Automata Theory :: Finite Automata

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Deterministic Finite Automata

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A **deterministic finite automaton**, short **DFA**, consists of:

- a finite set Q of **states**
- a finite **input alphabet** Σ
- a **transition function** $\delta : Q \times \Sigma \to Q$
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Example DFA

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Understanding the transition function $\delta : Q \times \Sigma \to Q$

If the automaton in state q reads the symbol a,
then the resulting state is $\delta(q, a)$.

DFAs Reading Words

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

A **configuration** of M is a pair (q, w) with $q \in Q$ and $w \in \Sigma^*$.

So (q, w) means the automaton is in state q and reads word w.

The **step relation** $\vdash$ of M is defined on configurations by

$$(q, aw) \vdash (q', w) \quad \text{if } \delta(q, a) = q'$$

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We define $\vdash^*$ as the **reflexive transitive closure of** $\vdash$.

Continuing the above example, we have $(q_0, abba) \vdash^* (q_0, \lambda)$.

Transition Function in Table Notation

Example DFA

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Hint: transition function δ can be written in the form of a **table**:

δ	q_0	q_1
a	q_0	q_1
b	q_1	q_0

DFAs as Transition Graphs

A DFA can be visualised as a **transition graph**, consisting of:

- **states** are the **nodes** of the graph
  - **starting state** indicated by an **extra incoming arrow**
  - **final states** indicated by **double circle**
- **arrows** with labels from Σ: $q \xrightarrow{a} q'$ if $\delta(q, a) = q'$

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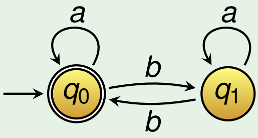

Exercise

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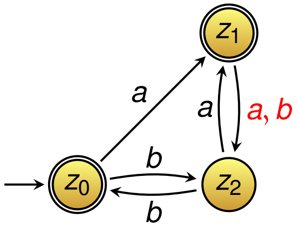

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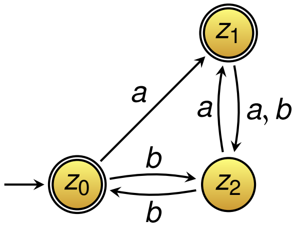

What is this DFA?

- states $Q =$
- alphabet $\Sigma =$
- transition function $\delta : Q \times \Sigma \to Q$:

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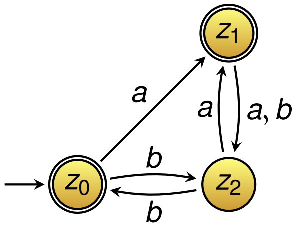

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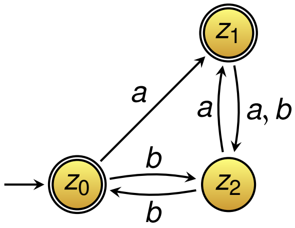

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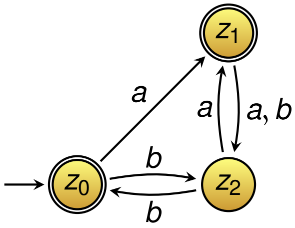

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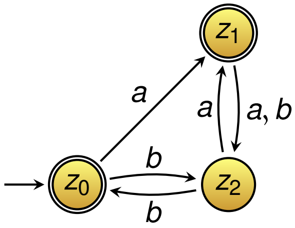

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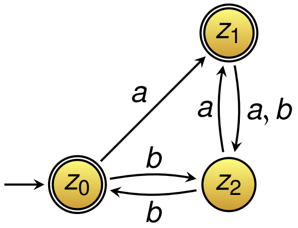

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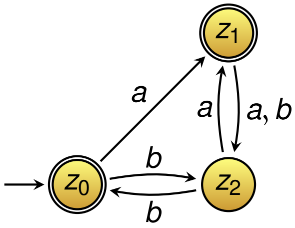

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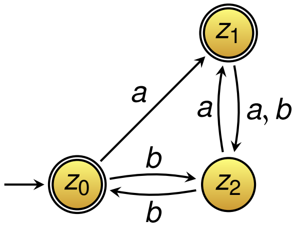

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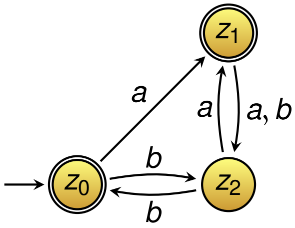

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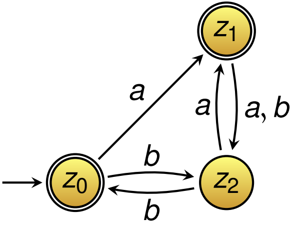

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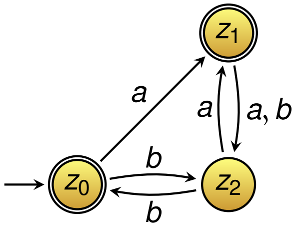

