Automata Theory :: Unrestricted Grammars

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Theorem

A language L is generated by an unrestricted grammar $\iff$ L is accepted by a Turing machine.

From Unrestricted Grammars to Turing Machines

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For every unrestricted G there is a Turing machine M such that

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M can do a **breadth-first search** for a derivation of w from S.

If a derivation is found, then w is accepted by M.

Then $L(M) = L(G)$.

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The variables are S, T, $\square$

and V^{α}_{γ}, $V^{\alpha}_{q\gamma}$ for every $\alpha \in \Sigma \cup \{\square\}$, $\gamma \in \Gamma$ and $q \in Q$.

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Step 1: guessing the word w

$$S \to V_{\Box}^{\Box} S \mid S V_{\Box}^{\Box} \mid T$$

$$T \to T V_{a}^{a} \mid V_{q_0 a}^{a} \qquad \text{for every } a \in \Sigma$$

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Step 1: guessing the word w

$$S \to V_\Box^\Box S \mid SV_\Box^\Box \mid T$$
$$T \to TV_a^a \mid V_{q_0 a}^a \qquad \text{for every } a \in \Sigma$$

After step 1, we have derived something of the form

$$V_\Box^\Box \cdots V_\Box^\Box V_{q_0 a_1}^{a_1} V_{a_2}^{a_2} V_{a_3}^{a_3} \cdots V_{a_n}^{a_n} V_\Box^\Box \cdots V_\Box^\Box$$

where $w = a_1 a_2 \cdots a_n$.

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where $w = a_1 a_2 \cdots a_n$.

Next, the TM is simulated using the lower line (the subscripts).

From Turing Machines to Unrestricted Grammars

Construction continued

Step 2: simulating the TM (in the subscripts)

$$V^{\alpha}_{qc} V^{\beta}_{\gamma} \to V^{\alpha}_{d} V^{\beta}_{q'\gamma} \qquad \text{if } \delta(q, c) = (q', d, R)$$

$$V^{\beta}_{\gamma} V^{\alpha}_{qc} \to V^{\beta}_{q'\gamma} V^{\alpha}_{d} \qquad \text{if } \delta(q, c) = (q', d, L)$$

for every $\alpha, \beta \in \Sigma \cup \{\square\}$ and $\gamma \in \Gamma$.

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for every $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$.

Step 3: If TM reaches accepting state, then generate w.
(From the superscripts left unchanged in step 2.)

$$V^{\alpha}_{q\gamma} \to \alpha \qquad \text{for every } q \in F$$

$$\beta V^{\alpha}_{\gamma} \to \beta\alpha$$

$$V^{\alpha}_{\gamma}\beta \to \alpha\beta$$

$$\Box \to \lambda$$

for every $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$.

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for every $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$.

Then $L(G) = L(M)$.

Example

Consider the TM with $\Sigma = \{a, b, c\}$, $\Gamma = \Sigma \cup \{\Box\}$, $F = \{q_2\}$ and

$$\delta(q_0, a) = (q_0, c, R) \qquad \delta(q_0, c) = (q_1, b, L)$$
$$\delta(q_0, b) = (q_0, b, R) \qquad \delta(q_1, b) = (q_2, a, R)$$

This TM accepts the language $L((a+b)^* bc(a+b+c)^*)$.

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The resulting grammar is:

$$S \rightarrow V_{\Box}^{\Box} S \mid S V_{\Box}^{\Box} \mid T$$
$$T \rightarrow T V_a^a \mid T V_b^b \mid T V_c^c \mid V_{q_0 a}^a \mid V_{q_0 b}^b \mid V_{q_0 c}^c$$

$$V_{q_0 a}^{\alpha} V_{\gamma}^{\beta} \rightarrow V_c^{\alpha} V_{q_0 \gamma}^{\beta} \qquad V_{q_2 \gamma}^{\alpha} \rightarrow \alpha$$
$$V_{q_0 b}^{\alpha} V_{\gamma}^{\beta} \rightarrow V_b^{\alpha} V_{q_0 \gamma}^{\beta} \qquad \beta V_{\gamma}^{\alpha} \rightarrow \beta\alpha$$
$$V_{\gamma}^{\beta} V_{q_0 c}^{\alpha} \rightarrow V_{q_1 \gamma}^{\beta} V_b^{\alpha} \qquad V_{\gamma}^{\alpha} \beta \rightarrow \alpha\beta$$
$$V_{q_1 b}^{\alpha} V_{\gamma}^{\beta} \rightarrow V_a^{\alpha} V_{q_2 \gamma}^{\beta} \qquad \Box \rightarrow \lambda$$

with $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$.

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$$T \to TV_a^a \mid TV_b^b \mid TV_c^c \mid V_{q_0 a}^a \mid V_{q_0 b}^b \mid V_{q_0 c}^c$$

$$V_{q_0 a}^\alpha V_\gamma^\beta \to V_c^\alpha V_{q_0 \gamma}^\beta \qquad V_{q_2 \gamma}^\alpha \to \alpha$$
$$V_{q_0 b}^\alpha V_\gamma^\beta \to V_b^\alpha V_{q_0 \gamma}^\beta \qquad \beta V_\gamma^\alpha \to \beta\alpha$$
$$V_\gamma^\beta V_{q_0 c}^\alpha \to V_{q_1 \gamma}^\beta V_b^\alpha \qquad V_\gamma^\alpha \beta \to \alpha\beta$$
$$V_{q_1 b}^\alpha V_\gamma^\beta \to V_a^\alpha V_{q_2 \gamma}^\beta \qquad \Box \to \lambda$$

with $\alpha, \beta \in \Sigma \cup \{\Box\}$ and $\gamma \in \Gamma$. Exercise: derive *abc*.