Automata Theory :: Pushdown Automata

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A pushdown automaton has a **stack** of unlimited size.

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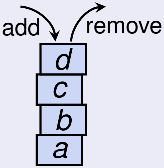
Next to the input alphabet Σ, there is now a **stack alphabet** Γ.

A **stack** is a finite sequence of elements from Γ:



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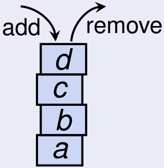

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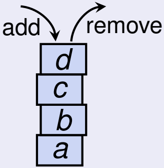

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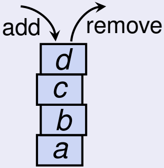

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The nondeterministic choice $\delta(q, \alpha, b)$ must always be finite!

Pushdown Automata

A **nondeterministic pushdown automaton** (**NPDA**) is a tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

- Q is a finite set of states
- Σ is a finite input alphabet
- Γ is a finite stack alphabet
- $\delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \to 2^{Q \times \Gamma^*}$

  the transition function, where $\delta(q, \alpha, b)$ is always finite
- $q_0 \in Q$ the starting state
- $z \in \Gamma$ the stack starting symbol
- $F \subseteq Q$ a set of final states

Initially, the stack content is z.

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If $(q', v) \in \delta(q, \alpha, b)$, this means that

- from state q with input αw and stack bu

the automaton can do a transition to

- state q' with input w and stack vu.

Language Accepted by a Pushdown Automaton

A **configuration** (q, w, u) of an NPDA consists of:

- current state $q \in Q$
- input word $w \in \Sigma^*$
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whenever $(q', v) \in \delta(q, \alpha, b)$.

We write $\vdash^*$ for computation (zero or more steps).

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The **language generated by** NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is

$$L(M) = \{\, w \in \Sigma^* \mid (q_0, w, z) \vdash^* (q', \lambda, u) \text{ where } q' \in F \,\}.$$

Note: no condition on the stack u at the end.

Drawing Pushdown Automata

The transition graph for a NPDA contains

for every $(q', v) \in \delta(q, \alpha, b)$ an arrow $q \xrightarrow{\alpha[b/v]} q'$

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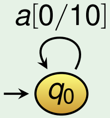

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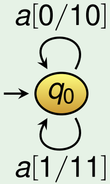

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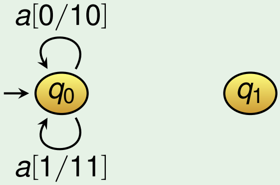

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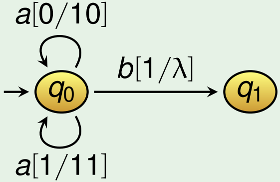

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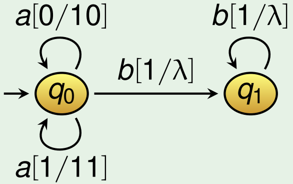

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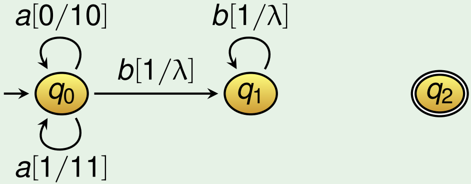

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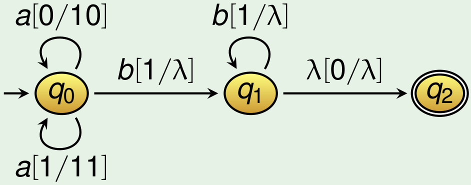

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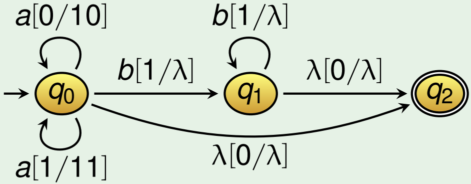
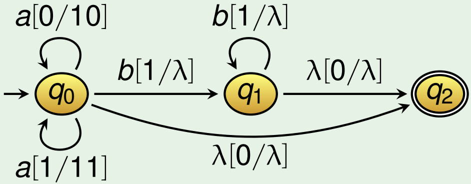


Example Computation



Stepwise reading of the word *aabb*:

$$(q_0, aabb, 0)$$

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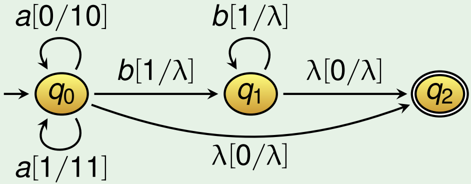

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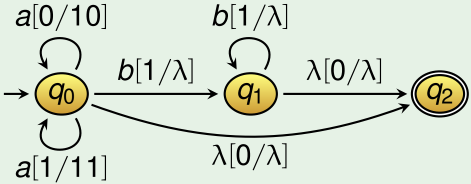

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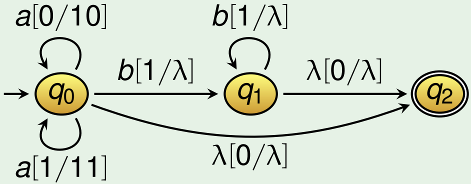

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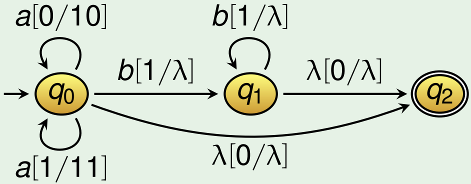

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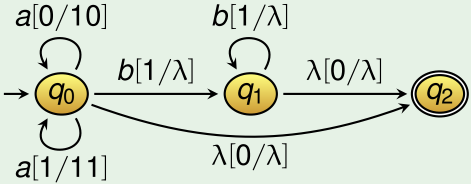

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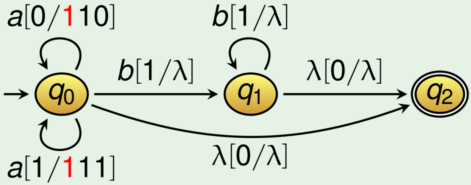

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Draw an NPDA M with $L(M) = \{ ww^R \mid w \in \{a, b\}^+ \}$.

Hint: define

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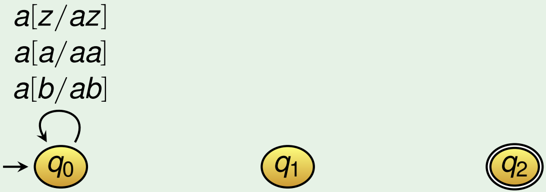

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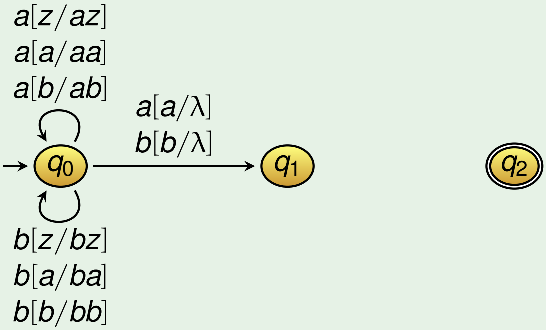

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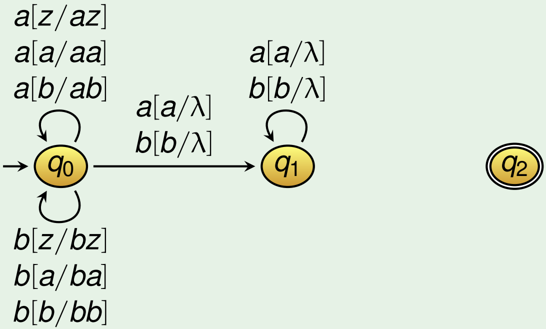

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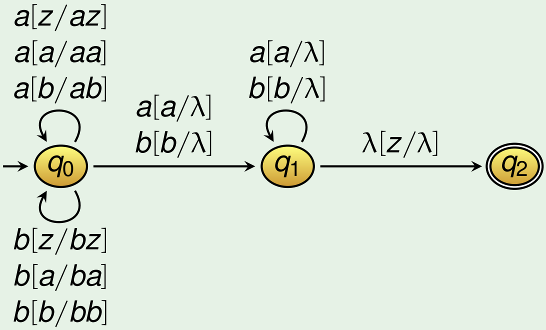

Exercises (3)

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No!

