Automata Theory :: Turing Machines

Jörg Endrullis

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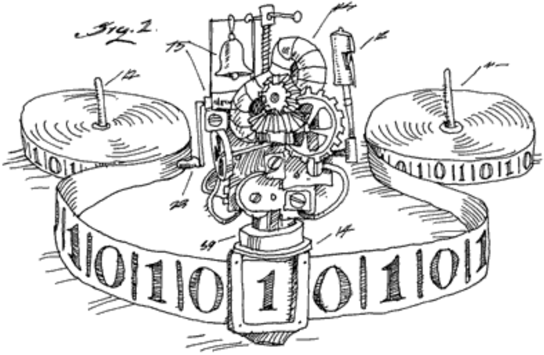
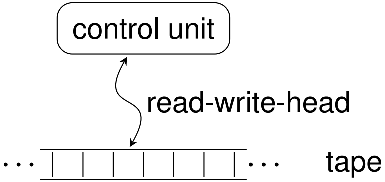
Turing Machines



Turing Machines

Turing machines can read and write the input word.

Input is written on a tape on which a read-write-head works.



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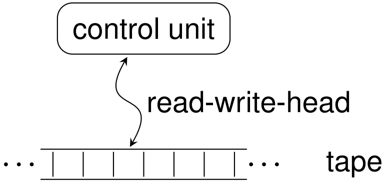

In each step:

- the read-write-head reads a symbol from the tape,
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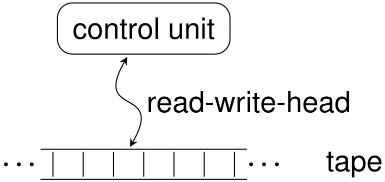

In each step:

- the read-write-head reads a symbol from the tape,
- overwrites the symbol, and
- moves one place to the left or right.

The tape is two-sided infinite: unlimited memory!

Turing Machines

We introduce a **blank symbol** $\square$. The initial tape content is

$$\cdots \square\square\square\square \text{ input word } \square\square\square\square \cdots$$

There is a finite set of states Q and a finite tape alphabet Γ.

Turing Machines

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The transition function δ has the form

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{ L, R \}$$

Here δ is a **partial function**: $\delta(q, a)$ may be undefined.

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$\delta(q, a) = (q', b, X)$ means: if

- the machine is in state q, and
- the head reads a from the tape

then

- then a is overwritten by b,
- the head moves 1 position left if $X = L$, right if $X = R$, and
- the machine switches to state q'.

Turing Machines

A **deterministic Turing machine**, short TM, is a 7-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$$

where

- Q is a finite set of states,
- $\Sigma \subseteq \Gamma \setminus \{ \Box \}$ a finite input alphabet,
- Γ a finite tape alphabet,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{ L, R \}$ a partial transition function,
- q_0 the starting state,
- $\Box \in \Gamma$ the blank symbol,
- $F \subseteq Q$ a set of final (accepting) states.

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Assumption: $\delta(q, a)$ is undefined for every $q \in F$ and $a \in \Gamma$.

So the computation stops when reaching a final state.

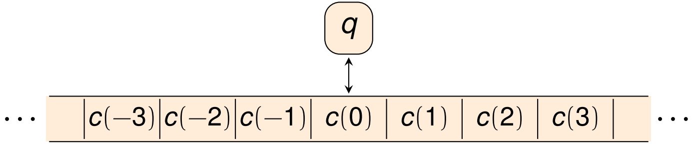
Turing Machine Configuration

A **configuration** (q, c) of a Turing machine consists of

- a state $q \in Q$, and
- a function $c : \mathbb{Z} \to \Gamma$, the **tape content.**

The non-blank positions $\{z \in \mathbb{Z} \mid c(z) \neq \square\}$ are finite.

The head of the machine stands on $c(0)$.



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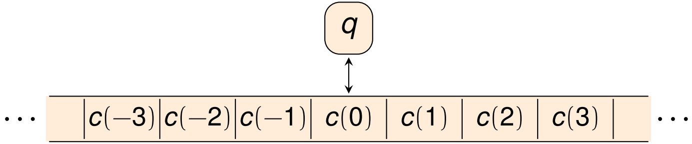


Let $n, m \in \mathbb{N}$ (exist for every configuration) such that

$$\forall i < -n.\ c(i) = \square \qquad \text{and} \qquad \forall i > m.\ c(i) = \square$$

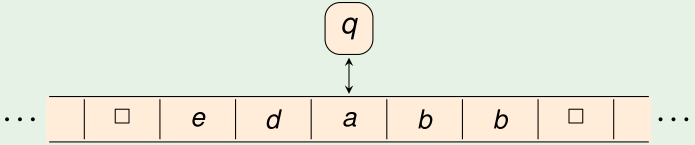
Then we **denote the configuration by the finite word**

$$c(-n)c(-n+1)\cdots c(-1)\ {\color{red}q}\ c(0)c(1)\cdots c(m)$$

Turing Machine Configuration

So configurations are denoted by words from $\Gamma^* \times Q \times \Gamma^*$.

For instance, the configuration



can be denoted by

$$ed\, q\, abb$$

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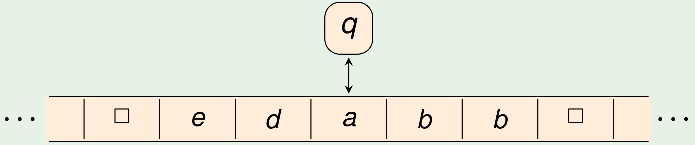

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The words

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denote the same configuration.

