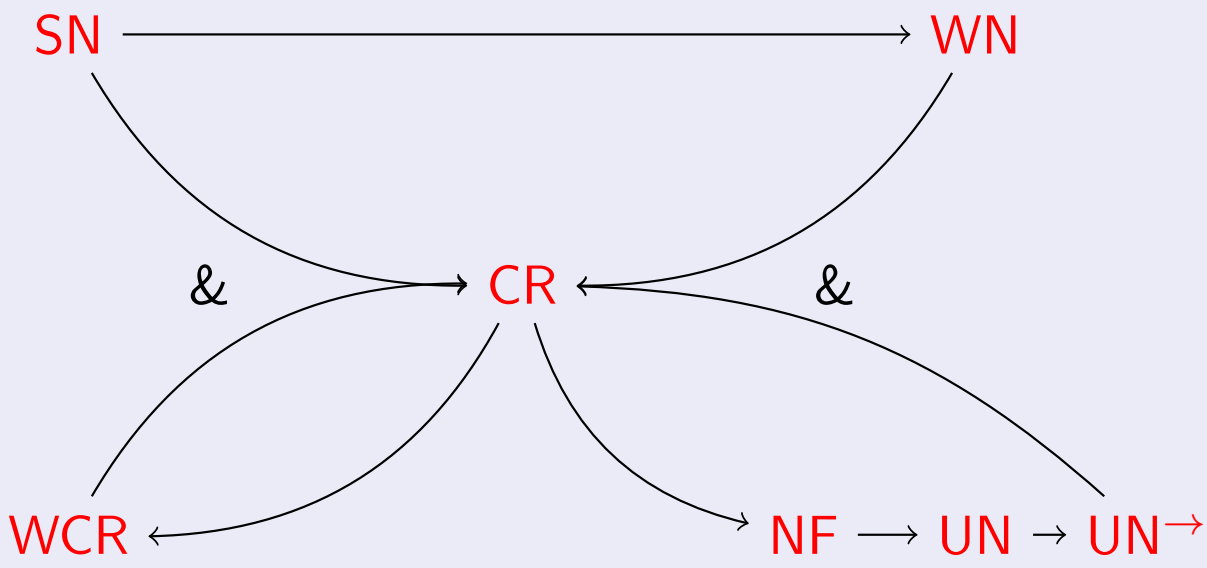

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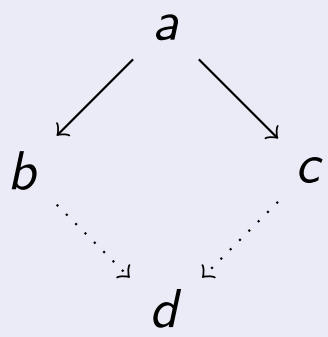

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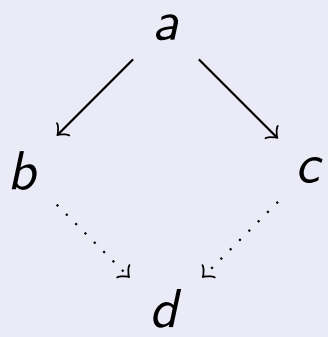

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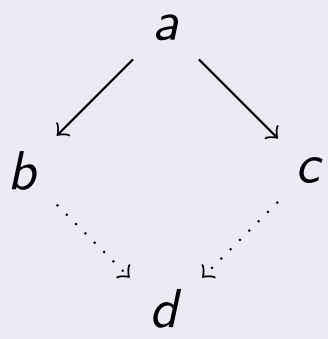

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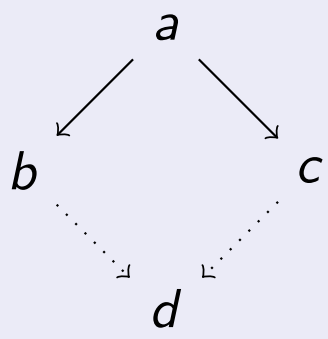