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Paths in DFAs

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

For a word $w = a_1 \cdots a_n$, $n \geq 0$, we write

$$q_0 \overset{w}{\longrightarrow\!\!\!\rightarrow} q_n$$

if there are states $q_1, \ldots, q_{n-1}$ such that

$$q_0 \xrightarrow{a_1} q_1 \qquad q_1 \xrightarrow{a_2} q_2 \qquad \ldots \qquad q_{n-1} \xrightarrow{a_n} q_n$$

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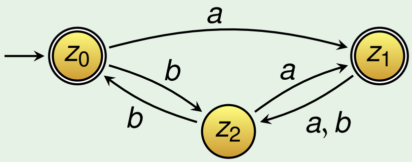

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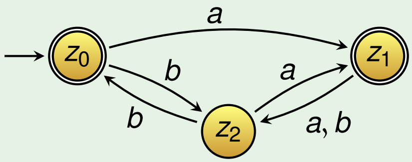

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Theorem: $q \xrightarrow{w}\!\!\!\!\rightarrow q' \iff (q, w) \vdash^* (q', \lambda)$.

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The **language accepted by** DFA $M = (Q, \Sigma, \delta, q_0, F)$ is

$$\begin{aligned} L(M) &= \{\, w \in \Sigma^* \mid (q_0, w) \vdash^* (q, \lambda) \text{ with } q \in F \,\} \\ &= \{\, w \in \Sigma^* \mid q_0 \xrightarrow{w}\!\!\twoheadrightarrow q \text{ with } q \in F \,\} \end{aligned}$$

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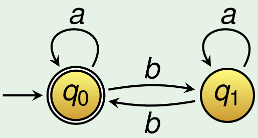

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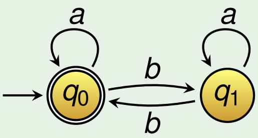

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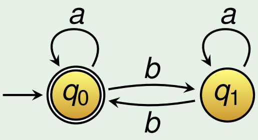

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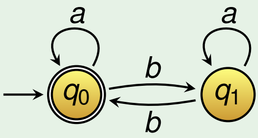
A language L is **regular** if there exists a DFA M with $L(M) = L$.

Exercise (1)

Let M be the following DFA:



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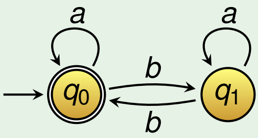

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Answer:

$L(M)$ consists of all words over the alphabet $\{ a, b \}$ that contain an even number of b's.

Exercise (2)

Show that the following language is regular:

$$\{\lambda\}$$

Construct a deterministic finite automaton for the language.

Exercise (3)

Show that the following language is regular:

$$\{\, a^n b \mid n \geq 0 \,\}$$

Construct a deterministic finite automaton for the language.

Exercise (4)

Show that the following language is regular:

$$\{ a^{2n+1} \mid n \geq 0 \} \cup \{ b^{2n} \mid n \geq 0 \}$$

Construct a deterministic finite automaton for the language.

DFAs are Deterministic

Recall that δ is a function from $Q \times \Sigma$ to Q.

DFAs are deterministic:

For every state $q \in Q$ and every symbol $a \in \Sigma$, the state q has **precisely one outgoing arrow** with label a.

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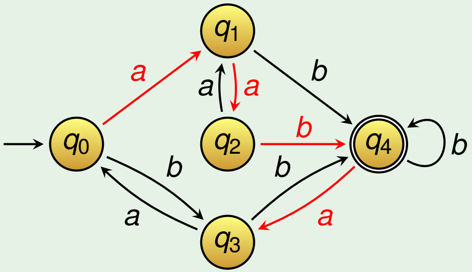
Hence, for every input word, there is precisely one path from the starting state through the transition graph.

The following picture shows the path for *aaba*:



Exercise (5)

Construct deterministic finite automata for the languages:

$$\{\, w \in \{a,b\}^* \mid w \text{ contains the subword } bab \,\}$$

and

$$\{\, w \in \{a,b\}^* \mid w \text{ does } \textbf{not} \text{ contain the subword } bab \,\}$$

Regular Languages: Complement

Theorem

If L is a regular language, then $\overline{L}$ is also regular.

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  $$w \in L \iff w \in L(M) \iff q \in F$$

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$$w \in L \iff w \in L(M) \iff q \in F$$
$$w \in \overline{L} \iff w \in \overline{L(M)} \iff q \in (Q \setminus F) \iff w \in L(N)$$

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Then it follows that $L(N) = L(M_1) \cup L(M_2) = L_1 \cup L_2$.

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Change the product construction to show that

- $L_1 \cap L_2$ is regular, and
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Intuition: a DFA has only a finite memory (the states).

We will later prove this using the **pumping lemma**.