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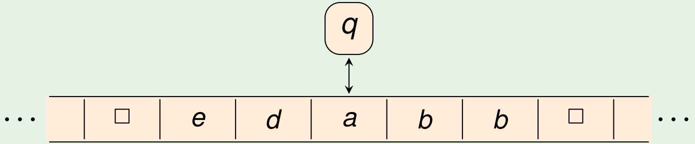

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The words

$$ed\,q\,abb\square \quad\approx\quad \square ed\,q\,abb \quad\approx\quad \square\square ed\,q\,abb\square \quad\cdots$$

denote the same configuration.

We write $w \approx v$ if w and v denote the same configuration.

Turing Machine Computations

The **computation steps** $\vdash$ on configurations are defined by:

$$v q a w \vdash v b q' w \qquad \text{if } \delta(q, a) = (q', b, R)$$
$$v c q a w \vdash v q' c b w \qquad \text{if } \delta(q, a) = (q', b, L)$$

where $v, w \in \Gamma^*$, $a, c \in \Gamma$ and $q \in Q$.

We write $\vdash^*$ for a computation of zero or more steps.

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The **computation steps** $\vdash$ on configurations are defined by:

$$vqaw \vdash vbq'w \qquad \text{if } \delta(q, a) = (q', b, R)$$
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Assume that (δ is undefined in all other case)

$$\delta(q_0, a) = (q_0, a, R) \quad \delta(q_1, a) = (q_1, b, L) \quad \delta(q_0, \Box) = (q_1, c, L)$$

Then we have steps:

$$q_0 aa$$

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A configuration $vqaw$ is a **halting state** if $\delta(q, a)$ is undefined.

Drawing Turing Machines

The transition graph for a TMs contains

an arrow $q \xrightarrow{a/b\ X} q'$ whenever $\delta(q, a) = (q', b, X)$

The Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ with $\Sigma = \{a, b\}$, $\Gamma = \{a, b, \Box\}$, $Q = \{q_0, q_1, q_2\}$, $F = \{q_2\}$ and

$$\delta(q_0, a) = (q_1, b, R) \qquad \delta(q_1, a) = (q_0, b, R)$$
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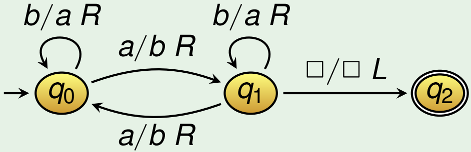
can be visualised as



Turing Machines and Languages

The **language** $L(M)$ accepted by TM $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ is

$$\{\, w \in \Sigma^* \;\mid\; q_0 w \vdash^* uqv \;\text{ for some } q \in F,\; u, v \in \Gamma^* \,\}$$

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If $w \notin L(M)$ this can have two causes:

- the execution halts in a configuration vqw with $q \notin F$, or
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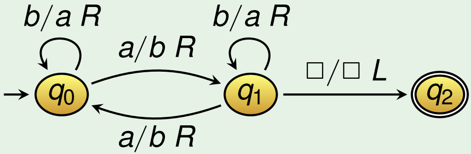

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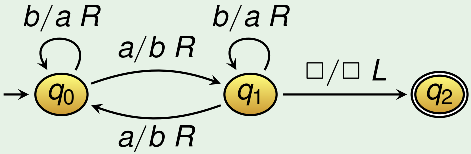

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The set of words over $\Sigma = \{ a, b \}$ with an odd number of a's.

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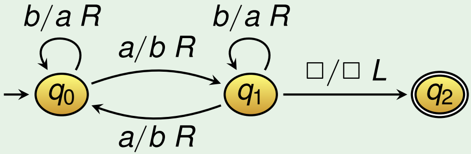

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A language is **recursively enumerable** if it is accepted by a TM.

Example

We construct a TM M with $L(M) = \{ a^n b^n c^n \mid n \geq 1 \}$.

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- $\Sigma = \{ a, b, c \}$ and $\Gamma = \{ a, b, c, 0, 1, 2, \Box \}$
- q_0: Read a, replace by 0, move right and switch to q_1.
- q_1: Keep moving right until we read b.
  Replace b by 1, move right and switch to q_2.
- q_2: Keep moving right until we read c.
  Replace c by 2, move left and switch to q_3.
- q_3: Keep moving left until we read 0.
  Move right and switch back to q_0.
- If we read 1 in q_0, switch to q_4.
- q_4: Keep moving right to check whether there are a's, b's or c's left. If not, then go to final state q_5.