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Exercise. ∎

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Outline

- Overview
- Examples
- Abstract Rewrite Systems
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Well-Founded Induction

given

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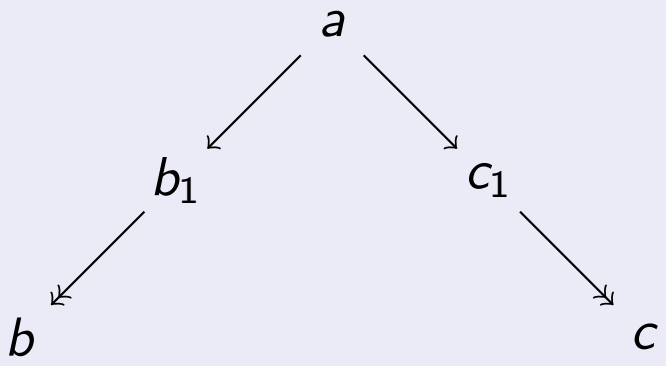
$$\Big(\forall a\colon \big(\forall b\colon a \to b \implies \mathrm{P}(b)\big) \implies \mathrm{P}(a)\Big) \implies \forall a\colon \mathrm{P}(a)$$

Newman's Lemma

$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(\mathcal{A})$

Proof.



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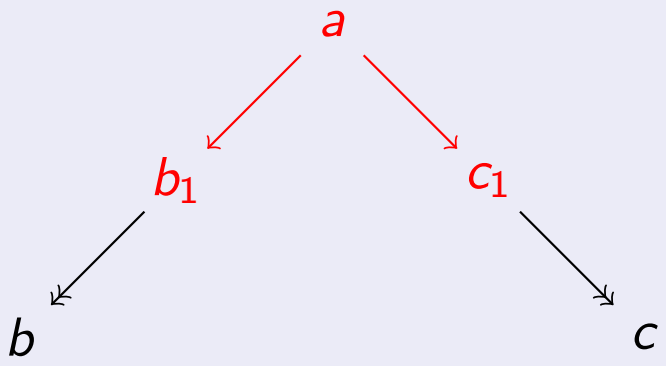

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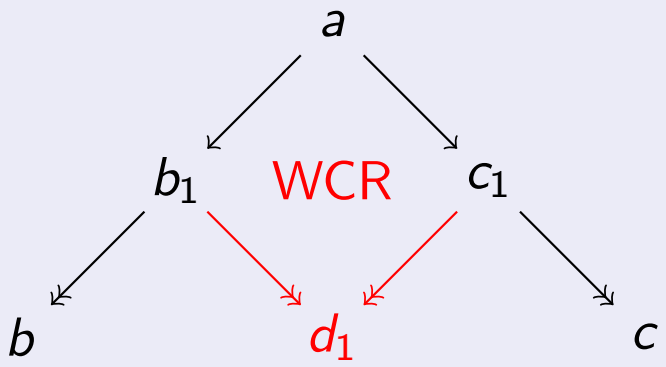

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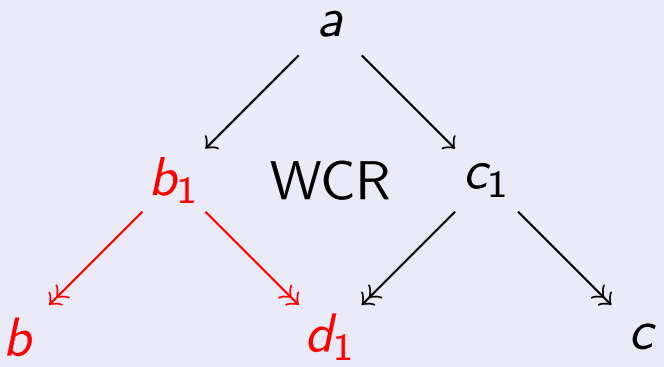

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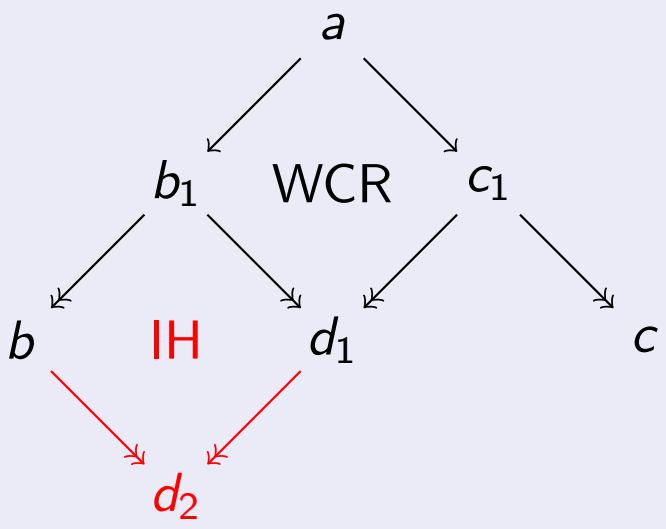

