Logic and Modelling

— Natural Deduction for Propositional Logic —

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Natural Deduction

$$\alpha_1, \ldots, \alpha_n \vdash \beta$$

means: there exists a **natural deduction derivation** with

- premises $\alpha_1, \ldots, \alpha_n$, and
- conclusion β.

Natural deduction is a **formal system** with **strict formal rules**!

Rules for $\wedge$ and $\vee$

Introduction of $\wedge$

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta} \; \wedge_i$$

(If you have derived α and β, then you can conclude $\alpha \wedge \beta$.)

Elimination of $\wedge$

$$\frac{\alpha \wedge \beta}{\alpha} \; \wedge_{e_1} \qquad \frac{\alpha \wedge \beta}{\beta} \; \wedge_{e_2}$$

Rules for $\vee$

$$\frac{\alpha}{\alpha \vee \beta} \; \vee_{i_1} \qquad \frac{\beta}{\alpha \vee \beta} \; \vee_{i_2}$$

Derivation with Natural Deduction

Derivation with Natural Deduction

Can we derive $q \land \neg r$ from $(p \land q) \land \neg r$?

1	$(p \land q) \land \neg r$	premise
2	$p \land q$	$\land_{e_1}$ 1
3	q	$\land_{e_2}$ 2
4	$\neg r$	$\land_{e_2}$ 1
5	$q \land \neg r$	$\land_i$ 3, 4

Hence we have derived

$$(p \land q) \land \neg r \quad \vdash \quad q \land \neg r$$

Rules for $\neg\neg$ and $\rightarrow$

Rules for $\neg\neg$

$$\frac{\neg\neg\alpha}{\alpha} \quad \neg\neg_e \qquad\qquad \frac{\alpha}{\neg\neg\alpha} \quad \neg\neg_i$$

Elimination rules for $\rightarrow$

This rule is called "Modus Ponens" (MP):

$$\frac{\alpha \qquad \alpha \rightarrow \beta}{\beta} \quad \rightarrow_e \ (or \ \ MP)$$

This rule is called "Modus Tollens" (MT):

$$\frac{\alpha \rightarrow \beta \qquad \neg\beta}{\neg\alpha} \quad MT$$

Derivation with Natural Deduction

Can we derive q from $\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r$?

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg\neg p$	$\neg\neg_i$ 2
5	$\neg q \rightarrow r$	$\rightarrow_e$ 4,1
6	$\neg\neg q$	MT 5,3
7	q	$\neg\neg_e$ 6

Hence we have derived

$$\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r \quad \vdash \quad q$$

Introduction of $\rightarrow$

Introduction rule for $\rightarrow$

$$\frac{\boxed{\begin{array}{c}\alpha \\ \vdots \\ \beta\end{array}}}{\alpha \rightarrow \beta} \quad \rightarrow_i$$

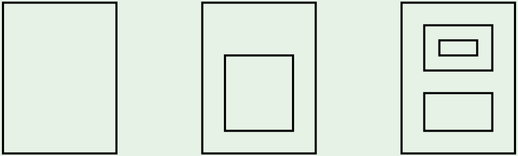
Derivation of $p \rightarrow q \vdash \neg q \rightarrow \neg p$:

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	$\rightarrow_i$ 2–3

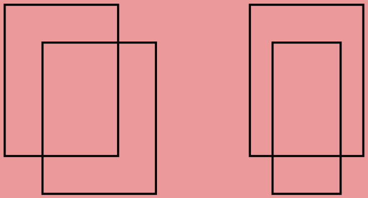
Block Structures

Allowed block structures

Blocks are allowed to be nested inside each other:



Blocks are not allowed to intersect:



Block Structure

When applying a rule

$$\frac{\alpha_1 \qquad \ldots \qquad \alpha_n}{\beta},$$

the $\alpha_1, \ldots, \alpha_n$ must be **in the scope**, that is, must have been derived in the current block or a surrounding block.

(Compare with scopes of variables in programming languages.)

1	$p \rightarrow q$	premise
2	p	assumption
3	q	$\rightarrow_e$ 1,2
4	$q \vee q$	$\vee_i$ 3 **This is not allowed!!!**

Derivation with Natural Deduction

Can we derive $\neg r \rightarrow \neg q$ from $p \rightarrow (q \rightarrow r),\ p$?