This language is not context-free.

Acceptance with Empty Stack

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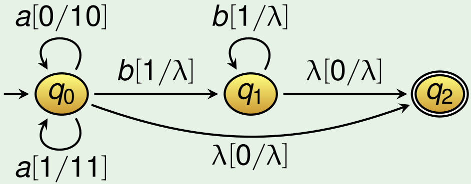
All automata we have seen so far had the following property:

$$(q_0, w, z) \vdash^* (q', w', u') \quad \Longrightarrow \quad (q' \in F \iff u' = \lambda)$$

They reach an accepting state if and only if the stack is empty.



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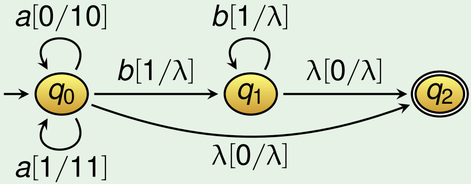


Acceptance with Empty Stack

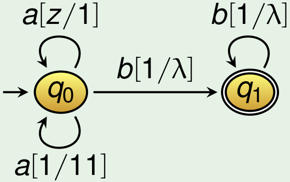
Empty stack language of NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is

$$L_\lambda(M) = \{ w \in \Sigma^* \mid (q_0, w, z) \vdash^* (q', \lambda, \lambda) \}.$$

(No need for final states in this definition.)

Example

Consider the following NPDA M:



What is the language accepted by this automaton?

$$L(M) =$$

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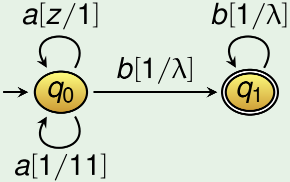

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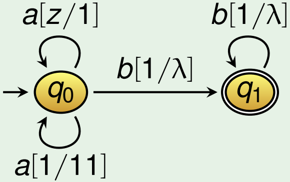

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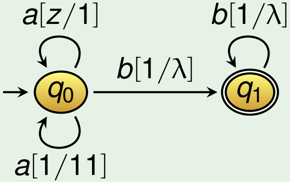

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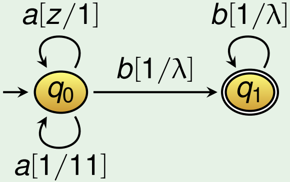

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For a language L the following two are equivalent:

- There is an NDPA M with $L(M) = L$.
- There is an NDPA M with $L_\lambda(M) = L$.

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Every NPDA M be transformed into NPDA N such that

- it has **a single final state** $F = \{ q_f \}$,
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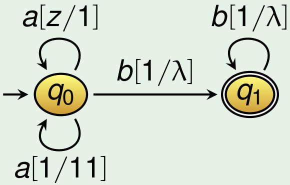

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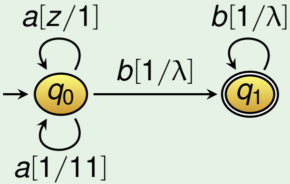

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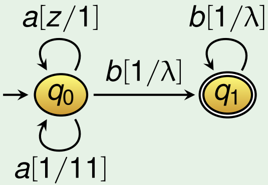

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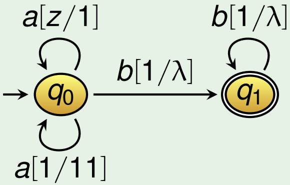

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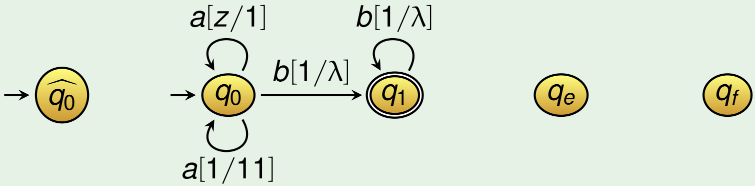

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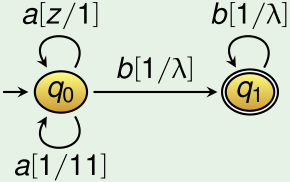

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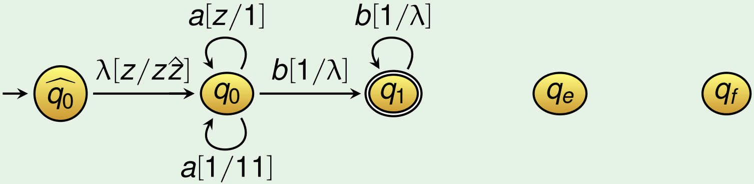

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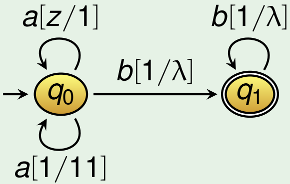

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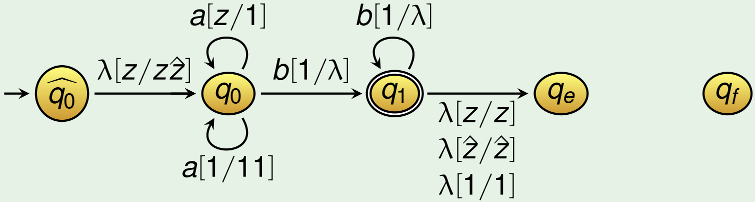

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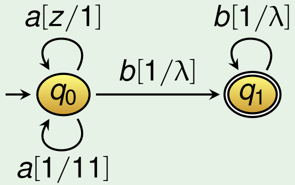

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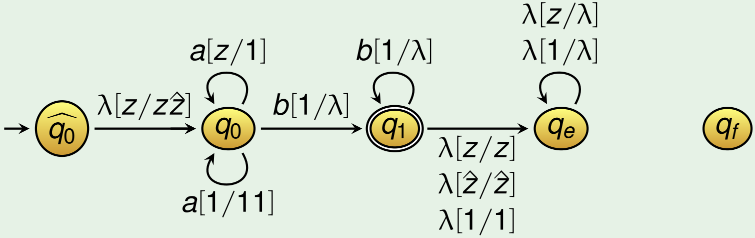

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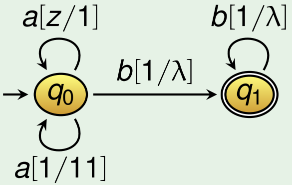