induction hypothesis $\mathrm{CR}(b_1)$

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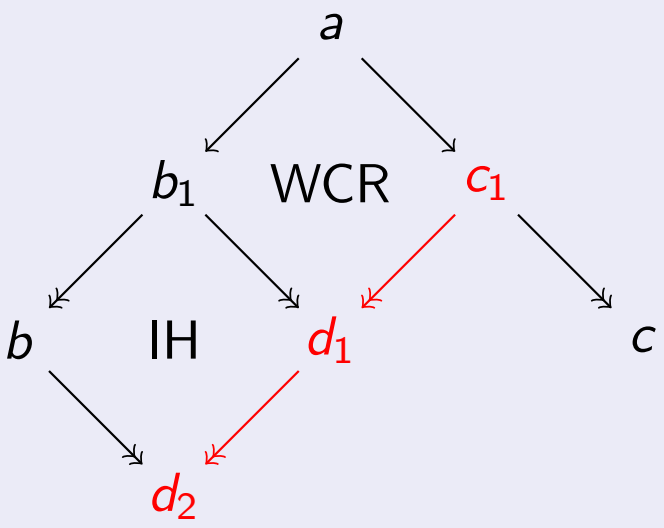

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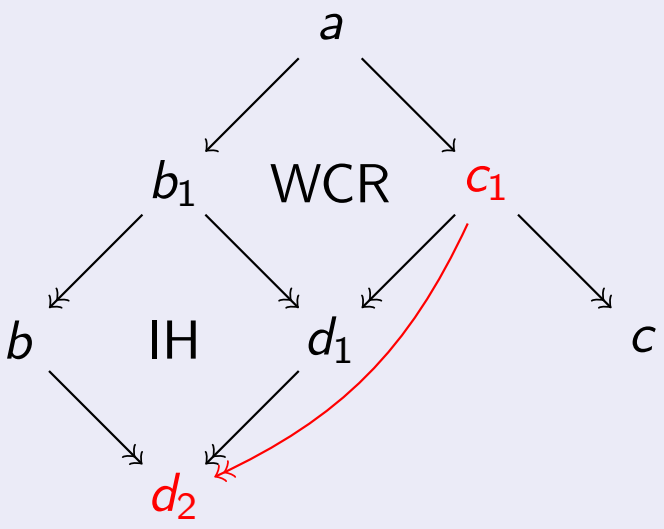

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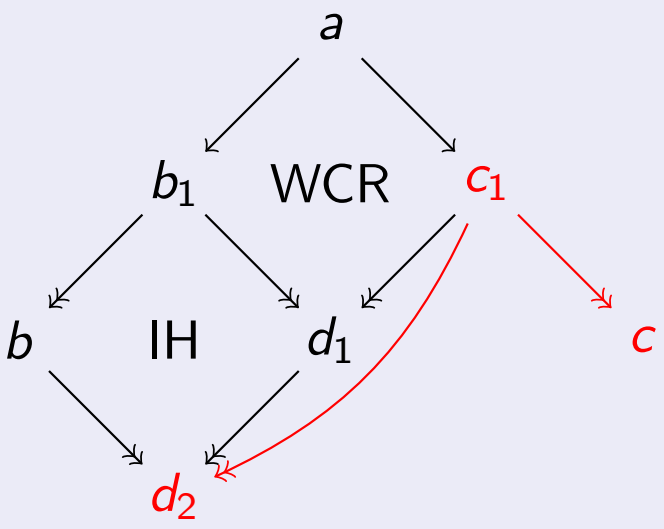