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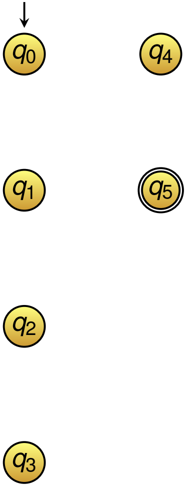

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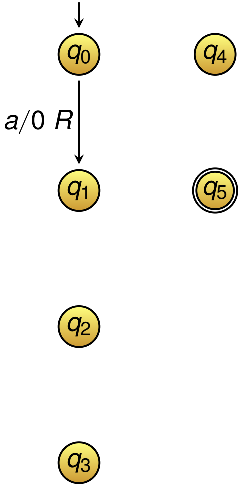

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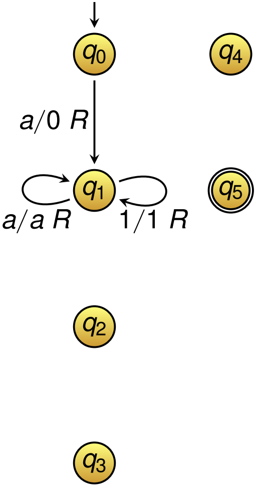

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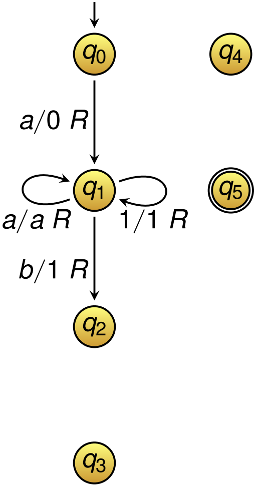

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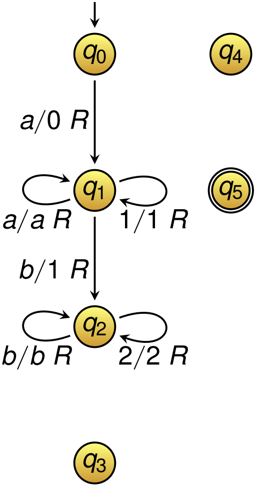

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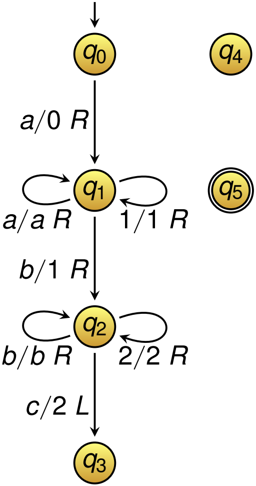

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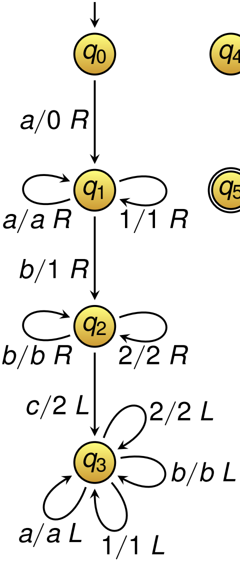

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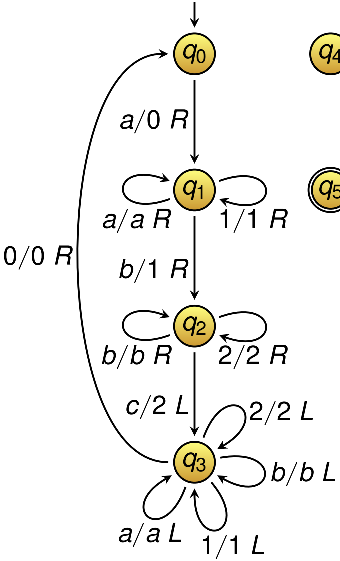

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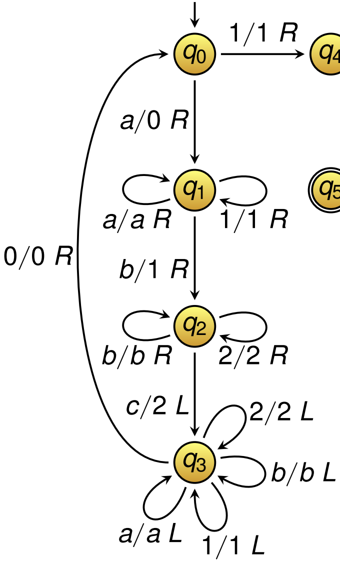

1/1 R
q0
q4
a/0 R
q1
q5
a/a R
1/1 R
0/0 R
b/1 R
q2
b/b R
2/2 R
c/2 L
2/2 L
q3
b/b L
a/a L
1/1 L

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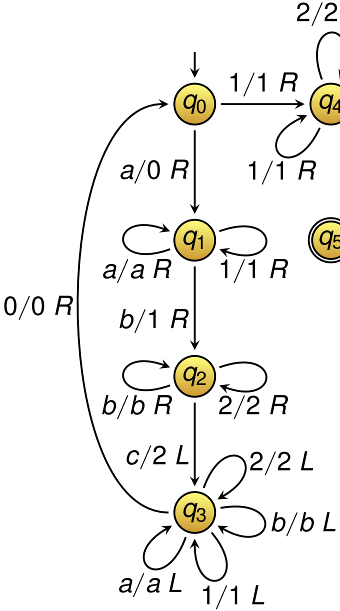