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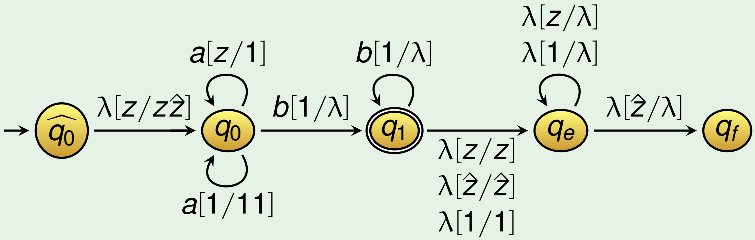

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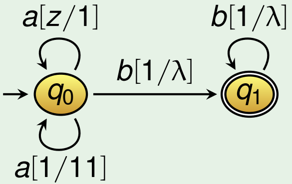

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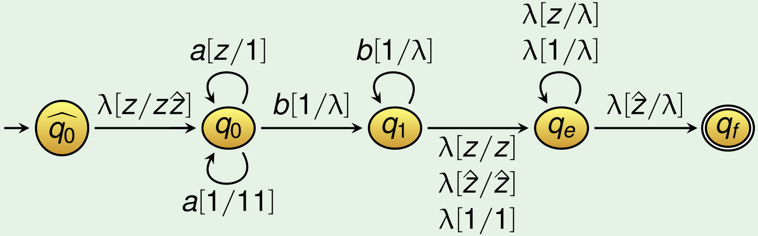

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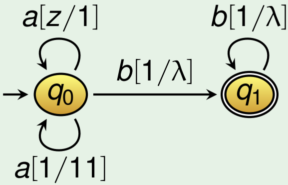

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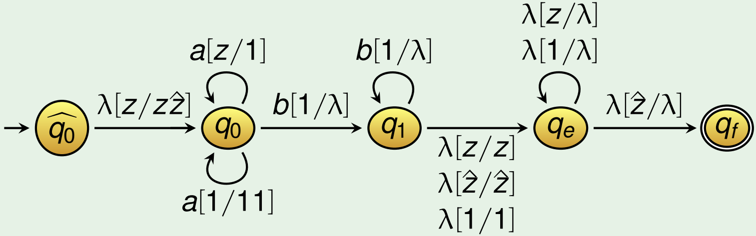

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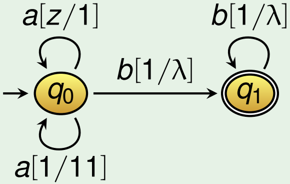

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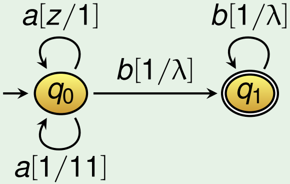

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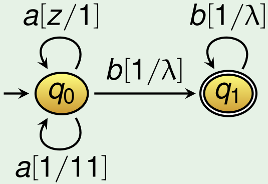

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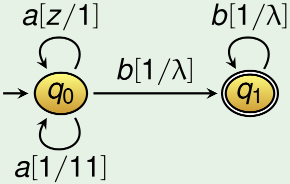

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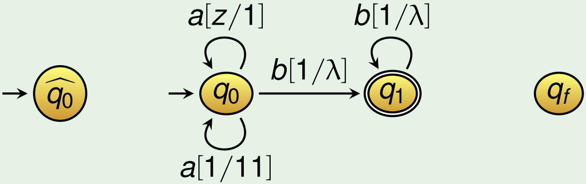

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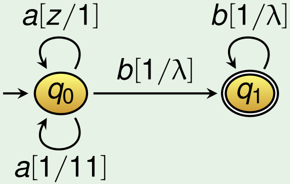

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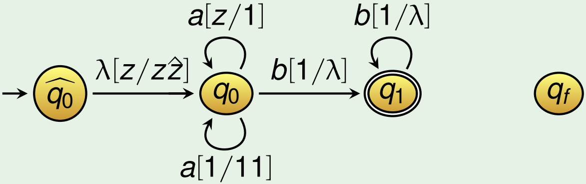

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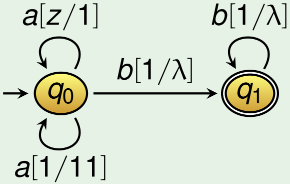

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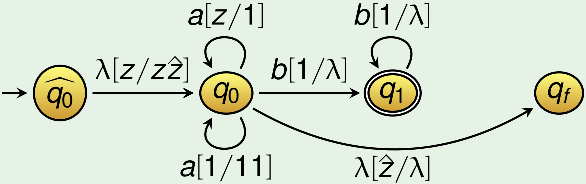

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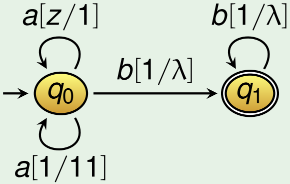

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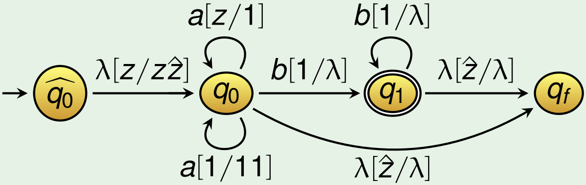

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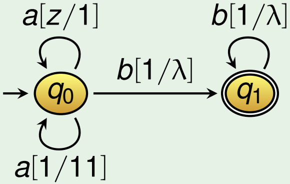

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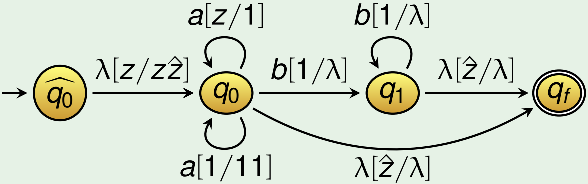

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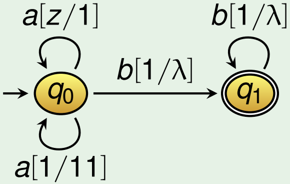

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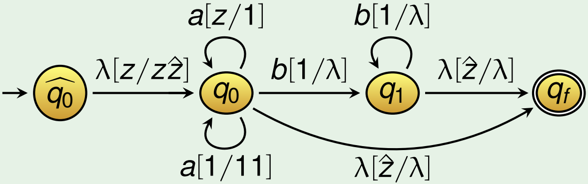

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Pushdown Automata & Context-Free Languages

Context-Free Languages and NPDA's

Theorem

A language L is context-free
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Proof.

We need to prove two directions:

- $(\Rightarrow)$ Translate context-free grammars into NPDA's.
- $(\Leftarrow)$ Translate NPDA's into context-free grammars.

□

From Context-Free Grammars to NPDA's

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$$Q = \{ q_0, q_1, q_2 \} \qquad \Sigma = T$$

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We add transitions simulating a leftmost derivation:

$$q_0 \xrightarrow{\lambda[z/Sz]} q_1$$
$$q_1 \xrightarrow{\lambda[A/x]} q_1 \quad \text{for every } A \to x \in P$$
$$q_1 \xrightarrow{a[a/\lambda]} q_1 \quad \text{for every } a \in T$$
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Then $L(M) = L(G)$.