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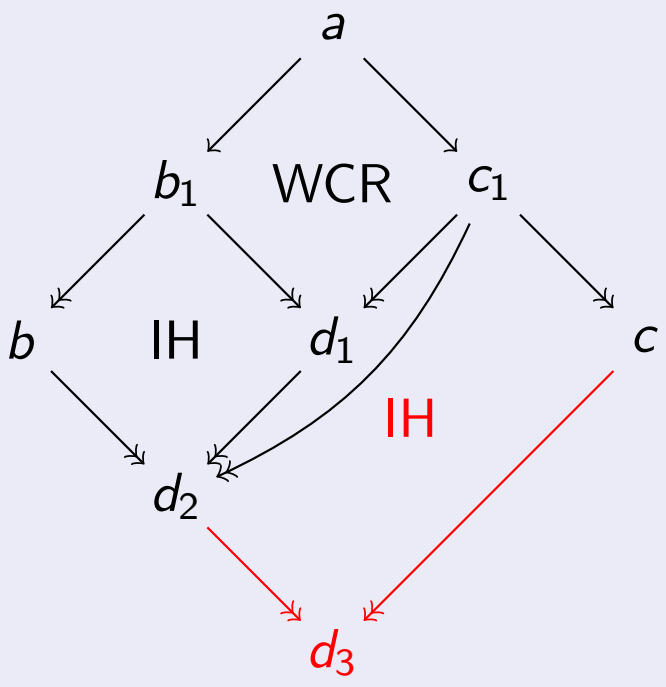

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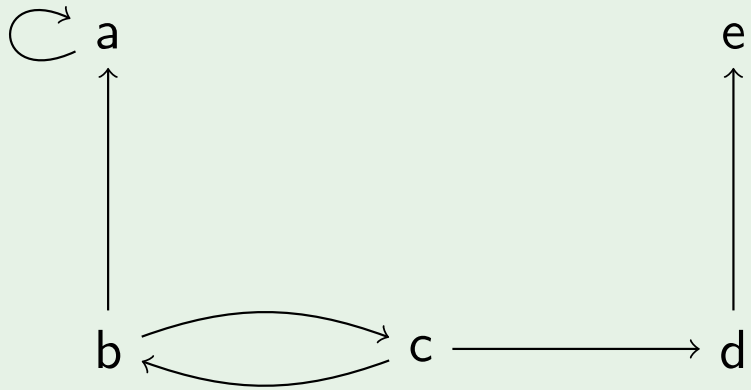
An ARS has the property if all its elements have the respective property.

Example



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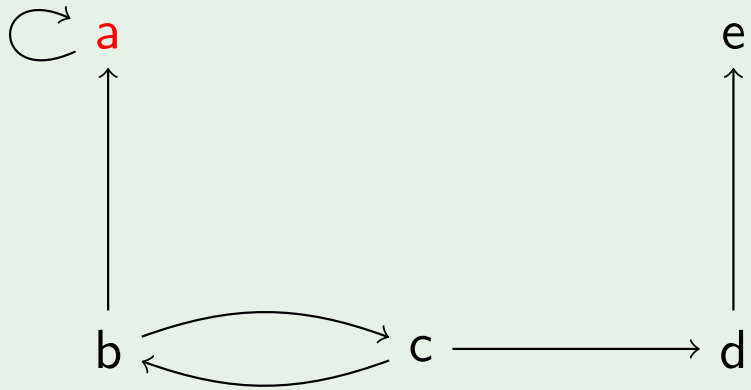

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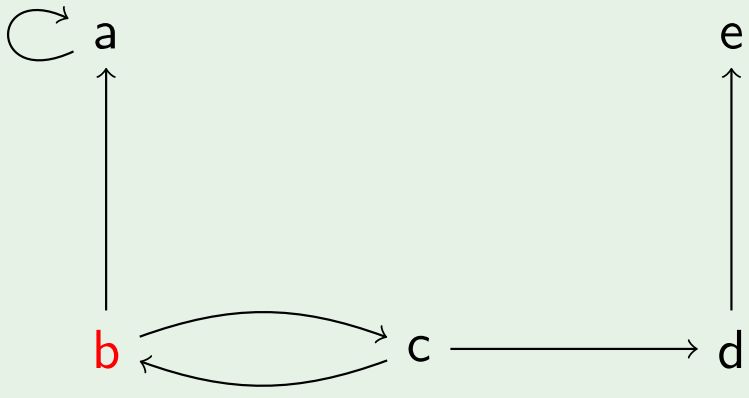

