# Automata Theory :: Complexity

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# Big O Notation

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n^a &\in O(b^n) && \text{for all } a > 0 \text{ and } b > 1 \\
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By definition $\log_a a^n = n$. This implies $a^{\log_a n} = n$, and hence

$$a^{\log_a b \cdot \log_b n} = \left(a^{\log_a b}\right)^{\log_b n} = b^{\log_b n} = n$$

Hence $\log_a b \cdot \log_b n = \log_a n$.

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A Turing machine $M$ has

**time complexity** $O(g)$

if there exists $f \in O(g)$ such that $M$ runs in time $f$.

# Complexity Classes P and NP

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Equivalently, $M$ has time complexity $O(n^k)$ for some $k$.

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Clearly P $\subseteq$ NP, but it is unknown whether P $=$ NP.

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The questions if a number is **not prime** is in NP.

Surprisingly, last question in P. (Agrawal, Kayal, Saxena, 2002)

# Satisfiability in Propositional Logic

A formula of propositional logic consists of

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| `true` | conjunction $\wedge$ | variables |
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### Theorem

Satisfiability of formulas of propositional logic is in NP.

### Proof.

We can construct a nondeterministic Turing machine that

- guesses an assignment of `true` and `false` to the variables,
- evaluates the formula (in polynomial time), and

accepts if the evaluation is `true`. □

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Let $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ be decision problems (languages).

Then $L_1$ is **polynomial-time reducible** to $L_2$ if there exists a **polynomial-time computable** function $f : \Sigma_1^* \to \Sigma_2^*$ such that:

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Let $f : \Sigma_1^* \to \Sigma_2^*$ and $g : \Sigma_2^* \to \Sigma_3^*$ be polynomial-time reductions.
The composition $g \circ f : \Sigma_1^* \to \Sigma_3^*$ is a polynomial-time reduction.

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## NP-completeness

A language $L \in \mathrm{NP}$ is **NP-complete** if every language in NP is polynomial time reducible to $L$.

# Examples of NP-complete Problems

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The **bounded tiling problem** is NP-complete.

…and many more questions

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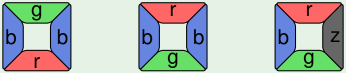
Given a finite collection of **types** of $1 \times 1$ **tiles** with a **colour** on each side. (There are infinitely many tiles of each type.)



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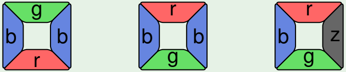


**Bonded tiling problem**: the input is $n \in \mathbb{N}$, a finite collection of types of tiles, the first row of $n$ tiles.

Is it possible to tile an $n \times n$ field (with the given first row)?

When connecting tiles, the touching side must have the same colour. Tiles must not be rotated.

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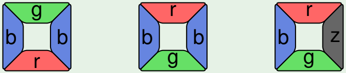


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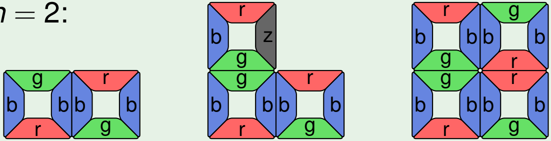
Example $n = 2$:



first row incomplete tiling correct tiling

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We can construct a nondeterministic Turing machien that

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continued on the next slide. . .

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## Proof continued... (the starting row)

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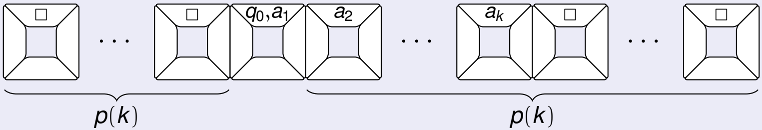

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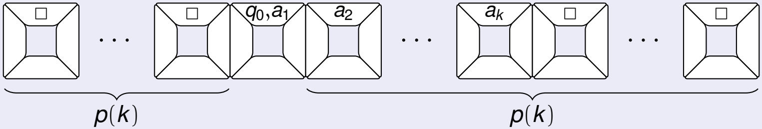
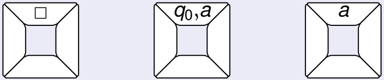