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	assumption
4	$q \rightarrow r$	$\rightarrow_e$ 2,1
5	$\neg q$	MT 4,3
6	$\neg r \rightarrow \neg q$	$\rightarrow_i$ 3–5

Hence we have derived

$$p \rightarrow (q \rightarrow r),\ p \quad \vdash \quad \neg r \rightarrow \neg q$$

Special Cases

1	p	assumption
2	$p \to p$	$\to_i$ 1–1

This is a derivation of

$$\vdash \ p \to p$$

Copy Rule

Copy rule

$$\frac{\alpha}{\alpha} \quad \textit{copy}$$

Lets try to prove $\vdash p \rightarrow (q \rightarrow p)$!

1	p	assumption
2	q	assumption
3	p	copy 1
4	$q \rightarrow p$	$\rightarrow_i$ 2–3
5	$p \rightarrow (q \rightarrow p)$	$\rightarrow_i$ 1–4

This concludes the derivation.

Rules for ¬ and ⊥

Rules for ¬ and ⊥

$$\frac{\alpha \qquad \neg\alpha}{\bot}\ \neg_e$$

$$\frac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \bot\end{array}}}{\neg\alpha}\ \neg_i$$

$$\frac{\bot}{\alpha}\ \bot_e$$

Example

Prove $\neg\neg p \rightarrow (\neg q \rightarrow r),\ p,\ \neg r \vdash q$ **without** $\neg\neg_i$ and MT.

1	$\neg\neg p \rightarrow (\neg q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$\neg p$	assumption
5	$\bot$	$\neg_e$ 2,4
6	$\neg\neg p$	$\neg_i$ 4–5
7	$\neg q \rightarrow r$	$\rightarrow_e$ 6,1
8	$\neg q$	assumption
9	r	$\rightarrow_e$ 8,7
10	$\bot$	$\neg_e$ 9,3
11	$\neg\neg q$	$\neg_i$ 8–10
12	q	$\neg\neg_e$ 11

MT and $\neg\neg_i$ as "derived rules"

The $\neg\neg_i$ rule derives $\neg\neg\alpha$ from α.

We can derive it using other rules as follows:

1	α	premise
2	$\neg\alpha$	assumption
3	$\bot$	$\neg_e$ 1,2
4	$\neg\neg\alpha$	$\neg_i$ 2–3

Thus: $\alpha \vdash \neg\neg\alpha$.

MT and $\neg\neg_i$ as "derived rules"

The Modus Tollens rule derives $\neg\alpha$ from $\alpha \to \beta$ and $\neg\beta$.

We can derive it using other rules as follows:

1	$\alpha \to \beta$	premise
2	$\neg\beta$	premise
3	α	assumption
4	β	$\to_e$ 3,1
5	$\bot$	$\neg_e$ 2,4
6	$\neg\alpha$	$\neg_i$ 3–5

Thus: $\alpha \to \beta, \neg\beta \vdash \neg\alpha$.

This shows that $\neg\neg_i$ and MT are **not needed**, but sometimes help to make derivations easier or shorter.

Elimination of $\vee$

Elimination of $\vee$

$$\frac{\alpha \vee \beta \qquad \boxed{\begin{array}{c}\alpha \\ \vdots \\ \gamma\end{array}} \qquad \boxed{\begin{array}{c}\beta \\ \vdots \\ \gamma\end{array}}}{\gamma} \; \vee_e$$

Example

Prove $p \lor \neg q,\ \neg p \to q \ \vdash\ p$!

1	$p \lor \neg q$	premise
2	$\neg p \to q$	premise
3	p	assumption
4	$\neg q$	assumption
5	$\neg\neg p$	MT 2,4
6	p	$\neg\neg_e$ 5
7	p	$\lor_e$ 1, 3–3, 4–6

Example

Use $\bot_e$ to prove $\neg p \vee q \vdash p \rightarrow q$!