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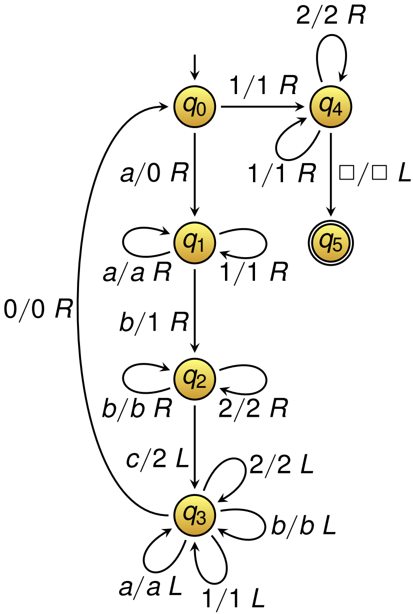
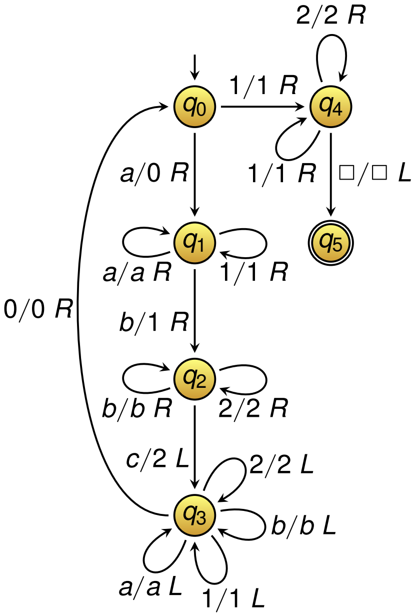
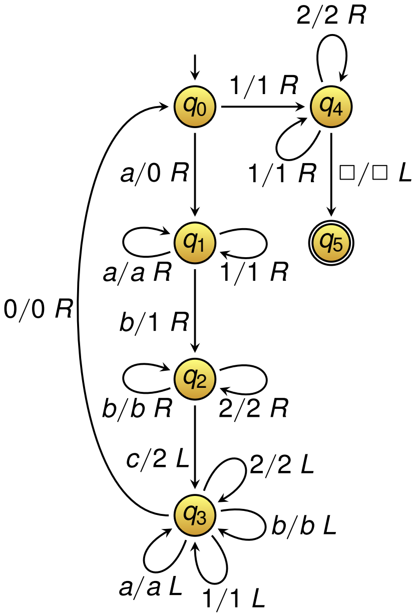

2/2 R
1/1 R
q_0
q_4
a/0 R
1/1 R
□/□ L
q_1
q_5
a/a R
1/1 R
0/0 R
b/1 R
q_2
b/b R
2/2 R
c/2 L
2/2 L
q_3
b/b L
a/a L
1/1 L