Tiles for building the first row (for every $a \in \Sigma$):



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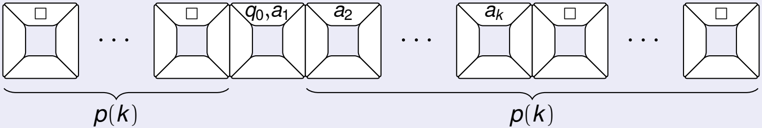


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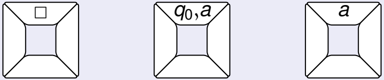


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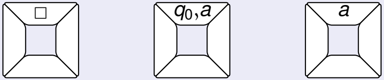

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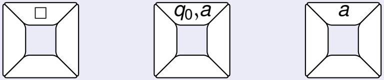


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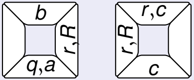


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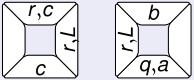


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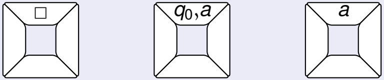


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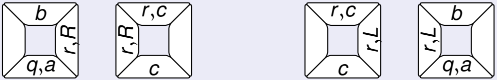


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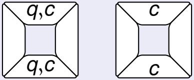
Tiles for leaving the tape unchanged (for every $q \in F$, $c \in \Gamma$):



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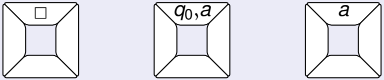


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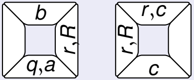
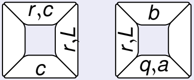
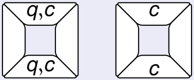



for $(r, b, R) \in \delta(q, a)$




for $(r, b, L) \in \delta(q, a)$

Tiles for leaving the tape unchanged (for every $q \in F$, $c \in \Gamma$):

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# Bounded Tiling Problem is NP-complete

## Proof continued. . .

Then, for input $x = a_1 \cdots a_k$ and with the indicated starting row:

$$n \times n \text{ field can be tiled} \quad \iff \quad x \in L(M)$$

# Bounded Tiling Problem is NP-complete

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Every tiling simulates a computation of $M$ on input $x$.

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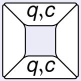


which exists only for $q \in F$.

# Bounded Tiling Problem is NP-complete

**Proof continued...**

Then, for input $x = a_1 \cdots a_k$ and with the indicated starting row:

$$n \times n \text{ field can be tiled} \iff x \in L(M)$$

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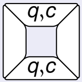


which exists only for $q \in F$.

Tiling can be finished

$$\iff M \text{ has an accepting computation for input } x.$$

# Example

Consider the TM $M$ with $\Sigma = \{a, b\}$, $\Gamma = \Sigma \cup \{\Box\}$, $F = \{q_1\}$ and

$$\delta(q_0, a) = \{(q_0, b, R)\} \qquad \delta(q_0, b) = \{(q_1, b, L)\}$$

Note that $L(M) = L(a^* b(a+b)^*) = L((a+b)^* b(a+b)^*)$

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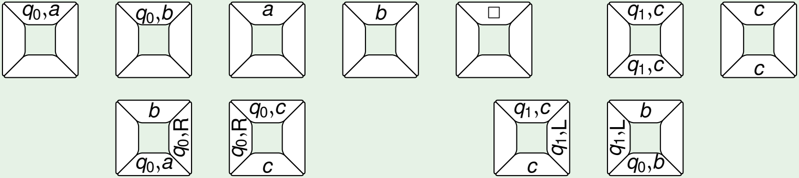


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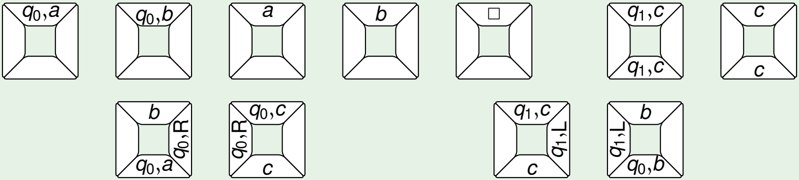