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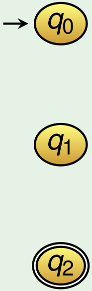

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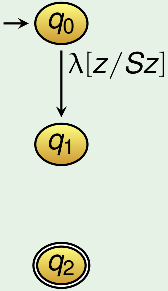

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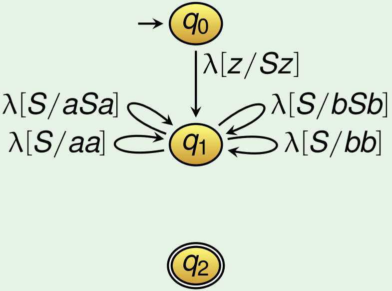

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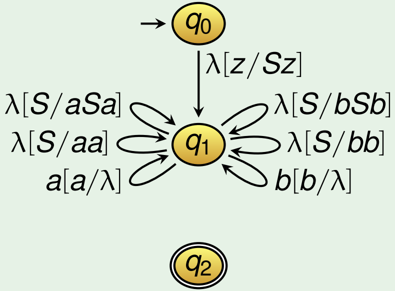

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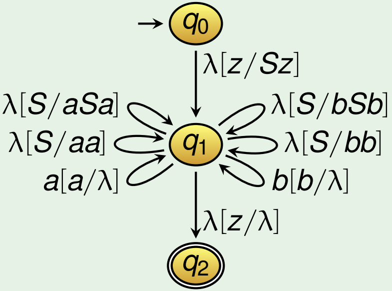

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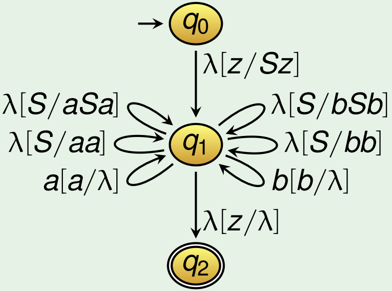

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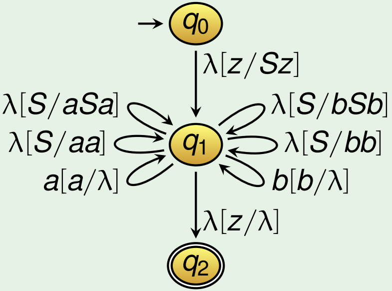

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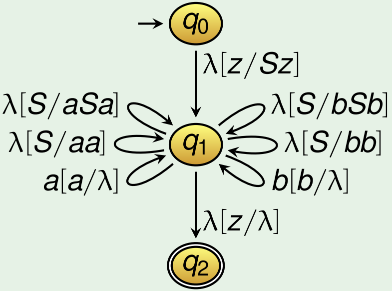

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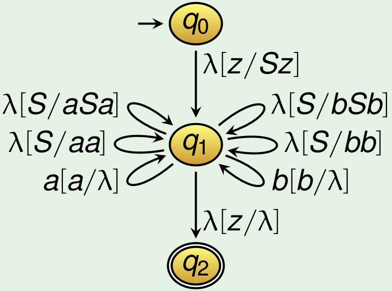

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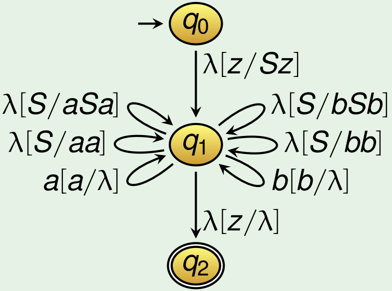

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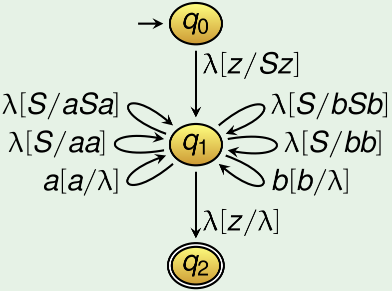

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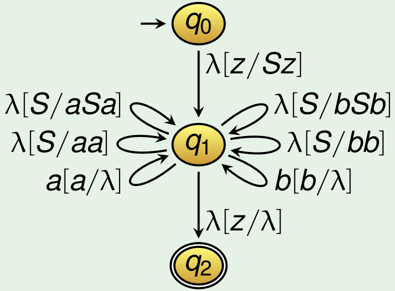

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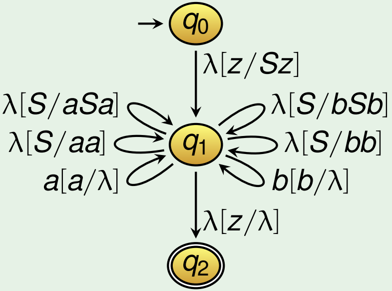

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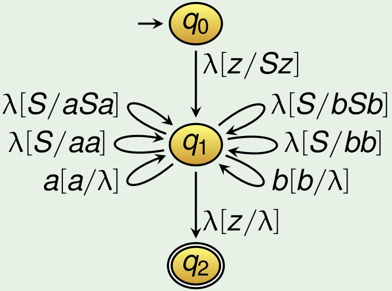

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Then we have $L(G) = L(M)$.

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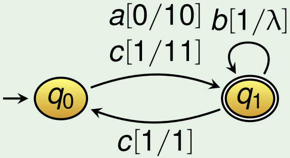

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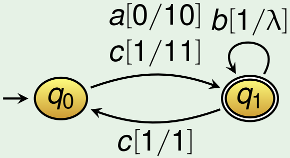

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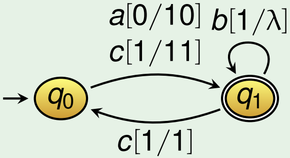

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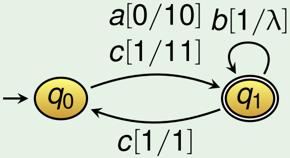

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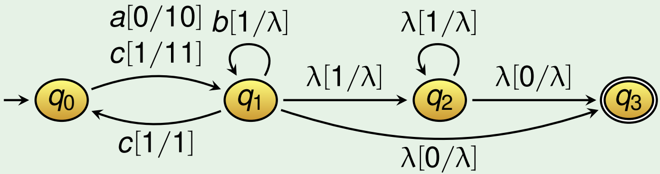

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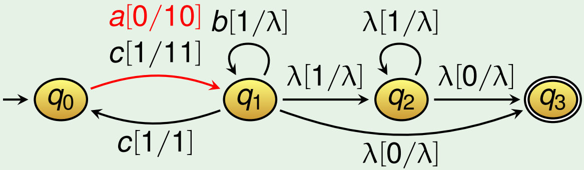

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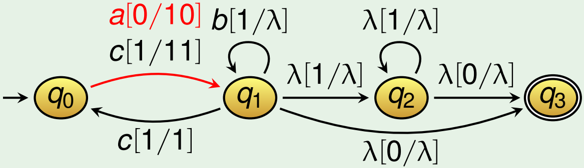

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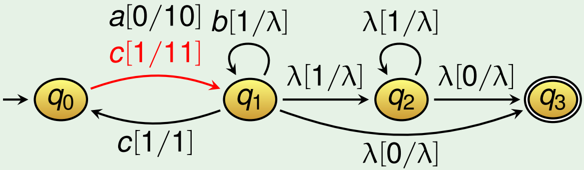

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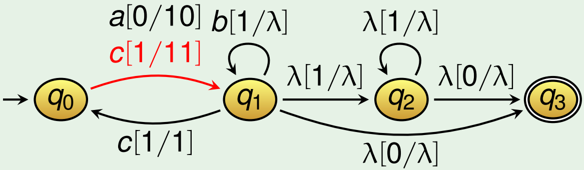