Example



q_0 *aabbcc*

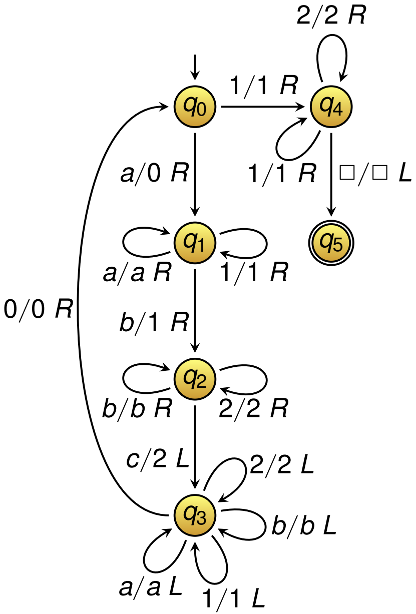
Example



$q_0\, aabbcc$

$\vdash 0 q_1\, abbcc$

Example

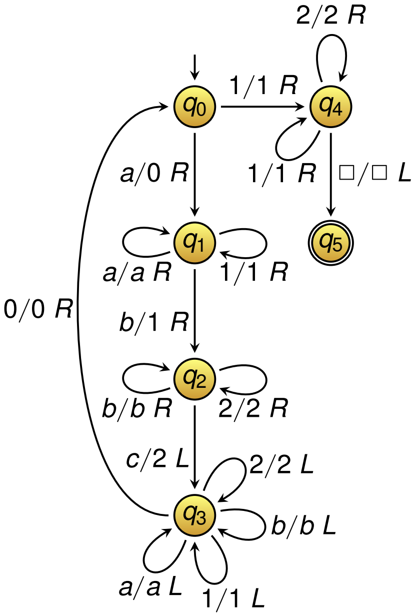


$q_0\, aabbcc$

$\vdash 0 q_1\, abbcc$

$\vdash 0 a q_1\, bbcc$

Example



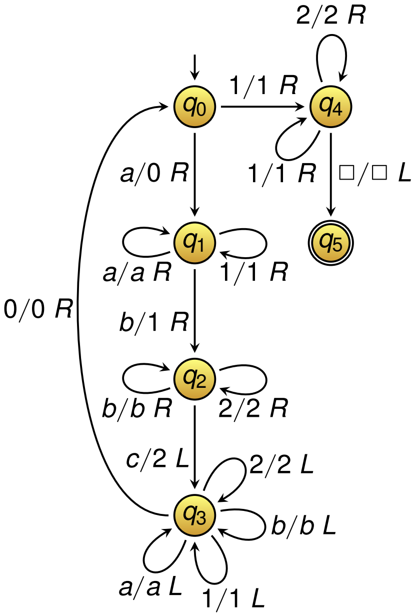
$q_0 aabbcc$

$\vdash 0 q_1 abbcc$

$\vdash 0 a q_1 bbcc$

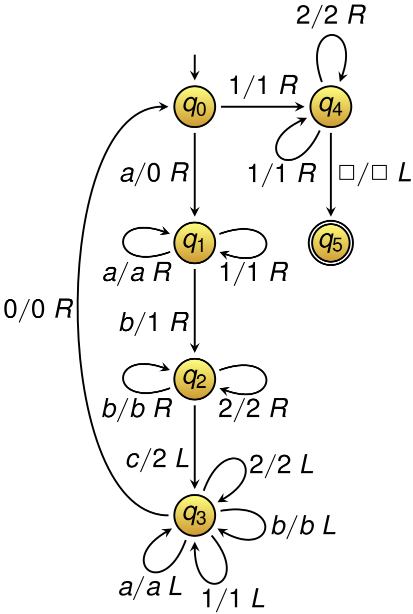
$\vdash 0 a 1 q_2 bcc$

Example



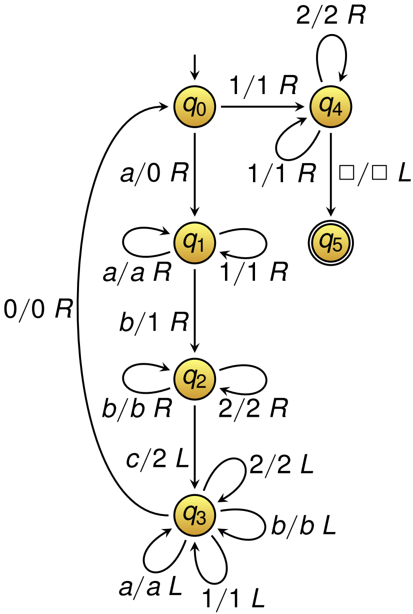
$$q_0\, aabbcc$$
$$\vdash 0 q_1\, abbcc$$
$$\vdash 0 a q_1\, bbcc$$
$$\vdash 0 a 1 q_2\, bcc$$
$$\vdash 0 a 1 b q_2\, cc$$

Example



$$q_0\,aabbcc$$
$$\vdash 0q_1\,abbcc$$
$$\vdash 0aq_1\,bbcc$$
$$\vdash 0a1\,q_2\,bcc$$
$$\vdash 0a1bq_2\,cc$$
$$\vdash 0a1\,q_3\,b2c$$

Example



$q_0\, aabbcc$

$\vdash 0 q_1\, abbcc$

$\vdash 0a q_1\, bbcc$

$\vdash 0a1 q_2\, bcc$

$\vdash 0a1b q_2\, cc$

$\vdash 0a1 q_3\, b2c$

$\vdash 0a q_3\, 1b2c$

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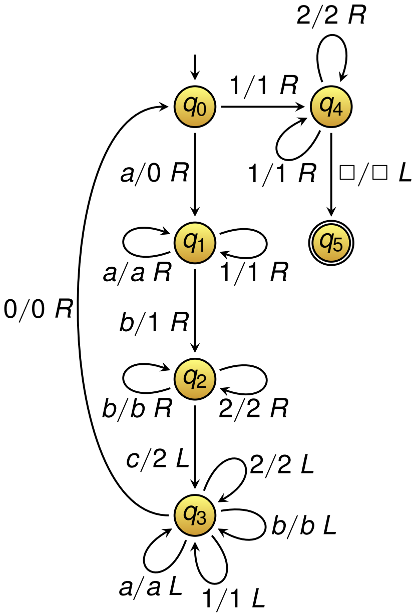
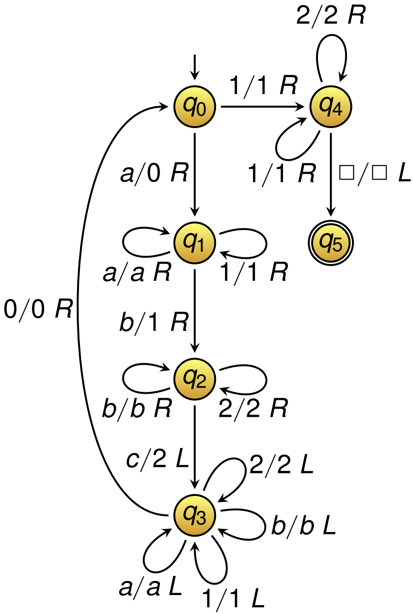


$$
\begin{aligned}
& q_0\, aabbcc \\
\vdash\, & 0 q_1\, abbcc \\
\vdash\, & 0a q_1\, bbcc \\
\vdash\, & 0a1 q_2\, bcc \\
\vdash\, & 0a1b q_2\, cc \\
\vdash\, & 0a1 q_3\, b2c \\
\vdash\, & 0a q_3\, 1b2c \\
\vdash\, & 0 q_3\, a1b2c
\end{aligned}
$$