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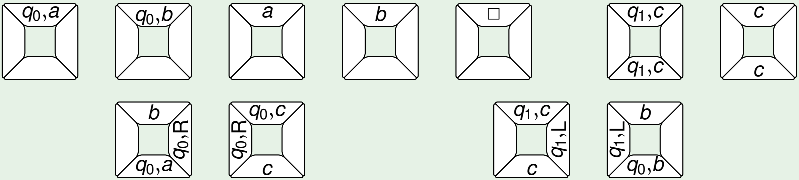


Consider the input word $aaa \notin L(M)$. Then $n = 2p(3) + 1 = 7$.

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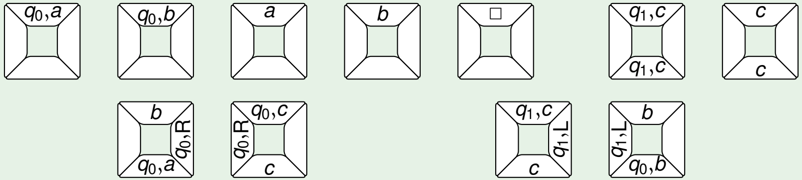


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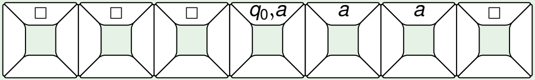

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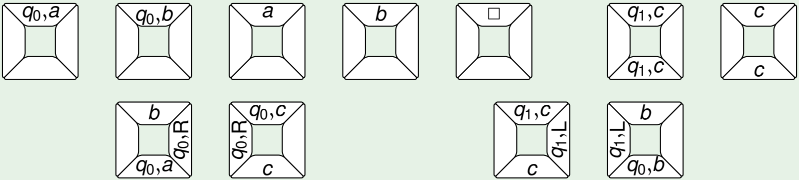


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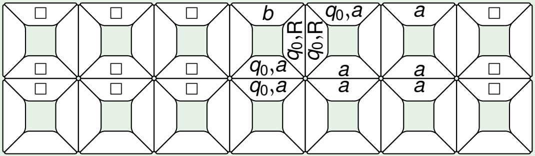

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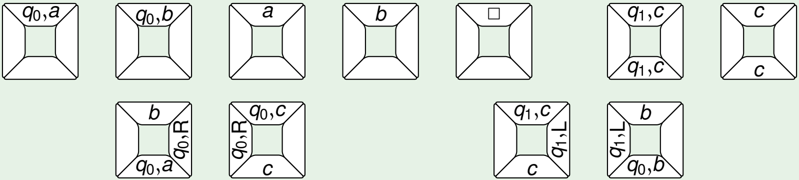


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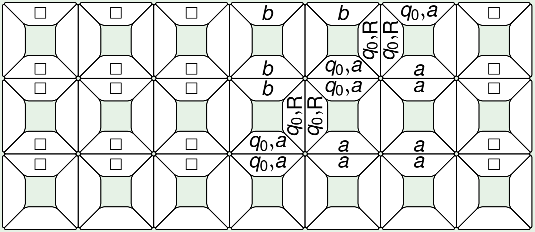

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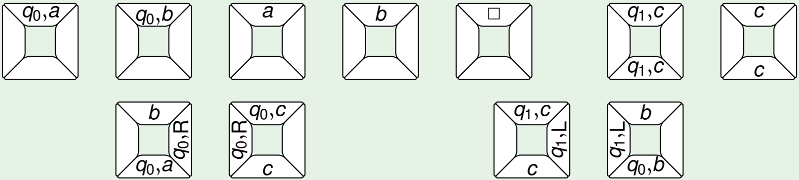


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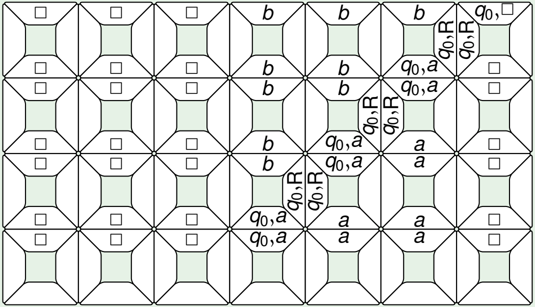