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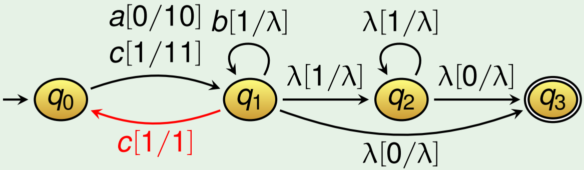

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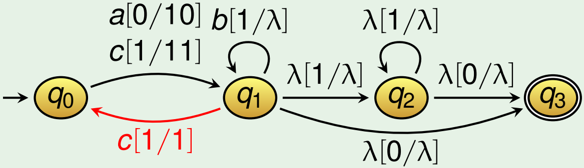

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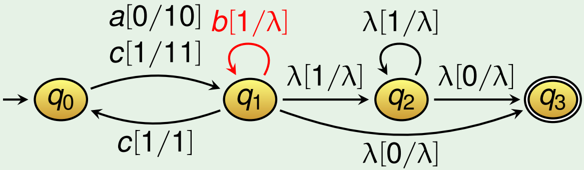

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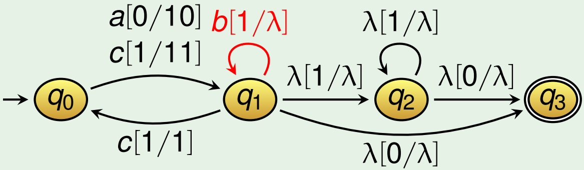

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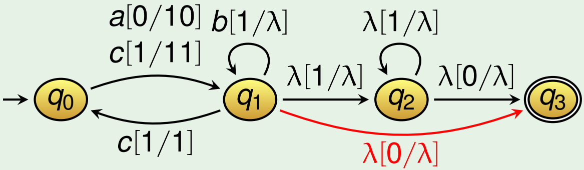

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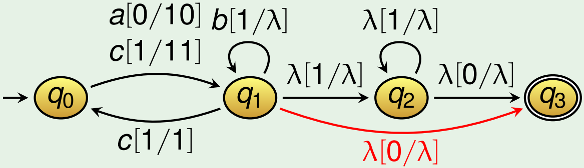

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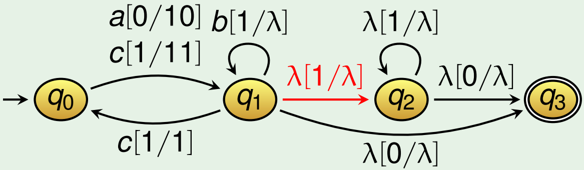

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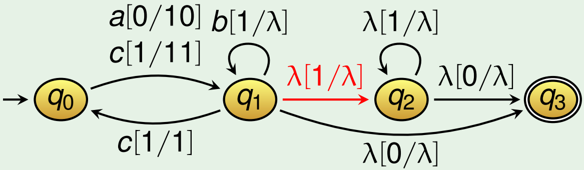

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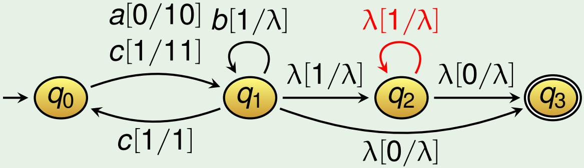

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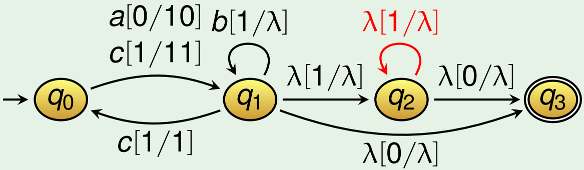
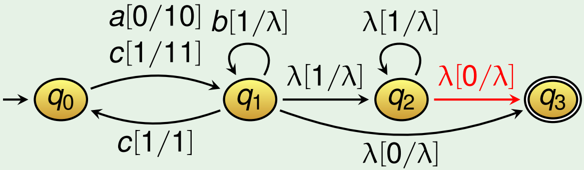


The resulting context-free grammar is:

$$(q_0\, 0\, r_2) \xrightarrow{1} a\,(q_1\, 1\, r_1)\,(r_1\, 0\, r_2) \qquad (q_1\, 0\, q_3) \xrightarrow{5} \lambda$$
$$(q_0\, 1\, r_2) \xrightarrow{2} c\,(q_1\, 1\, r_1)\,(r_1\, 1\, r_2) \qquad (q_1\, 1\, q_2) \xrightarrow{6} \lambda$$
$$(q_1\, 1\, r_1) \xrightarrow{3} c\,(q_0\, 1\, r_1) \qquad (q_2\, 1\, q_2) \xrightarrow{7} \lambda$$
$$(q_1\, 1\, q_1) \xrightarrow{4} b$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$.

Example



The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$

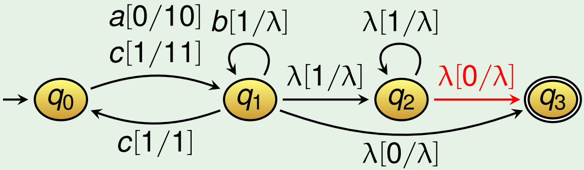
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$

$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$

$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$.

Example

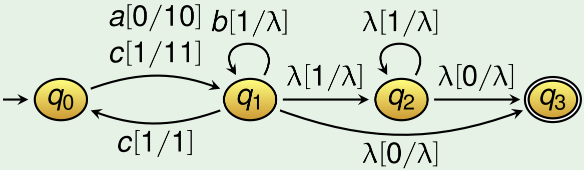


The resulting context-free grammar is:

$$
\begin{aligned}
(q_0\,0\,r_2) &\xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & (q_1\,0\,q_3) &\xrightarrow{5} \lambda \\
(q_0\,1\,r_2) &\xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & (q_1\,1\,q_2) &\xrightarrow{6} \lambda \\
(q_1\,1\,r_1) &\xrightarrow{3} c\,(q_0\,1\,r_1) & (q_2\,1\,q_2) &\xrightarrow{7} \lambda \\
(q_1\,1\,q_1) &\xrightarrow{4} b & (q_2\,0\,q_3) &\xrightarrow{8} \lambda
\end{aligned}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$.

Example

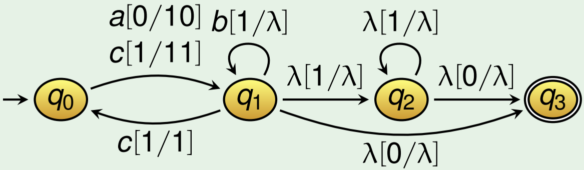


The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & & (q_1\,1\,q_2) \xrightarrow{6} \lambda \\
(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) & & (q_2\,1\,q_2) \xrightarrow{7} \lambda \\
(q_1\,1\,q_1) \xrightarrow{4} b & & (q_2\,0\,q_3) \xrightarrow{8} \lambda
\end{array}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

Example



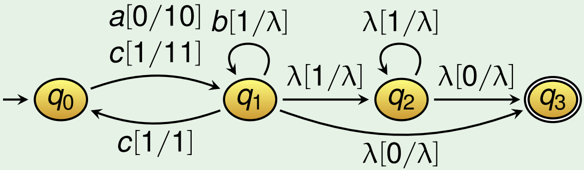
The resulting context-free grammar is:

$$(q_0\, 0\, r_2) \xrightarrow{1} a\,(q_1\, 1\, r_1)\,(r_1\, 0\, r_2) \qquad (q_1\, 0\, q_3) \xrightarrow{5} \lambda$$
$$(q_0\, 1\, r_2) \xrightarrow{2} c\,(q_1\, 1\, r_1)\,(r_1\, 1\, r_2) \qquad (q_1\, 1\, q_2) \xrightarrow{6} \lambda$$
$$(q_1\, 1\, r_1) \xrightarrow{3} c\,(q_0\, 1\, r_1) \qquad (q_2\, 1\, q_2) \xrightarrow{7} \lambda$$
$$(q_1\, 1\, q_1) \xrightarrow{4} b \qquad (q_2\, 0\, q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\, 0\, q_3)$.