1	$\neg p \vee q$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\bot$	$\neg_e$ 2,3
5	q	$\bot_e$ 4
6	q	assumption
7	q	$\vee_e$ 1, 3–5, 6–6
8	$p \rightarrow q$	$\rightarrow_i$ 2–7

Proof by Contradiction

Assume that you have derived

$$\begin{array}{c}\neg\alpha \\ \vdots \\ \bot\end{array}$$

Then also

$$\begin{array}{ll}\boxed{\begin{array}{c}\neg\alpha \\ \vdots \\ \bot\end{array}} & \\ \neg\neg\alpha & \neg_i \\ \alpha & \neg\neg_e\end{array}$$

This is known as **Proof by Contradiction (PBC)**!

Also known as **Reductio ad Absurdum (RAA)**!

Proof by Contradiction as a Rule

Proof by Contradiction Rule

$$\frac{\boxed{\begin{array}{c}\neg\alpha \\ \vdots \\ \bot\end{array}}}{\alpha} \quad \textit{PBC} \quad (\textit{or} \ \textit{RAA})$$

We now can derive the rule $\neg\neg_e$:

1	$\neg\neg\alpha$	premise
2	$\neg\alpha$	assumption
3	$\bot$	$\neg_e$ 2,1
4	α	PBC 2–3

Law of Excluded Middle

Law of Excluded Middle Rule

$$\frac{}{\alpha \vee \neg\alpha} \quad \textit{LEM}$$

(The rule does not have premises.)

Show that $p \rightarrow q \vdash \neg p \vee q$:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\rightarrow_e$ 3,1
5	$\neg p \vee q$	$\vee_{i_2}$ 4
6	$\neg p$	assumption
7	$\neg p \vee q$	$\vee_{i_1}$ 7
8	$\neg p \vee q$	$\vee_e$ 2, 3–5, 6–7

Law of Excluded Middle is Derivable

The LEM rule $\vdash \alpha \vee \neg\alpha$ is derivable:

1	$\neg(\alpha \vee \neg\alpha)$	assumption
2	α	assumption
3	$\alpha \vee \neg\alpha$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	$\neg\alpha$	$\neg_i$ 2–4
6	$\alpha \vee \neg\alpha$	$\vee_{i_2}$ 5
7	$\bot$	$\neg_e$ 6,1
8	$\alpha \vee \neg\alpha$	PBC 1–7

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \rightarrow q$	$\rightarrow_i$ 3–4
6	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 5
7	$\neg q$	assumption
8	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 6
9	$\neg q \vee (p \rightarrow q)$	$\vee_e$ 1, 2–6, 7–8

Example from a Previous Exam

Show that $\vdash \neg q \vee (p \rightarrow q)$ with PBC instead of LEM:

1	$\neg(\neg q \vee (p \rightarrow q))$	assumption
2	$\neg q$	assumption
3	$\neg q \vee (p \rightarrow q)$	$\vee_{i_1}$ 2
4	$\bot$	$\neg_e$ 3,1
5	q	PBC 2–4
6	p	assumption
7	q	copy 5
8	$p \rightarrow q$	$\rightarrow_i$ 6–7
9	$\neg q \vee (p \rightarrow q)$	$\vee_{i_2}$ 8
10	$\bot$	$\neg_e$ 9,1
11	$\neg q \vee (p \rightarrow q)$	PBC 1–11

More Exam Preparation Tasks

Exam Exercises

Try to derive yourself:

- $p \vee q,\ \neg p \ \vdash\ q$
- $p \rightarrow (q \rightarrow r) \ \vdash\ q \rightarrow (p \rightarrow r)$
- $(p \rightarrow q) \rightarrow r \ \vdash\ p \rightarrow (q \rightarrow r)$
- $p \vee (q \wedge r) \ \vdash\ p \vee q$
- $a \vee b,\ a \rightarrow c,\ \neg d \rightarrow \neg b \ \vdash\ c \vee d$
- $(a \rightarrow b) \wedge (b \rightarrow a) \ \vdash\ (a \wedge b) \vee (\neg a \wedge \neg b)$
- $a \wedge (b \vee c) \ \vdash\ (a \wedge b) \vee (a \wedge c)$
- $\vdash\ ((p \rightarrow q) \rightarrow p) \rightarrow p$