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- $Q = \{\, q_w \mid w \in \Sigma^*,\ |w| \leq N \,\} \cup \{\, q_\perp \,\}$
- $F = \{\, q_w \mid w \in L \}$
- the transition function δ is defined by

$$\delta(q_w, a) = \begin{cases} q_{wa} & \text{if } |wa| \leq N, \\ q_\perp & \text{if } |wa| > N \end{cases} \qquad \delta(q_\perp, a) = q_\perp$$

  for every $w \in \Sigma^*$ with $|w| \leq N$ and $a \in \Sigma$

Nondeterministic Finite Automata

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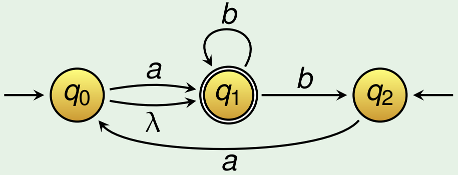
NFAs are defined like DFAs, except that NFAs allow for:

- **Multiple starting states.**
- **Any number of outgoing arrows** with the same label.
- **Empty steps**: arrows labelled λ (do not consume input).



Note that:

- both q_0 and q_2 are starting states
- the state q_1 has two outgoing arrows with label b
- there is an empty step from q_0 to q_1

Nondeterministic Finite Automata

A **nondeterministic finite automaton**, short **NFA**, consists of:

- a finite set Q of states
- a finite input alphabet Σ
- a transition function $\delta : Q \times (\Sigma \cup \{\lambda\}) \to 2^Q$
- a set $S \subseteq Q$ of starting states
- a set $F \subseteq Q$ of final states

Here 2^Q is the set of all subsets of Q: $2^Q = \{ X \mid X \subseteq Q \}$.

The NFA on the preceding slide is $M = (Q, \Sigma, \delta, S, F)$ where

$$Q = \{ q_0, q_1, q_2 \}$$
$$\Sigma = \{ a, b \}$$
$$S = \{ q_0, q_2 \}$$
$$F = \{ q_1 \}$$

δ	q_0	q_1	q_2
a	$\{ q_1 \}$	$\varnothing$	$\{ q_0 \}$
b	$\varnothing$	$\{ q_1, q_2 \}$	$\varnothing$
λ	$\{ q_1 \}$	$\varnothing$	$\varnothing$

NFAs Reading Words

Let $M = (Q, \Sigma, \delta, S, F)$ be a NFA.

The **step relation** $\vdash$ of M is defined on configurations by

$$(q, \alpha w) \vdash (q', w) \quad \text{if } q' \in \delta(q, \alpha) \text{ with } \alpha \in \Sigma \cup \{\lambda\}$$

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Note that if $\alpha = \lambda$, then

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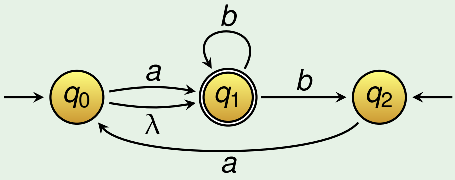

$$(q_0, abbab) \vdash (q_1, bbab) \vdash (q_1, bab) \vdash (q_2, ab)$$
$$\vdash (q_0, b) \vdash (q_1, b) \vdash (q_1, \lambda)$$

Paths in NFAs

Let $M = (Q, \Sigma, \delta, S, F)$ be a NFA.

For a word w, we write

$$q \overset{w}{\twoheadrightarrow} q'$$

if $w = \alpha_1 \cdots \alpha_n$ for some $\alpha_1, \ldots, \alpha_n \in (\Sigma \cup \{\lambda\})$ and there are states $q_1, \ldots, q_{n-1}$ such that

$$q \overset{\alpha_1}{\to} q_1 \qquad q_1 \overset{\alpha_2}{\to} q_2 \qquad q_2 \overset{\alpha_3}{\to} q_3 \qquad \ldots \qquad q_{n-1} \overset{\alpha_n}{\to} q'$$

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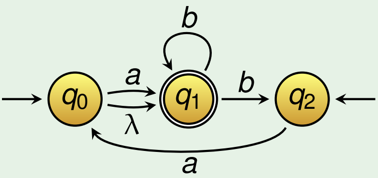

$$q_0 \overset{\lambda}{\twoheadrightarrow} q_0 \qquad q_1 \overset{ba}{\twoheadrightarrow} q_0$$

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Paths in NFAs

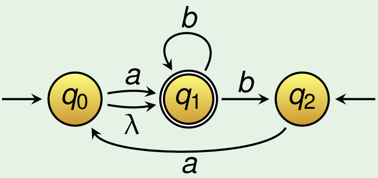
Let $M = (Q, \Sigma, \delta, S, F)$ be a NFA.