$(q_0, accb,\ \ 0)$

Example



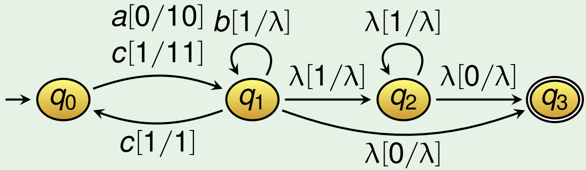
The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$(q_0, accb,\ 0)$$
$$\vdash (q_1, ccb,\ 10)$$

Example



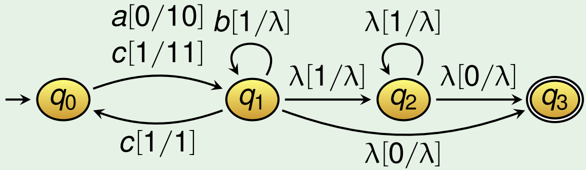
The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & & (q_1\,1\,q_2) \xrightarrow{6} \lambda \\
(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) & & (q_2\,1\,q_2) \xrightarrow{7} \lambda \\
(q_1\,1\,q_1) \xrightarrow{4} b & & (q_2\,0\,q_3) \xrightarrow{8} \lambda
\end{array}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{ll}
 & (q_0, accb,\ 0) \\
\vdash & (q_1, ccb,\ 10) \\
\vdash & (q_0, cb,\ 10)
\end{array}
$$

Example



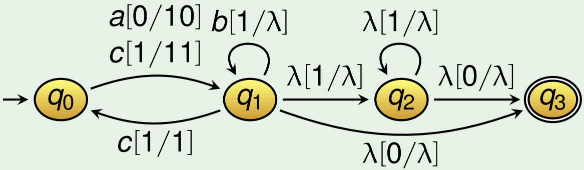
The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & & (q_1\,1\,q_2) \xrightarrow{6} \lambda \\
(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) & & (q_2\,1\,q_2) \xrightarrow{7} \lambda \\
(q_1\,1\,q_1) \xrightarrow{4} b & & (q_2\,0\,q_3) \xrightarrow{8} \lambda
\end{array}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{ll}
 & (q_0, accb, \; 0) \\
\vdash & (q_1, ccb, \; 10) \\
\vdash & (q_0, cb, \; 10) \\
\vdash & (q_1, b, \; 110)
\end{array}
$$

Example



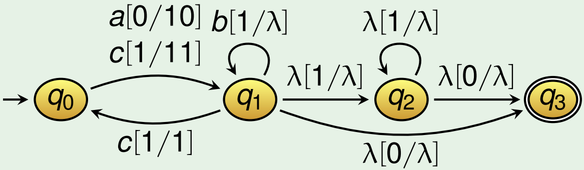
The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & & (q_1\,1\,q_2) \xrightarrow{6} \lambda \\
(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) & & (q_2\,1\,q_2) \xrightarrow{7} \lambda \\
(q_1\,1\,q_1) \xrightarrow{4} b & & (q_2\,0\,q_3) \xrightarrow{8} \lambda
\end{array}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{rlr}
& (q_0, accb, & 0) \\
\vdash & (q_1, ccb, & 10) \\
\vdash & (q_0, cb, & 10) \\
\vdash & (q_1, b, & 110) \\
\vdash & (q_1, \lambda, & 10)
\end{array}
$$

Example



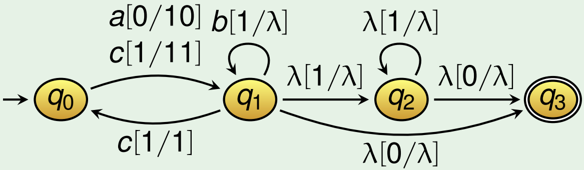
The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$\begin{aligned} & (q_0, accb, 0) \\ \vdash\ & (q_1, ccb, 10) \\ \vdash\ & (q_0, cb, 10) \\ \vdash\ & (q_1, b, 110) \\ \vdash\ & (q_1, \lambda, 10) \\ \vdash\ & (q_2, \lambda, 0) \end{aligned}$$

Example



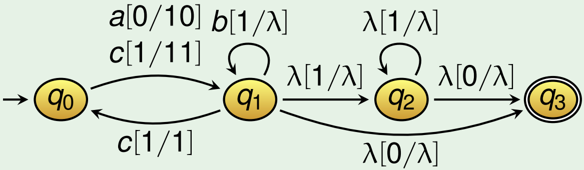
The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$\begin{aligned}
& (q_0, accb, 0) \\
\vdash\ & (q_1, ccb, 10) \\
\vdash\ & (q_0, cb, 10) \\
\vdash\ & (q_1, b, 110) \\
\vdash\ & (q_1, \lambda, 10) \\
\vdash\ & (q_2, \lambda, 0) \\
\vdash\ & (q_3, \lambda, \lambda)
\end{aligned}$$

Example



The resulting context-free grammar is:

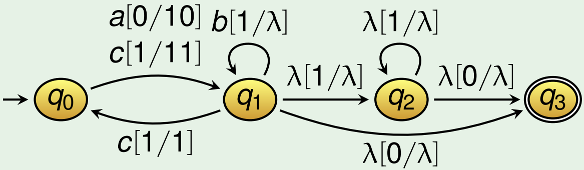
$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$(q_0, accb, 0)$$
$$\vdash (q_1, ccb, 10)$$
$$\vdash (q_0, cb, 10)$$
$$\vdash (q_1, b, 110)$$
$$\vdash (q_1, \lambda, 10)$$
$$\vdash (q_2, \lambda, 0)$$
$$\vdash (q_3, \lambda, \lambda)$$

$(q_0\,0\,q_3)$

Example



The resulting context-free grammar is:

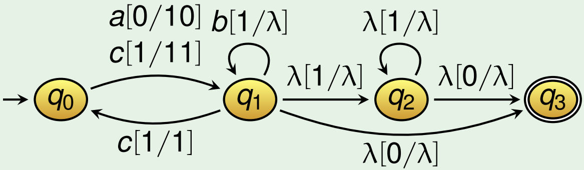
$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{ll}
 & (q_0, accb, \ 0) \\
\vdash & (q_1, ccb, \ 10) \\
\vdash & (q_0, cb, \ 10) \\
\vdash & (q_1, b, \ 110) \\
\vdash & (q_1, \lambda, \ 10) \\
\vdash & (q_2, \lambda, \ 0) \\
\vdash & (q_3, \lambda, \ \lambda)
\end{array}
$$

$$(q_0\,0\,q_3) \Rightarrow \underline{a\,(q_1\,1\quad)\,(\quad 0\,q_3)}$$

Example



The resulting context-free grammar is:

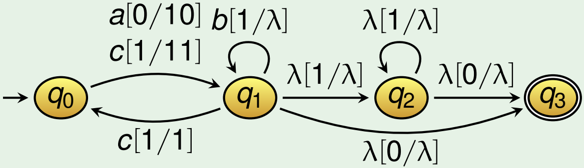
$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$(q_0, accb, 0)$$
$$\vdash (q_1, ccb, 10)$$
$$\vdash (q_0, cb, 10)$$
$$\vdash (q_1, b, 110)$$
$$\vdash (q_1, \lambda, 10)$$
$$\vdash (q_2, \lambda, 0)$$
$$\vdash (q_3, \lambda, \lambda)$$

$$(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)}$$

Example



The resulting context-free grammar is:

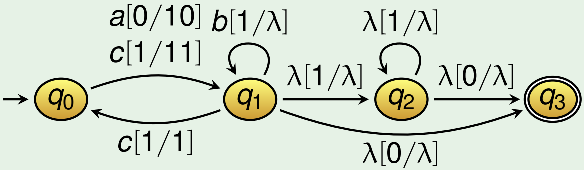
$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$(q_0, accb, 0)$$
$$\vdash (q_1, ccb, 10)$$
$$\vdash (q_0, cb, 10)$$
$$\vdash (q_1, b, 110)$$
$$\vdash (q_1, \lambda, 10)$$
$$\vdash (q_2, \lambda, 0)$$
$$\vdash (q_3, \lambda, \lambda)$$

$$(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \Rightarrow a \qquad (q_2\,0\,q_3)$$

Example



The resulting context-free grammar is:

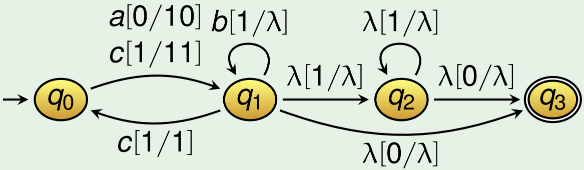
$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$(q_0, accb,\ 0)$$
$$\vdash (q_1, ccb,\ 10)$$
$$\vdash (q_0, cb,\ 10)$$
$$\vdash (q_1, b,\ 110)$$
$$\vdash (q_1, \lambda,\ 10)$$
$$\vdash (q_2, \lambda,\ 0)$$
$$\vdash (q_3, \lambda,\ \lambda)$$

$$(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} ac\,\underline{(q_0\,1\,q_2)}\,(q_2\,0\,q_3)$$

Example



The resulting context-free grammar is:

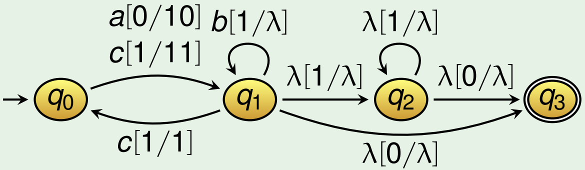
$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
$$(q_1\,1\,q_1) \xrightarrow{4} b \qquad (q_2\,0\,q_3) \xrightarrow{8} \lambda$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$\begin{aligned} & (q_0, accb, 0) \\ \vdash\ & (q_1, ccb, 10) \\ \vdash\ & (q_0, cb, 10) \\ \vdash\ & (q_1, b, 110) \\ \vdash\ & (q_1, \lambda, 10) \\ \vdash\ & (q_2, \lambda, 0) \\ \vdash\ & (q_3, \lambda, \lambda) \end{aligned}$$

$$(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} ac\,\underline{(q_0\,1\,q_2)}\,(q_2\,0\,q_3)$$
$$\Rightarrow ac \qquad (q_2\,0\,q_3)$$

Example



The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & & (q_1\,1\,q_2) \xrightarrow{6} \lambda \\
(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) & & (q_2\,1\,q_2) \xrightarrow{7} \lambda \\
(q_1\,1\,q_1) \xrightarrow{4} b & & (q_2\,0\,q_3) \xrightarrow{8} \lambda
\end{array}
$$

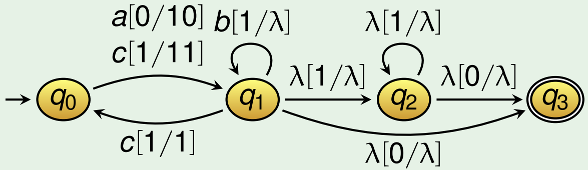
for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{rl}
& (q_0, accb, \ 0) \\
\vdash & (q_1, ccb, \ 10) \\
\vdash & (q_0, cb, \ 10) \\
\vdash & (q_1, b, \ 110) \\
\vdash & (q_1, \lambda, \ 10) \\
\vdash & (q_2, \lambda, \ 0) \\
\vdash & (q_3, \lambda, \ \lambda)
\end{array}
$$

$$
(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} ac\,\underline{(q_0\,1\,q_2)}\,(q_2\,0\,q_3)
$$

$$
\Rightarrow acc\,\underline{(q_1\,1\quad)\,(\quad 1\,q_2)}\,(q_2\,0\,q_3)
$$

Example



The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$

$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$

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for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

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& (q_0, accb, \; 0) \\
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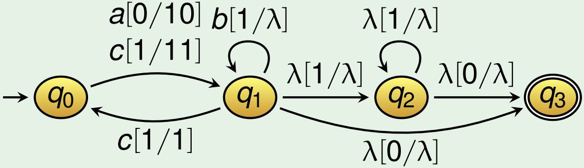
\vdash\; & (q_1, \lambda, \; 10) \\
\vdash\; & (q_2, \lambda, \; 0) \\
\vdash\; & (q_3, \lambda, \; \lambda)
\end{aligned}$$

$$(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} a\underline{c\,(q_0\,1\,q_2)}\,(q_2\,0\,q_3)$$

$$\overset{2}{\Rightarrow} ac\underline{c\,(q_1\,1\,q_1)\,(q_1\,1\,q_2)}\,(q_2\,0\,q_3)$$

Example



The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) & & (q_1\,1\,q_2) \xrightarrow{6} \lambda \\
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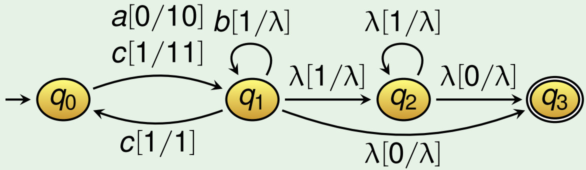
(q_1\,1\,q_1) \xrightarrow{4} b & & (q_2\,0\,q_3) \xrightarrow{8} \lambda
\end{array}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{ll}
& (q_0, accb, \; 0) \\
\vdash & (q_1, ccb, \; 10) \\
\vdash & (q_0, cb, \; 10) \\
\vdash & (q_1, b, \; 110) \\
\vdash & (q_1, \lambda, \; 10) \\
\vdash & (q_2, \lambda, \; 0) \\
\vdash & (q_3, \lambda, \; \lambda)
\end{array}
$$

$$
\begin{array}{l}
(q_0\,0\,q_3) \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} ac\,\underline{(q_0\,1\,q_2)}\,(q_2\,0\,q_3) \\
\overset{2}{\Rightarrow} acc\,\underline{(q_1\,1\,q_1)\,(q_1\,1\,q_2)}\,(q_2\,0\,q_3)
\end{array}
$$

