Automata Theory :: Undecidability

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Decidability

A **decision problem** P is a language $P \subseteq \Sigma^*$.

The problem P is called

- **decidable** if the P is recursive, otherwise **undeciable**,
- **semidecidable** if the P is recursively enumerable.

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- algorithm that always halts
- always answers yes or no

Semidecidable problem:

- algorithm halts (eventually) it the answer is yes ($w \in P$),
- may or may not halt if the answer is no ($w \notin P$).

(Problem: one cannot know how long to wait for an answer.)

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(Semidecidable: execute M on w and wait.)

The following question not decidable and not semidecidable:

Universal halting problem

Does TM M reach a halting state on all $w \in \Sigma^*$? (Input: M.)

(The complement is also not semidecidable.)

The Halting Problem (1936)

The **halting problem** is: given

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- a word x,

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Theorem

The halting problem H is undecidable.

(The language H is not recursive.)

The Halting Problem is Undecidable - Proof 1

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Proof.

Assume the halting problem was decidable. Then there is a Turing machine $\mathcal{H}$ that, given (M, x) decides if M halts on x.

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Let M be a deterministic Turing machine and x a word.

We can decide $x \in L(M)$ as follows:

- If according to $\mathcal{H}$, M does not halt on x, then $x \notin L(M)$.
- If according to $\mathcal{H}$, M halts on x, then execute M on x to see whether $x \in L(M)$.

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The algorithm always terminates, so $L(M)$ is recursive.

Contradiction: not every recursively enumerable language is recursive. □

The Halting Problem is Undecidable - Proof 2

The Halting Problem is Undecidable

Assume there would be a program T with the behaviour:

- input: a program M
- output: *yes* if M terminates on input M, *no* otherwise

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p = read input;
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(both parts together form program T')

What happens if we run T' with input T'?

- initial part T decides whether T' terminates on input T'

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program T:

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p = read input;
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program T' (program T followed by):

```
if result = yes
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What happens if we run T' with input T'?

- initial part T decides whether T' terminates on input T'
- if the result is yes, then T' runs forever

The Halting Problem is Undecidable

Assume there would be a program T with the behaviour:

- input: a program M
- output: *yes* if M terminates on input M, *no* otherwise

program T { p = read input; ⋮ print result; }

{ **if** result = yes **then** loop forever **else** terminate } program T'

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p = read input;
   ⋮
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if result = yes
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What happens if we run T' with input T'?

- initial part T decides whether T' terminates on input T'
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program T' (program T followed by the second block)

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Thus T cannot exist!

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- if the result is no, then T' terminates Contradiction

Thus T cannot exist! The halting problem is undecidable!

Theorem of Rice

Theorem of Rice (1951)

A property of a class K is **trivial** if it holds for **all** or **no** $k \in K$.

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Proof.

Assume that $P(\varnothing)$ (if not, take $\neg P$).

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Assume that $P(\varnothing)$ (if not, take $\neg P$).

Let L_0 be a recursively enumerable language with $\neg P(L_0)$.

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Construct a TM M_x such that M_x accepts y if $x \in L$ and $y \in L_0$.

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Construct a TM M_x such that M_x accepts y if $x \in L$ and $y \in L_0$.

$$L(M_x) = \emptyset \quad \text{if } x \notin L \qquad\qquad L(M_x) = L_0 \quad \text{if } x \in L$$

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Then $x \notin L \iff P(L(M_x))$.

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Then $x \notin L \iff P(L(M_x))$.

Contradiction: decidability of $P \implies L$ recursive. □

Theorem of Rice: Example

For recursively enumerable languages L, the following questions are undecidable:

1. Is $a \in L$?
2. Is L finite?

Post Correspondence Problem

Post Correspondence Problem (1946)

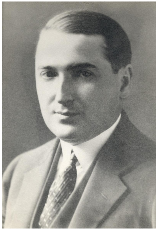
Post Correspondence Problem (PCP)

Given n pairs of words:

$$(w_1, v_1), \ \ldots, \ (w_n, v_n)$$

Are there indices $i_1, i_2 \ldots, i_k$ ($k \geq 1$) s.t.

$$w_{i_1} w_{i_2} \cdots w_{i_k} \quad = \quad v_{i_1} v_{i_2} \cdots v_{i_k} \quad ?$$



Emil Post
(1897-1954)

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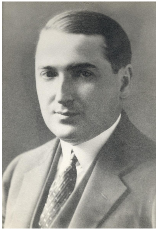

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Exercise

Find a solution for

$$(w_1, v_1) = (01, 100)$$
$$(w_2, v_2) = (1, 011)$$
$$(w_3, v_3) = (110, 1)$$

Modified Post Correspondence Problem

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Are there indices $i_1, i_2 \ldots, i_k$ ($k \geq 1$) such that

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We define (where F and E are fresh):

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\begin{array}{llllll}
w_1 & = & F & v_1 & = & FS \Rightarrow \\
w_2 & = & \Rightarrow wE & v_2 & = & E \\
\vdots & & x & \vdots & & y \qquad (x \to y \in P) \\
 & & a & & & a \qquad (a \in T) \\
 & & A & & & A \qquad (A \in V) \\
 & & \Rightarrow & & & \Rightarrow
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This MPCP has a solution $\iff$ $w \in L(G)$.

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This MPCP has a solution $\iff$ $w \in L(G)$.

Contradiction: If the MPCP was decidable, then $w \in L(G)$ was decidable for unrestricted grammars G! □

Example

$$S \to AA \qquad A \to aB \mid Bb \qquad BB \to aa$$

This grammar with $w = aaab$ translates to the following MPCP:

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i	w_i	v_i
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Example derivation: $S \Rightarrow AA \Rightarrow aBA \Rightarrow aBBb \Rightarrow aaab$.

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$FS\Rightarrow$	AA	$\Rightarrow$	aB	A	$\Rightarrow$	a	B	Bb	$\Rightarrow$

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Post Correspondence Problem

Theorem

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Proof.

Given an MPCP X: $(w_1, v_1), \ldots, (w_n, v_n)$ where

$$w_i = a_{i1} \cdots a_{im_i} \quad \text{and} \quad v_i = b_{i1} \cdots b_{ir_i} \quad (\text{with } m_i + r_i > 0)$$

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As the MPCP is undecidable, so must be the PCP. □

Example

Consider the following instance of the MPCP:

$$w_1 = 11 \qquad w_2 = 1$$
$$v_1 = 1 \qquad v_2 = 11$$

It reduces to the following PCP problem:

$$y_0 = @\$1\$1\$ \qquad y_1 = 1\$1\$ \qquad y_2 = 1\$ \qquad y_3 = \#$$
$$z_0 = @\$1 \qquad z_1 = \$1 \qquad z_2 = \$1\$1 \qquad z_3 = \$\#$$

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Example solution MPCP:

$$w_1 w_2 = 111 = v_1 v_2$$

Corresponding solution PCP:

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In general: the original MPCP instance has a solution
$\iff$ the resulting PCP instance has a solution

Undecidable Properties of Context-Free Languages

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Undecidable properties of context-free languages:

- empty intersection
- ambiguity
- palindromes
- equality
- …

Empty Intersection of Context-Free Languages

Theorem

The question $L_1 \cap L_2 = \varnothing$ **?** for context-free languages L_1, L_2 is undecidable.

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for $1 \leq i \leq n$. Here #, $\langle$ and $\rangle$ are fresh symbols.

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for $1 \leq i \leq n$. Here #, $\langle$ and $\rangle$ are fresh symbols. Then

$$L(G_1) = \{w_j \cdots w_k \# \langle k \rangle \cdots \langle j \rangle \mid 1 \leq j, \ldots, k \leq n\}$$
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$L(G_1) \cap L(G_2) = \varnothing \iff$ the PCP X has no solution. □

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Then G is ambiguous $\iff$ the PCP X has a solution. □

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$L(G)$ contains a palindrome $\iff$ PCP X has a solution. $\square$

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The question $L = \Sigma^*$? (and hence $L_1 = L_2$?) for context-free languages L (L_1, L_2) is undecidable.

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Given a PCP X: $(w_1, v_1), \ldots, (w_n, v_n)$. Define G_1 and G_2:

$$S_1 \to w_i S_1 \langle i \rangle \mid w_i \# \langle i \rangle$$
$$S_2 \to v_i S_2 \langle i \rangle \mid v_i \# \langle i \rangle$$

as before. Then

$$\begin{aligned}
\text{PCP } X \text{ has no solution} &\iff L(G_1) \cap L(G_2) = \varnothing \\
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Equality of Context-Free Languages

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Equality of Context-Free Languages (2)

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The words in $L(G_1)$ are of the form

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All these words are of the shape

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Exercise

Give context-free grammars for L_{smaller}, L_{larger} and L_{equal}.

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- halting problem,
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- non-empty intersection of context-free languages,
- ambiguity of context-free grammars.

There exist algorithms for these problems that always halt if the answer is yes, but may or may not halt if the answer is no.

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In 1970 **Yuri Matiyasevich** proved that this is undecidable.