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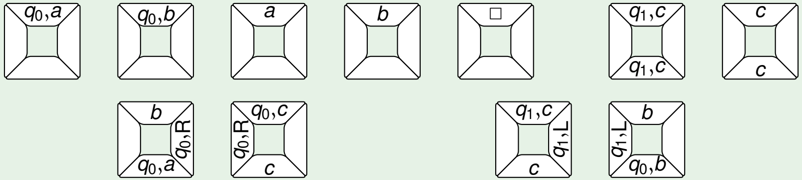


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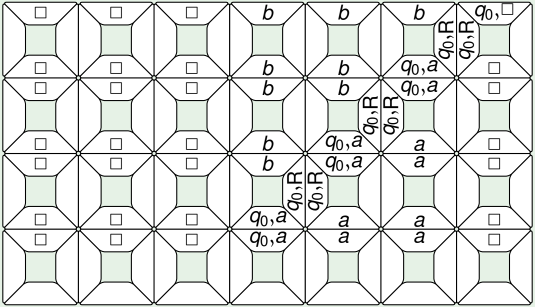


The tiling cannot be completed.

# Example continued

Consider the input word $aab \in L(M)$. Then $n = 2p(3) + 1 = 7$.

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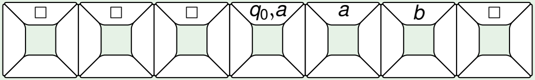

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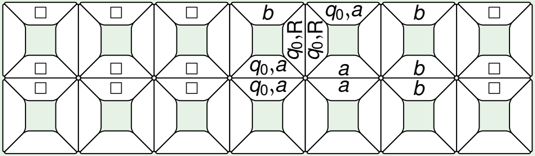

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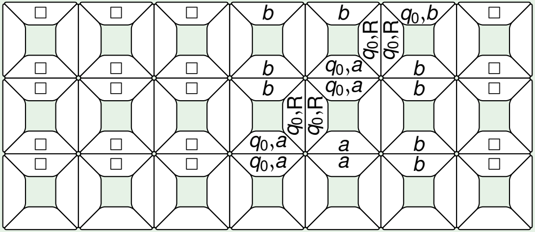

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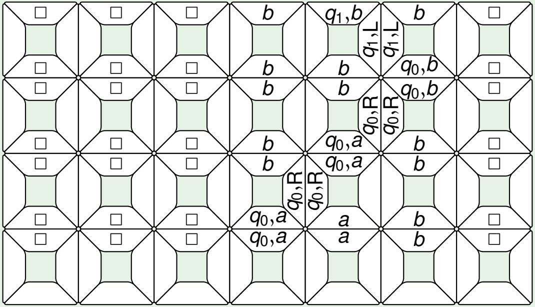

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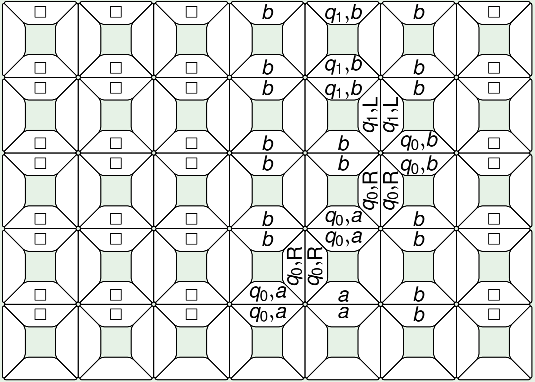

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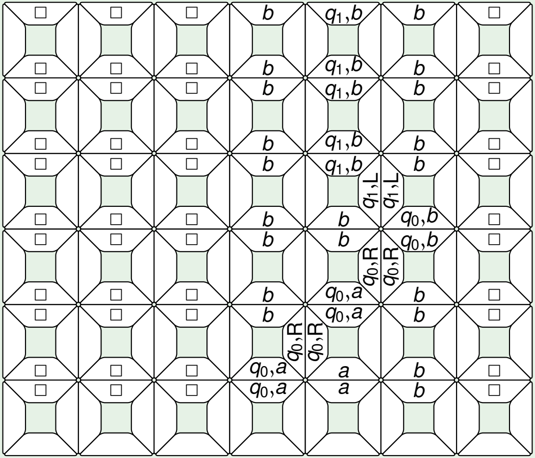

