Automata Theory :: Minimal DFAs

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Minimal DFAs (Hopcroft, 1971)

Goal

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$.

Construct the (unique) **minimal DFA** $\widehat{M}$ with $L(M) = L(\widehat{M})$.

(Here minimal is with respect to the number of states.)

Construction

Step 1: Remove all unreachable states from M.

Step 2: Partition Q in indistinguishable states.

Step 3: Read off the minimal DFA.

Step 1: Remove all unreachable states from M.

Remove all states $q \in Q$ for which there is no path from q_0 to q.

Minimal DFAs (2)

States $q_1, q_2 \in Q$ are **distinguishable** if there exists $w \in \Sigma^*$ s.t.

$$q_1 \xrightarrow{w}\!\!\!\!\twoheadrightarrow q_1' \in F \qquad\qquad q_2 \xrightarrow{w}\!\!\!\!\twoheadrightarrow q_2' \notin F\ ,$$

or vice versa.

Step 2: Partition Q in indistinguishable states.

We construct the partitioning stepwise:

- Initial partitioning is $\{\, Q \setminus F,\ F \,\}$.
- If there are partitions R and S such that

  $$\delta(q, a) \in S \qquad \text{and} \qquad \delta(q', a) \notin S,$$

  for some $a \in \Sigma$ and $q, q' \in R$, then we split R in

  $$\{\, q \in R \mid \delta(q, a) \in S \,\} \qquad \{\, q \in R \mid \delta(q, a) \notin S \,\}$$

We keep splitting until no more split is possible.

Minimal DFAs (3)

Step 3: Read off the minimal DFA.

Let $Q_1, \ldots, Q_n$ be the final partition of Q.

These are the states of the minimal DFA $\widehat{M}$.

The transitions (arrows) of $\widehat{M}$ are:

$$Q_i \xrightarrow{a} Q_j \quad \Longleftrightarrow \quad q \xrightarrow{a} q' \text{ for some } q \in Q_i,\ q' \in Q_j$$

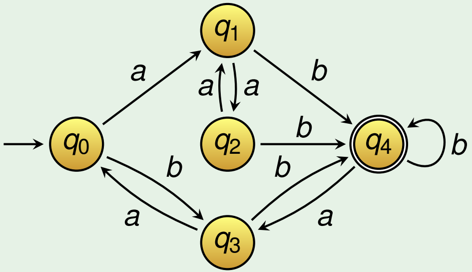
The starting state is the set that contains q_0.

The final states are the subsets of F.

Worst-case time complexity: $O(|\Sigma| \cdot |Q|^2)$, since

- There are maximal $|Q| - 1$ splits.
- Every split costs maximal $O(|\Sigma| \cdot |Q|)$.

Exercise: DFA Minimisation



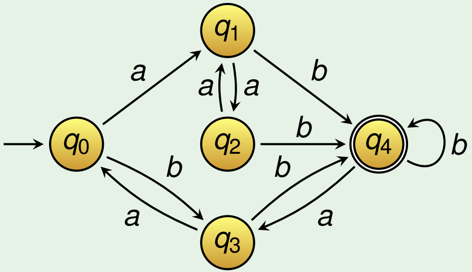
1. All states are reachable (nothing to remove).
2. Initial partitioning: $\{Q \setminus F, F\} = \{\{q_0, q_1, q_2, q_3\}, \{q_4\}\}$

   Splitting $R = \{q_0, q_1, q_2, q_3\}$ with $S = \{q_4\}$ and letter $b \in \Sigma$.
   New partitioning: $\{\{q_0\}, \{q_1, q_2, q_3\}, \{q_4\}\}$.

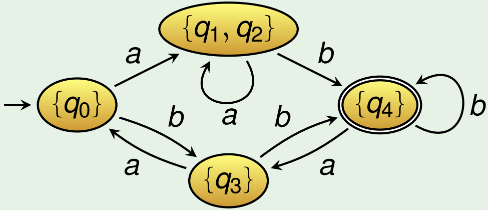
   Splitting $R = \{q_1, q_2, q_3\}$ with $S = \{q_0\}$ and letter $a \in \Sigma$.
   New partitioning: $\{\{q_0\}, \{q_1, q_2\}, \{q_3\}, \{q_4\}\}$.

   Nothing more to split!

Exercise: DFA Minimisation



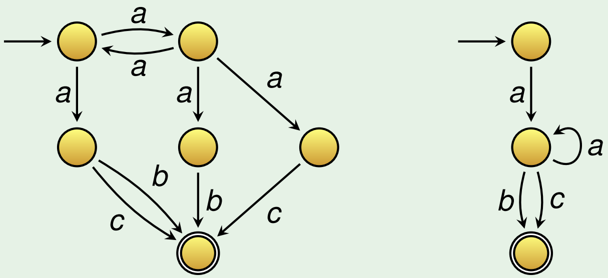
2. Final partitioning: $\{\{q_0\},\{q_1,q_2\},\{q_3\},\{q_4\}\}$.
3. Reading off the minimal DFA:



Minimising of NFAs

Minimising of NFAs is very difficult.

Example



Theorem

Minimising of NFAs is PSpace-complete.

The definition of PSpace-complete follows later.

Lexical Analysis

Lexical Analysis

Lexical analysis converts a sequence of **characters** into a sequence of **tokens**.

Programs that do lexical analysis are **lexers** or **tokenizers**.

For example the expression

```
sum = 15 + 2
```

could be converted to the sequence of tokens

token	token category
`sum`	identifier
`=`	assignment
`15`	integer literal
`+`	operator
`2`	integer literal

Allows to write parsers on the more abstract level of tokens.

Lexical Analysis

How to get from characters to tokens?

- Regular expressions $r_1, \dots, r_n$ express the pattern.

  Every regular expression corresponds to a token.
- Lexical analysis repeatedly searches the **longest prefix** of the input that is matched by one of the regular expressions. This prefix is transformed into a token.

For **improved performance**:

- Regular expressions are translated **minimal DFAs**.

Parser/lexer generators like

- JavaCC
- LEX

generate the lexer automatically. Thereby regular expressions or grammars are converted to minimal DFAs.