Example



$q_0\, aabbcc$

$\vdash 0 q_1\, abbcc$

$\vdash 0a q_1\, bbcc$

$\vdash 0a1 q_2\, bcc$

$\vdash 0a1b q_2\, cc$

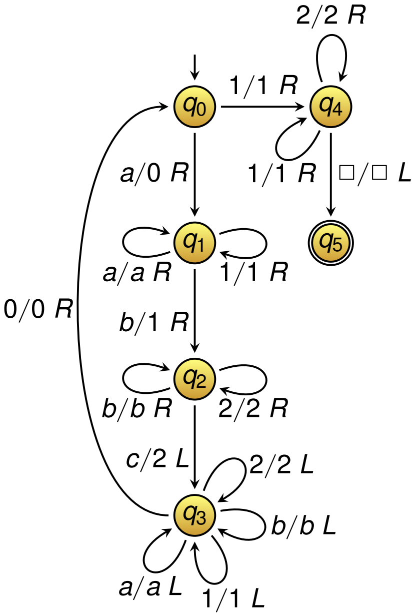
$\vdash 0a1 q_3\, b2c$

$\vdash 0a q_3\, 1b2c$

$\vdash 0 q_3\, a1b2c$

$\vdash q_3\, 0a1b2c$

Example



$q_0\,aabbcc$

$\vdash 0q_1\,abbcc$

$\vdash 0aq_1\,bbcc$

$\vdash 0a1q_2\,bcc$

$\vdash 0a1bq_2\,cc$

$\vdash 0a1q_3\,b2c$

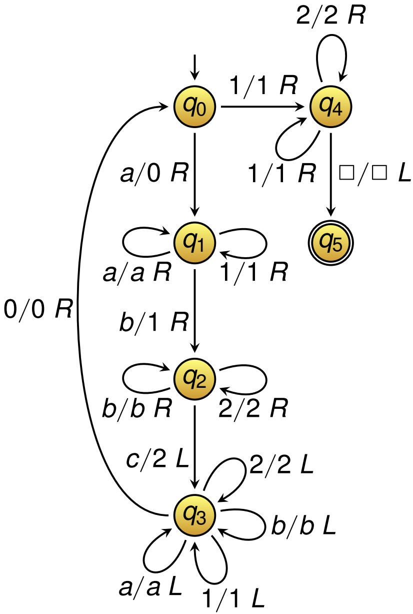
$\vdash 0aq_3\,1b2c$

$\vdash 0q_3\,a1b2c$

$\vdash q_3\,0a1b2c$

$\vdash 0q_0\,a1b2c$

Example



$q_0 aabbcc$

$\vdash 0 q_1 abbcc$

$\vdash 0 a q_1 bbcc$

$\vdash 0 a 1 q_2 bcc$

$\vdash 0 a 1 b q_2 cc$

$\vdash 0 a 1 q_3 b 2 c$

$\vdash 0 a q_3 1 b 2 c$

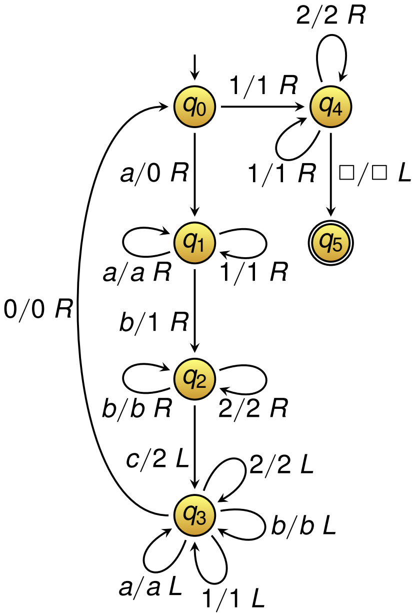
$\vdash 0 q_3 a 1 b 2 c$

$\vdash q_3 0 a 1 b 2 c$

$\vdash 0 q_0 a 1 b 2 c$

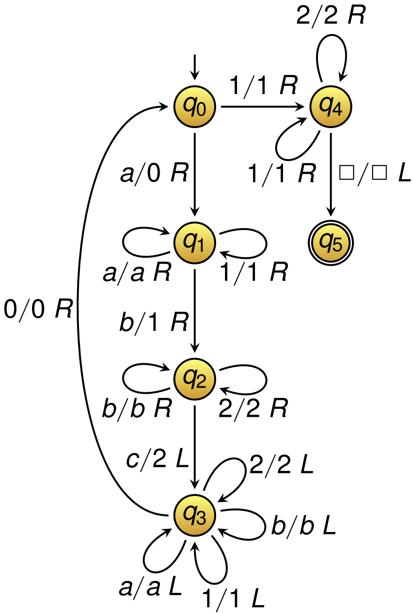
$\vdash^* 00 q_0 1122$

Example



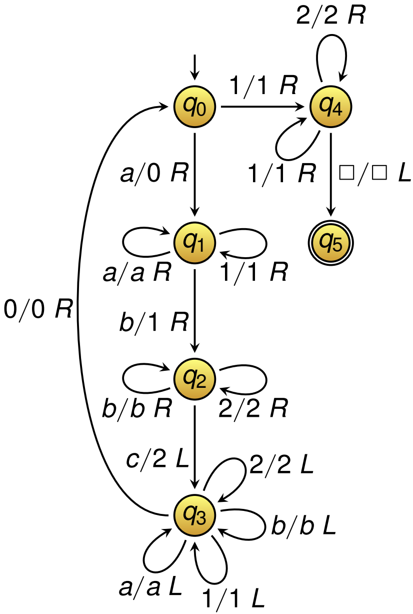
$$q_0 aabbcc$$
$$\vdash 0 q_1 abbcc$$
$$\vdash 0 a q_1 bbcc$$
$$\vdash 0 a 1 q_2 bcc$$
$$\vdash 0 a 1 b q_2 cc$$
$$\vdash 0 a 1 q_3 b 2 c$$
$$\vdash 0 a q_3 1 b 2 c$$
$$\vdash 0 q_3 a 1 b 2 c$$
$$\vdash q_3 0 a 1 b 2 c$$
$$\vdash 0 q_0 a 1 b 2 c$$
$$\vdash^* 00 q_0 1122$$
$$\vdash 001 q_4 122$$