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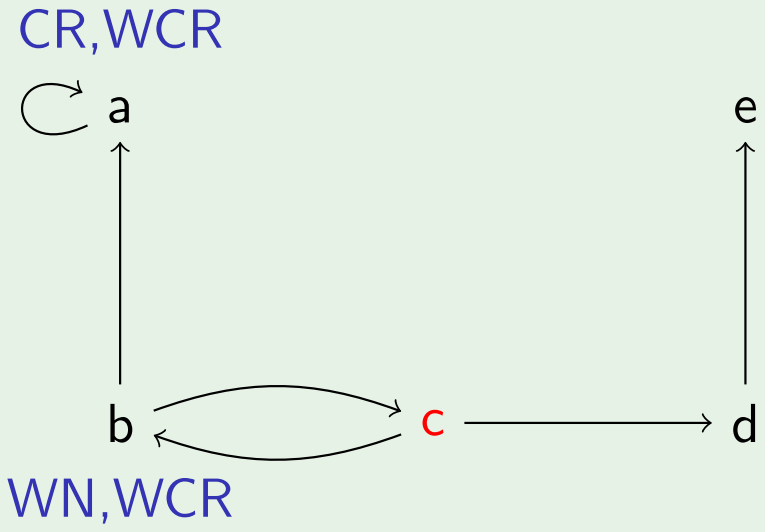

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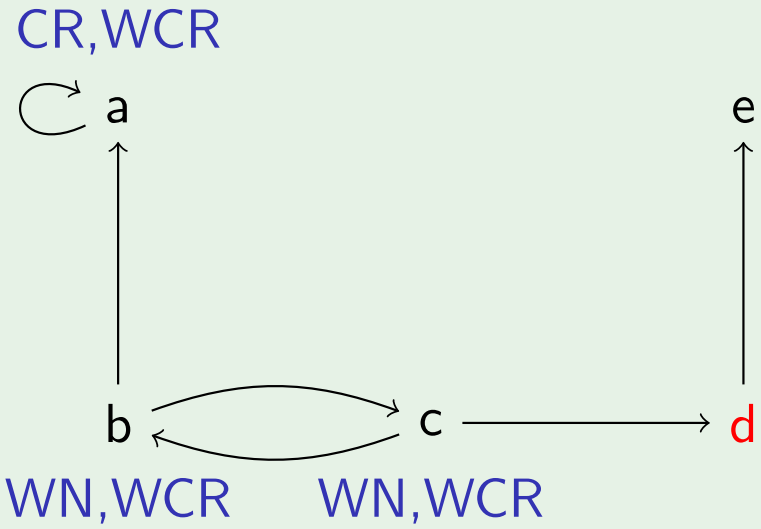

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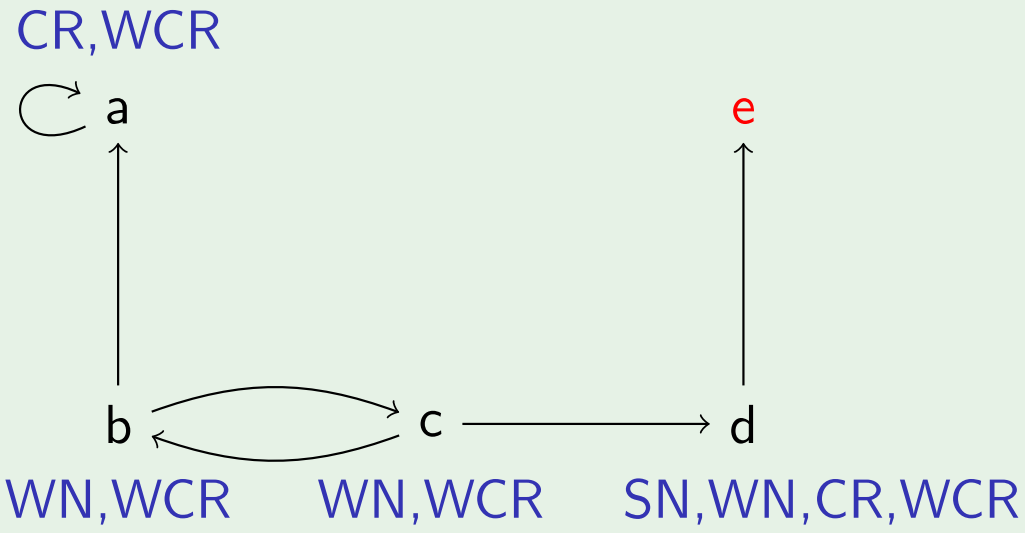