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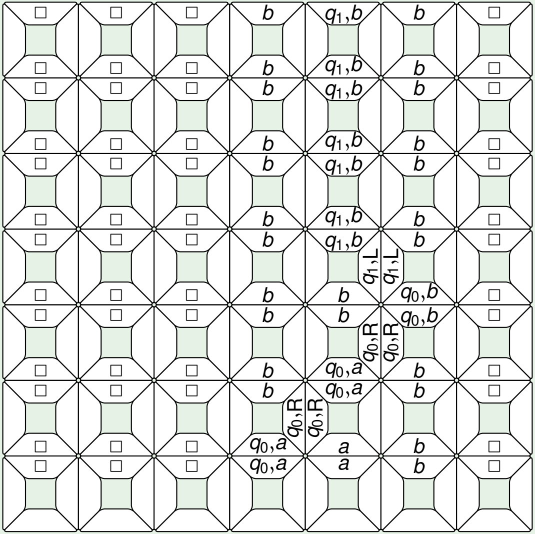


Complete tiling of the $7 \times 7$ field.

# Satisfiability Problem

# Satisfiability Problem is NP-complete

## Theorem of Cook

The satisfiability problem in propositional logic is NP-complete.

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We introduce Boolean variables $x_{rct}$ for $1 \leq r, c \leq n$ and $t \in T$.

Intention: $x_{rct} = \texttt{true} \iff$ tile of type $t$ at row $r$ and column $c$.

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continued on the next slide...

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## Proof continued…

We define $\Phi$ to be the conjunction of the 4 formulas:

1. Fist row is $t_1 \cdots t_n$: $\bigwedge_{c=1}^{n} x_{1ct_k}$
2. At every position at most one tile type:
$$\bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigwedge_{t \neq t'} \neg(x_{rct} \wedge x_{rct'})$$
3. Neighbouring tiles must match (horizontal neighbours):
$$\bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n-1} \bigvee_{tt' \text{ matches}} (x_{rct} \wedge x_{r(c+1)t'})$$
4. Neighbouring tiles must match (vertical neighbours):
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Size of the formula is polynomial in $n$.

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There exists an $n \times n$ tiling with first row $t_1 \cdots t_n$
$\iff$ the propositional formula $\Phi$ is satisfiable.

# Satisfiability Problem is NP-complete

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There exists an $n \times n$ tiling with first row $t_1 \cdots t_n$

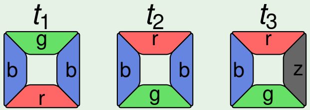
$\iff$ the propositional formula $\Phi$ is satisfiable.

Thus we have a polynomial-time reduction. $\square$

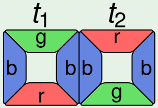
# Exercise

Reduce this bounded tiling problem to the satisfiability problem.

Types of tiles:




First row:



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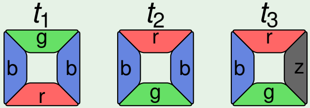


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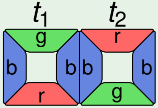


Then $\Phi$ is the conjunction of:

1. $x_{11t_1} \wedge x_{12t_2}$

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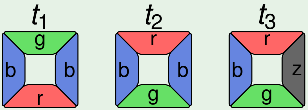


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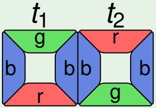


Then $\Phi$ is the conjunction of:

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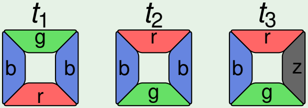
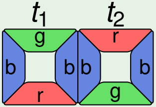



First row:

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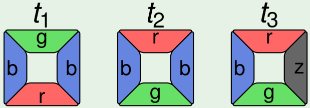
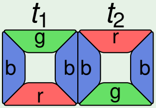
# Exercise

Reduce this bounded tiling problem to the satisfiability problem.