Example



$$
\begin{aligned}
& q_0\,aabbcc \\
\vdash\ & 0q_1\,abbcc \\
\vdash\ & 0aq_1\,bbcc \\
\vdash\ & 0a1\,q_2bcc \\
\vdash\ & 0a1bq_2\,cc \\
\vdash\ & 0a1\,q_3b2c \\
\vdash\ & 0aq_3\,1b2c \\
\vdash\ & 0q_3\,a1b2c \\
\vdash\ & q_3\,0a1b2c \\
\vdash\ & 0q_0\,a1b2c \\
\vdash^*\ & 00q_0\,1122 \\
\vdash\ & 001\,q_4\,122 \\
\vdash^*\ & 001122q_4
\end{aligned}
$$

Example



$q_0 aabbcc$

$\vdash 0 q_1 abbcc$

$\vdash 0 a q_1 bbcc$

$\vdash 0 a 1 q_2 bcc$

$\vdash 0 a 1 b q_2 cc$

$\vdash 0 a 1 q_3 b 2 c$

$\vdash 0 a q_3 1 b 2 c$

$\vdash 0 q_3 a 1 b 2 c$

$\vdash q_3 0 a 1 b 2 c$

$\vdash 0 q_0 a 1 b 2 c$

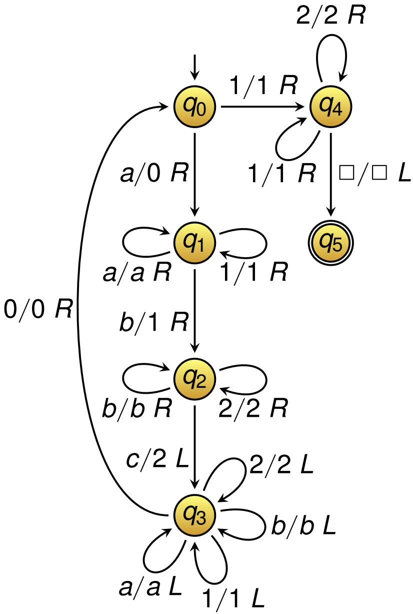
$\vdash^* 00 q_0 1122$

$\vdash 001 q_4 122$

$\vdash^* 001122 q_4$

$\vdash 00112 q_5 2$

Example



$q_0 aabbcc$

$\vdash 0 q_1 abbcc$

$\vdash 0 a q_1 bbcc$

$\vdash 0 a 1 q_2 bcc$

$\vdash 0 a 1 b q_2 cc$

$\vdash 0 a 1 q_3 b 2 c$

$\vdash 0 a q_3 1 b 2 c$

$\vdash 0 q_3 a 1 b 2 c$

$\vdash q_3 0 a 1 b 2 c$

$\vdash 0 q_0 a 1 b 2 c$

$\vdash^* 00 q_0 1122$

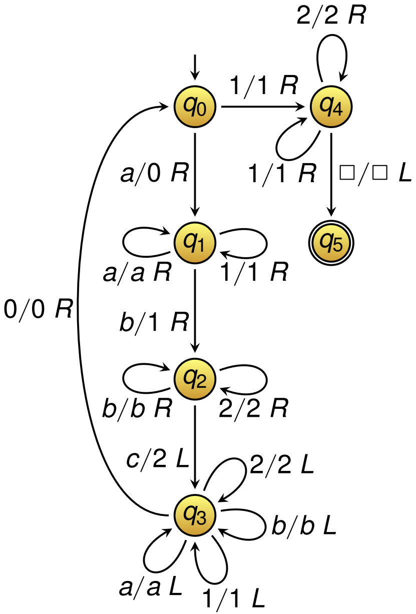
$\vdash 001 q_4 122$

$\vdash^* 001122 q_4$

$\vdash 00112 q_5 2$

$q_0 aabbbcc$

Example



$q_0\, aabbcc$

$\vdash 0 q_1\, abbcc$

$\vdash 0a q_1\, bbcc$

$\vdash 0a1 q_2\, bcc$

$\vdash 0a1b q_2\, cc$

$\vdash 0a1 q_3\, b2c$

$\vdash 0a q_3\, 1b2c$

$\vdash 0 q_3\, a1b2c$

$\vdash q_3\, 0a1b2c$

$\vdash 0 q_0\, a1b2c$

$\vdash^* 00 q_0\, 1122$

$\vdash 001 q_4\, 122$

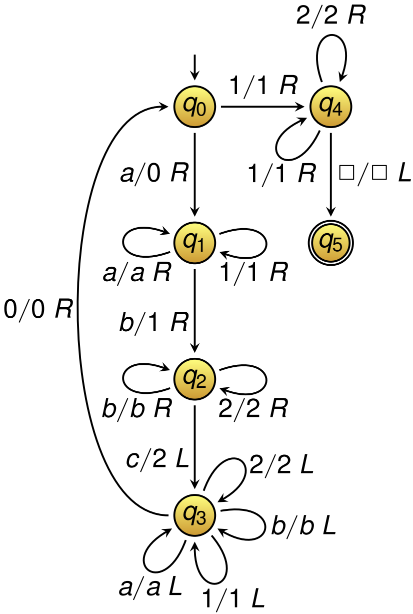
$\vdash^* 001122 q_4$

$\vdash 00112 q_5\, 2$

$q_0\, aabbbcc$

$\vdash^+ 0 q_0\, a1bb2c$

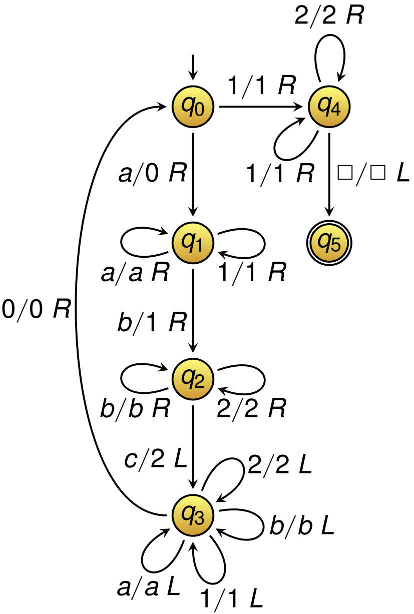
Example



$q_0 aabbcc$
$\vdash 0 q_1 abbcc$
$\vdash 0 a q_1 bbcc$
$\vdash 0 a 1 q_2 bcc$
$\vdash 0 a 1 b q_2 cc$
$\vdash 0 a 1 q_3 b 2 c$
$\vdash 0 a q_3 1 b 2 c$
$\vdash 0 q_3 a 1 b 2 c$
$\vdash q_3 0 a 1 b 2 c$
$\vdash 0 q_0 a 1 b 2 c$
$\vdash^* 00 q_0 1122$
$\vdash 001 q_4 122$
$\vdash^* 001122 q_4$
$\vdash 00112 q_5 2$

$q_0 aabbbcc$
$\vdash^+ 0 q_0 a 1 bb 2 c$
$\vdash^+ 00 q_0 11 b 22$

Example



$q_0\,aabbcc$

$\vdash 0q_1\,abbcc$

$\vdash 0aq_1\,bbcc$

$\vdash 0a1q_2\,bcc$

$\vdash 0a1bq_2\,cc$

$\vdash 0a1q_3\,b2c$

$\vdash 0aq_3\,1b2c$