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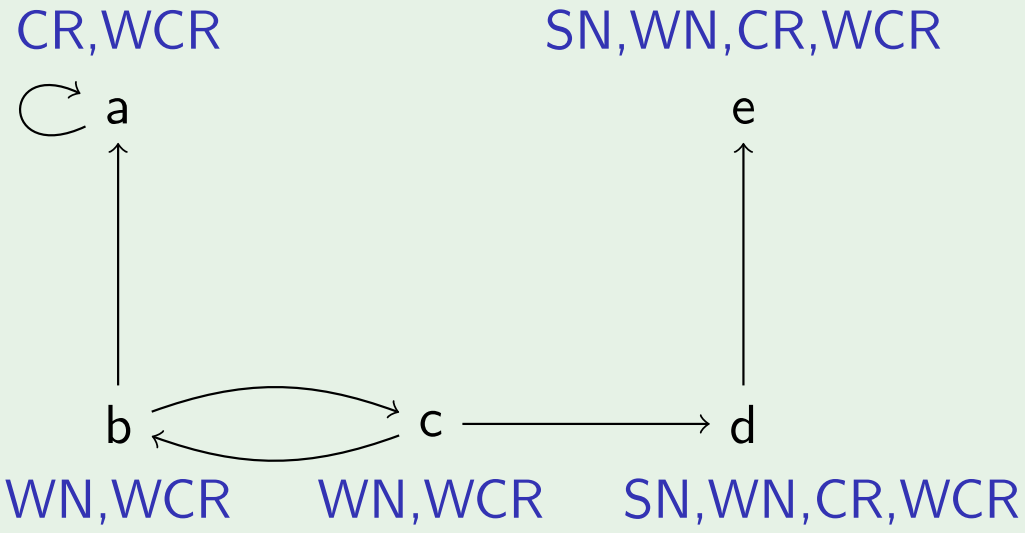

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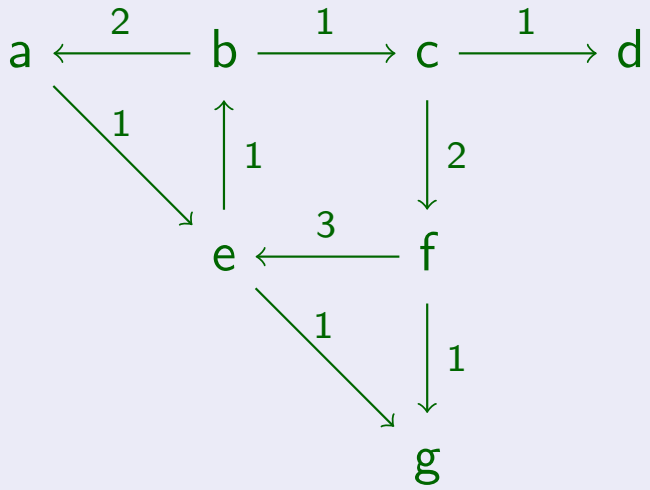

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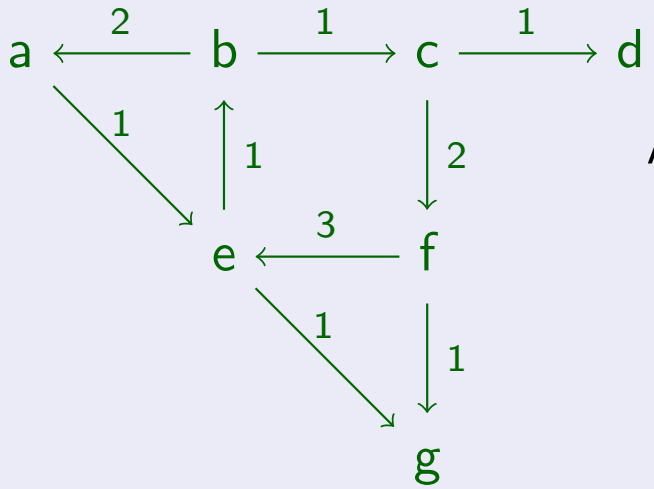