Example



The resulting context-free grammar is:

$$
\begin{array}{llll}
(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) & & (q_1\,0\,q_3) \xrightarrow{5} \lambda \\
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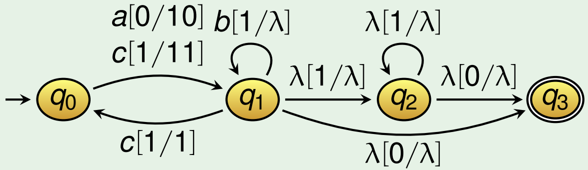
\end{array}
$$

for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{array}{rlr}
 & (q_0, accb, & 0) \\
\vdash & (q_1, ccb, & 10) \\
\vdash & (q_0, cb, & 10) \\
\vdash & (q_1, b, & 110) \\
\vdash & (q_1, \lambda, & 10) \\
\vdash & (q_2, \lambda, & 0) \\
\vdash & (q_3, \lambda, & \lambda)
\end{array}
$$

$$
\begin{aligned}
(q_0\,0\,q_3) &\overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} a\underline{c\,(q_0\,1\,q_2)}\,(q_2\,0\,q_3) \\
&\overset{2}{\Rightarrow} acc\,\underline{(q_1\,1\,q_1)\,(q_1\,1\,q_2)}\,(q_2\,0\,q_3) \\
&\overset{4}{\Rightarrow} acc\underline{b}\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)
\end{aligned}
$$

Example



The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$
$$(q_0\,1\,r_2) \xrightarrow{2} c\,(q_1\,1\,r_1)\,(r_1\,1\,r_2) \qquad (q_1\,1\,q_2) \xrightarrow{6} \lambda$$
$$(q_1\,1\,r_1) \xrightarrow{3} c\,(q_0\,1\,r_1) \qquad (q_2\,1\,q_2) \xrightarrow{7} \lambda$$
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for all $r_1, r_2 \in \{q_0, q_1, q_2, q_3\}$. Here $S = (q_0\,0\,q_3)$.

$$
\begin{aligned}
& (q_0, accb, 0) \\
\vdash\; & (q_1, ccb, 10) \\
\vdash\; & (q_0, cb, 10) \\
\vdash\; & (q_1, b, 110) \\
\vdash\; & (q_1, \lambda, 10) \\
\vdash\; & (q_2, \lambda, 0) \\
\vdash\; & (q_3, \lambda, \lambda)
\end{aligned}
$$

$$
\begin{aligned}
(q_0\,0\,q_3) &\overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} ac\,\underline{(q_0\,1\,q_2)}\,(q_2\,0\,q_3) \\
&\overset{2}{\Rightarrow} acc\,\underline{(q_1\,1\,q_1)\,(q_1\,1\,q_2)}\,(q_2\,0\,q_3) \\
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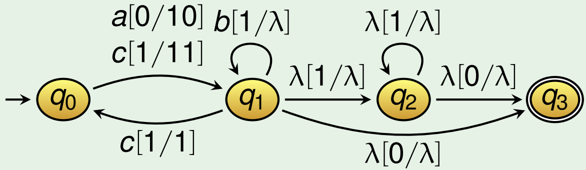
\end{aligned}
$$

Example



The resulting context-free grammar is:

$$(q_0\,0\,r_2) \xrightarrow{1} a\,(q_1\,1\,r_1)\,(r_1\,0\,r_2) \qquad (q_1\,0\,q_3) \xrightarrow{5} \lambda$$

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\vdash\ & (q_1, ccb, 10) \\
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\vdash\ & (q_1, b, 110) \\
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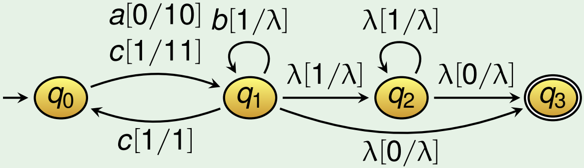
\end{aligned}
$$

$$
\begin{aligned}
(q_0\,0\,q_3) & \overset{1}{\Rightarrow} \underline{a\,(q_1\,1\,q_2)\,(q_2\,0\,q_3)} \overset{3}{\Rightarrow} a\underline{c\,(q_0\,1\,q_2)}\,(q_2\,0\,q_3) \\
& \overset{2}{\Rightarrow} acc\,\underline{(q_1\,1\,q_1)\,(q_1\,1\,q_2)}\,(q_2\,0\,q_3) \\
& \overset{4}{\Rightarrow} acc\underline{b}\,(q_1\,1\,q_2)\,(q_2\,0\,q_3) \overset{6}{\Rightarrow} accb\,(q_2\,0\,q_3)
\end{aligned}
$$

Example



The resulting context-free grammar is:

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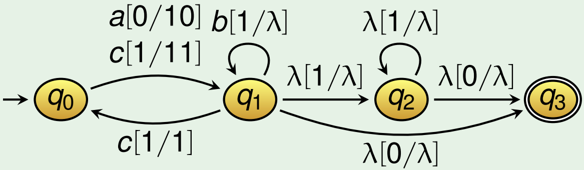
\end{aligned}
$$

Example



The resulting context-free grammar is:

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\end{aligned}
$$

$$
\begin{aligned}
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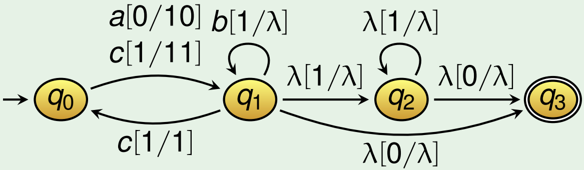
&\overset{4}{\Rightarrow} acc\underline{b}\,(q_1\, 1\, q_2)\,(q_2\, 0\, q_3) \overset{6}{\Rightarrow} accb\,(q_2\, 0\, q_3) \overset{8}{\Rightarrow} accb
\end{aligned}
$$

Example



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$$
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&\overset{2}{\Rightarrow} ac\underline{c\,(q_1\,1\,q_1)\,(q_1\,1\,q_2)}\,(q_2\,0\,q_3)\\
&\overset{4}{\Rightarrow} acc\underline{b}\,(q_1\,1\,q_2)\,(q_2\,0\,q_3) \overset{6}{\Rightarrow} accb\,(q_2\,0\,q_3) \overset{8}{\Rightarrow} accb
\end{aligned}
$$

Deterministic Pushdown Automata

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A **deterministic pushdown automaton** (**DPDA**) is an NPDA such that

- $\delta(q, \alpha, b)$ contains at most one element
- If $\delta(q, \lambda, b) \neq \varnothing$, then $\delta(q, a, b) = \varnothing$ for every $a \in \Sigma$.

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- $\delta(q, \alpha, b)$ contains at most one element
- If $\delta(q, \lambda, b) \neq \varnothing$, then $\delta(q, a, b) = \varnothing$ for every $a \in \Sigma$.

A language L is **deterministic context-free** if there exists a DPDA M with $L(M) = L$.

A deterministic context-free L allows for efficient parsing.

Exercises

Which of these languages are deterministic context-free?

- $\{ a^n b^n \mid n \geq 0 \}$
- $\{ ww^R \mid w \in \{a, b\}^+ \}$
- $\{ wcw^R \mid w \in \{a, b\}^+ \}$

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Conclusion

Not all context-free languages are deterministic context-free.

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Conclusion

Not all context-free languages are deterministic context-free.

Theorem

It is decidable if two DPDA's generate the same language.

(Géraud Sénizergues, 1997)