$\vdash 0q_3\,a1b2c$

$\vdash q_3\,0a1b2c$

$\vdash 0q_0\,a1b2c$

$\vdash^* 00q_0\,1122$

$\vdash 001q_4\,122$

$\vdash^* 001122q_4$

$\vdash 00112q_5\,2$

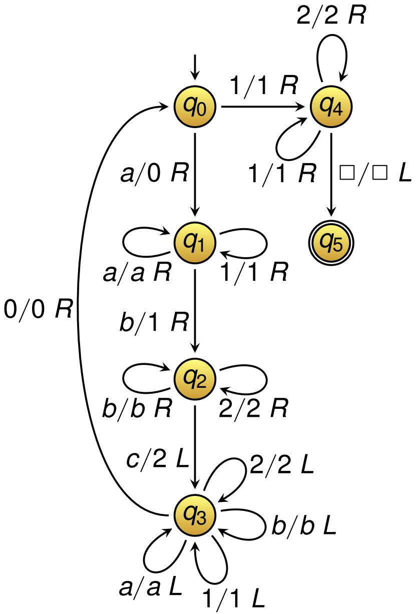
$q_0\,aabbbcc$

$\vdash^+ 0q_0\,a1bb2c$

$\vdash^+ 00q_0\,11b22$

$\vdash 001q_4\,1b22$

Example



$q_0 aabbcc$
$\vdash 0 q_1 abbcc$
$\vdash 0a q_1 bbcc$
$\vdash 0a1 q_2 bcc$
$\vdash 0a1b q_2 cc$
$\vdash 0a1 q_3 b2c$
$\vdash 0a q_3 1b2c$
$\vdash 0 q_3 a1b2c$
$\vdash q_3 0a1b2c$
$\vdash 0 q_0 a1b2c$
$\vdash^* 00 q_0 1122$
$\vdash 001 q_4 122$
$\vdash^* 001122 q_4$
$\vdash 00112 q_5 2$

$q_0 aabbbcc$
$\vdash^+ 0 q_0 a1bb2c$
$\vdash^+ 00 q_0 11b22$
$\vdash 001 q_4 1b22$
$\vdash 0011 q_4 b22$

Exercise

Construct a Turing machine accepting all words of **odd** length over the alphabet $\Sigma = \{a, b\}$.

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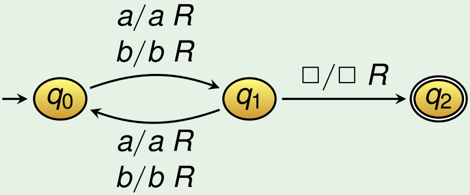

Multiple labels on an arrow are short for multiple transitions.

Extensions of Turing Machines

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A nondeterministic TM can be simulated by deterministic TM using **breadth-first search** (all computations in parallel).

The overhead in **time complexity** is believed to be an **exponential factor**.

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This thesis has stood the test of time.

Also computations of **quantum computers** can be simulated by a Turing machines.

Quantum computers can do certain computations faster than classical computers, but they do not change the limits of computability.

Alonzo Church & Alan Turing

 

Two of the founders of the **theory of computability.**

Alonzo Church (1903-1995) is inventor of the **λ-calculus.**

Alan Turing (1912-1954)

- introduced the **Turing machine**,
- invented the **Turing test**,
- key role in cracking the German **Enigma machine.**

Both proved **undecidability of validity in predicate logic**.

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Conclusion: not all languages are recursively enumerable.

Universal Turing Machine

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Theorem

There exists a universal Turing machine.