ARS $\mathcal{A} = \langle A, \rightarrow_1, \rightarrow_2, \rightarrow_3 \rangle$

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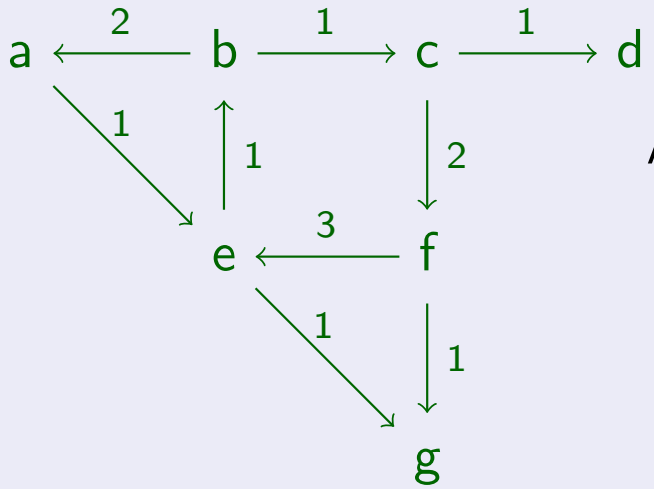

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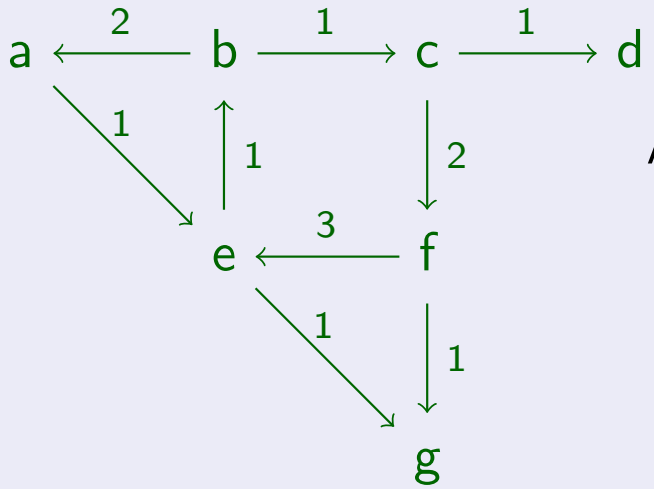

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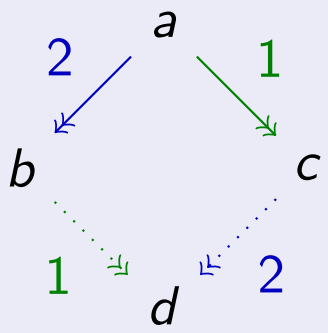

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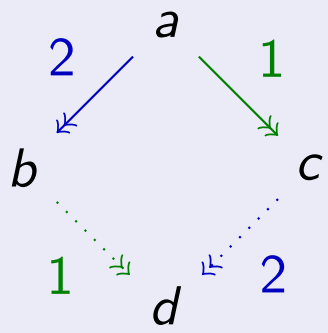

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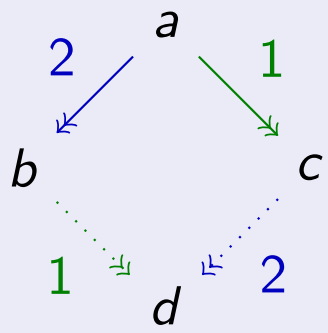

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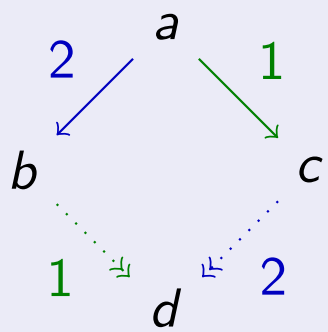

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