For a word w, we write

$$q \xrightarrow{w}\!\!\!\!\rightarrow q'$$

if $w = \alpha_1 \cdots \alpha_n$ for some $\alpha_1, \ldots, \alpha_n \in (\Sigma \cup \{\lambda\})$ and there are states $q_1, \ldots, q_{n-1}$ such that

$$q \stackrel{\alpha_1}{\rightarrow} q_1 \qquad q_1 \stackrel{\alpha_2}{\rightarrow} q_2 \qquad q_2 \stackrel{\alpha_3}{\rightarrow} q_3 \qquad \ldots \qquad q_{n-1} \stackrel{\alpha_n}{\rightarrow} q'$$



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Theorem: $q \xrightarrow{w}\!\!\!\!\rightarrow q' \iff (q, w) \vdash^* (q', \lambda)$.

NFAs Accepting Languages

The **language accepted by** NFA $M = (Q, \Sigma, \delta, S, F)$ is

$$L(M) = \{\, w \in \Sigma^* \mid (q_0, w) \vdash^* (q, \lambda) \text{ with } q_0 \in S,\ q \in F \,\}$$

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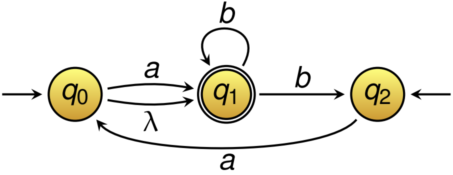

Paths are not unique!

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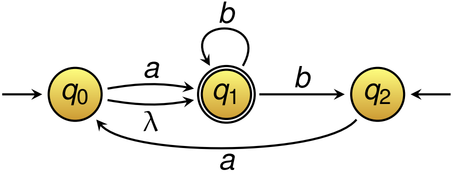

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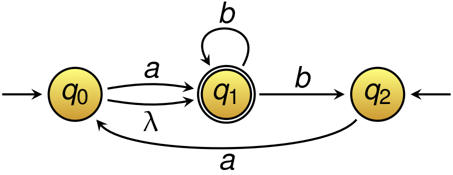

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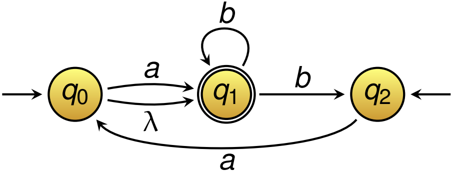

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One accepting path suffices! So *ab* is accepted.

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For every NFA M there is an NFA N such that $L(M) = L(N)$ and N has a **single starting state**.

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Convention

We denote NFAs $(Q, \Sigma, \delta, S, F)$ with a single starting state $S = \{ q_0 \}$ by $(Q, \Sigma, \delta, q_0, F)$.

DFAs and NFAs are Equally Expressive

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\begin{aligned}
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\delta'(X, a) &= \\
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\end{aligned}$$

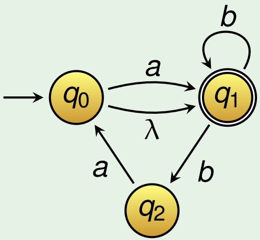
For every $w \in \Sigma^*$ and $X \subseteq Q$ it holds that

$$X \xrightarrow{w}\!\!\!\twoheadrightarrow X' \text{ in } N \quad \iff \quad X' = \{\, q' \mid q \in X,\ q \xrightarrow{w}\!\!\!\twoheadrightarrow q' \text{ in } M \,\}$$

From this property it follows that $L(N) = L(M)$.

Exercise

Given is the following NFA:



Construct a DFA that accepts the same language.

Regular Languages: Reversal

Theorem

If L is regular, then its reverse L^R is regular.

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Then we have

$$q \xrightarrow{w}\!\!\!\!\rightarrow q' \text{ in } M \quad \iff \quad q' \xrightarrow{w^R}\!\!\!\!\rightarrow q \text{ in } N$$

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Then we have

$$q \xrightarrow{w}\!\!\!\!\rightarrow q' \text{ in } M \quad \Longleftrightarrow \quad q' \xrightarrow{w^R}\!\!\!\!\rightarrow q \text{ in } N$$

Since starting and final states are swapped, it follows that

$$w \in L(M) \iff w^R \in L(N)$$

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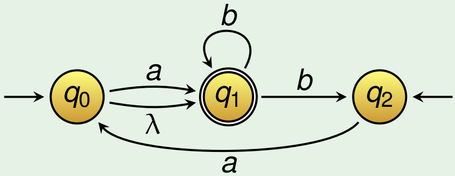

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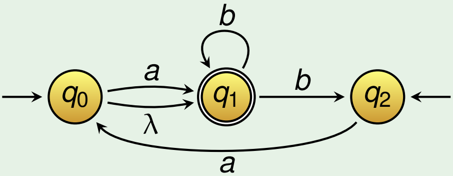

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