Automata Theory :: Pumping Lemma

Jörg Endrullis

Vrije Universiteit Amsterdam

Non-Regular Languages

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for some $\ell \geq 1$ and $q \in Q$.

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Then $q' \in F$ since $a^k b^k \in L$.

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Then $q' \in F$ since $a^k b^k \in L$. However, $q' \notin F$ since $a^{k+\ell} b^k \notin L$.

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We generalise the idea of the proof. . .

Pumping Lemma for Regular Languages (1959)

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Let L be a regular language. There exists $m > 0$ such that every $w \in L$ with $|w| \geq m$ can be written in the form

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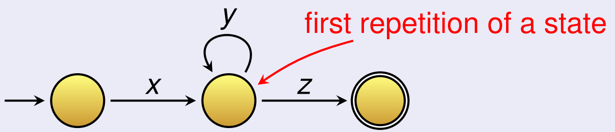
When M reads $w \in L$ with $|w| \geq m$, there must be a cycle



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&\quad\quad \exists x, y, z \text{ with } w = xyz,\ |xy| \leq m,\ |y| \geq 1. \\
&\quad\quad\quad \forall i \geq 0.\, xy^i z \in L
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Who wins the game when L is finite?

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