Types of tiles:




First row:




Then $\Phi$ is the conjunction of:

1. $x_{11t_1} \wedge x_{12t_2}$
2. $\neg(x_{11t_1} \wedge x_{11t_2}) \wedge \neg(x_{12t_1} \wedge x_{12t_2}) \wedge \neg(x_{21t_1} \wedge x_{21t_2}) \wedge \neg(x_{22t_1} \wedge x_{22t_2}) \wedge$
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For proving $\mathrm{P} = \mathrm{NP}$ it suffices to show that one NP-complete problem can be solved in deterministic polynomial time.

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**Theorem**

If an NP-complete problem is in co-NP, then NP $=$ co-NP.

Note that there are problems that are both in NP $\cap$ co-NP.

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Let $f, g : \mathbb{N} \to \mathbb{N}$.

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The function $f$ gives an upper bound on the number of visited cells on the tape in terms of the length of the input word.

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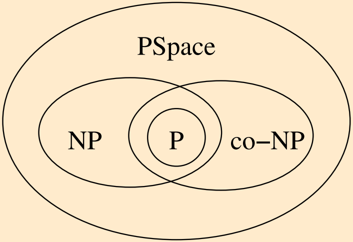
## Theorem of Savitch

$$\text{PSpace} = \text{NPSpace}$$

Actually, the theorem says something more general:

If $L$ is accepted by a nondeterministic TM in $f(n)$ space,
then $L$ is accepted by a deterministic TM in $f(n)^2$ space.

# PSpace-completeness




It is unknown whether these inclusions are strict.

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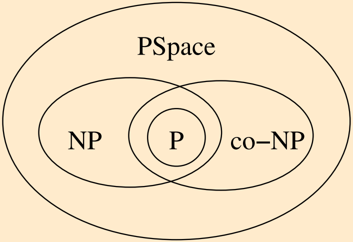


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A language $L \in$ PSpace is **PSpace-complete** if every language $L' \in$ PSpace is polynomial-time reducible to $L$.

$L(r) = \Sigma^*$ ? for regular expression $r$ is PSpace-complete.

# The Classes EXP, NEXP and EXPSpace

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A nondeterministic Turing machine $M$ is

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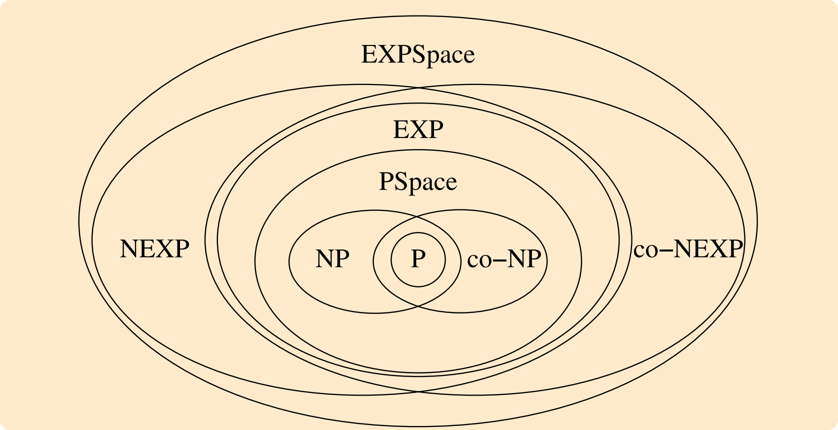
$$\text{P} \subseteq \text{NP} \subseteq \text{PSpace} \subseteq \text{EXP} \subseteq \text{NEXP} \subseteq \text{EXPSpace}$$

It is unknown whether these inclusions are strict. We know

$$\text{P} \neq \text{EXP} \quad \text{NP} \neq \text{NEXP} \quad \text{PSpace} \neq \text{EXPSpace} = \text{NEXPSpace}$$

PSpace $\subseteq$ EXP holds since a polynomial-space TM can at most take an exponential number of configurations.

# Complexity Hierarchy




The following inclusions are known to be strict:

$P \neq EXP \qquad NP \neq NEXP \qquad PSpace \neq EXPSpace$