Results on the transducer degrees, formalized in Lean 4

Claude (Anthropic) Jörg Endrullis (Vrije Universiteit Amsterdam)

October 2026

Abstract

We collect results on the hierarchy of finite state transducer degrees of infinite words that answer, or make progress on, open problems listed on the web page [7]. The new results were found with the help of Claude, except the strictness $Q < P$ (found by Marcel Horbach with ChatGPT [18]); a few known results from the literature on which they build (the normal form for transducts of spiralling streams, the atoms $\langle p_k \rangle$) are included as well. All of them are formalized in Lean 4 with Mathlib. For each result we give the statement as it is proved in Lean, a short proof sketch, and the Lean module and theorem name.

Contents

1 Introduction **2**

2 Transducts of spiralling streams **3**

3 The structure of the order **3**
3.1 Least upper bounds . . . 3
3.2 Greatest lower bounds and exact pairs . . . 4
3.3 Infinite chains . . . 4
3.4 A dense chain . . . 5
3.5 Embeddings of partial orders . . . 5
3.6 An interval isomorphic to periodic sequences . . . 5
3.7 The first-order theory . . . 6

4 Atoms **6**
4.1 General structure of atoms . . . 6
4.2 Atoms among quasi-polynomial streams . . . 7
4.3 Fast-growing spiralling functions . . . 7
4.4 Chacón and Toeplitz words . . . 7
4.5 Sturmian words with quadratic slope . . . 8
4.6 Thue–Morse and period doubling . . . 8
4.7 The Mephisto Waltz and prime generated streams . . . 9
4.8 Paperfolding, Rudin–Shapiro, squares and powers of two . . . 9

5 Polynomial degrees **10**
5.1 Other spiralling families . . . 12

6 Minimal covers and degrees with few degrees below **12**

7 Continuum many degrees with almost disjoint cones **13**

8 Morphic streams 13

9 Decidability 15

10 Mix-automatic sequences 15

11 Number representations 16

12 Finite automata 17

1 Introduction

Setting. A *stream* is an infinite word $\sigma \in A^{\mathbb{N}}$ over a finite alphabet A. A *finite state transducer* (FST) $T = (S, s_0, \delta, \lambda)$ is a deterministic finite automaton with input alphabet A whose transitions carry output words: $\delta : S \times A \to S$ and $\lambda : S \times A \to B^*$; outputs may be empty. T *transforms* σ into $\tau \in B^{\mathbb{N}}$ if the concatenation of the outputs along the run of T on σ is infinite and equals τ. We write $\sigma \geq \tau$ if some FST transforms σ into τ. This is a preorder; its equivalence classes ($\sigma \equiv \tau$ iff $\sigma \geq \tau \geq \sigma$) are the *(transducer) degrees*, partially ordered by $\geq$ [10, 14]. The least degree $\mathbf{0}$ consists of the ultimately periodic streams. A degree is an *atom* if it is non-zero and every degree below it is $\mathbf{0}$ or itself.

For $f : \mathbb{N} \to \mathbb{N}$ let $\langle f \rangle = 0^{f(0)}1\,0^{f(1)}1\,0^{f(2)}1\cdots$, and write $\langle f(n) \rangle$ for $\langle n \mapsto f(n) \rangle$. For functions $f_0, \ldots, f_{m-1}$, $fzip(f_0, \ldots, f_{m-1})(mn+i) = f_i(n)$. A function f is *spiralling* if $f(n) \to \infty$ and $f \bmod M$ is ultimately periodic for every $M \geq 1$. The polynomial atoms of [11, 12] are $\langle p_k \rangle$ with $p_k(n) = \sum_{i<k}(kn+i)^k$. M is the Thue–Morse word, P the period doubling word ($P(n) = 1$ iff $\nu_2(n+1)$ is even), Q is P with every third letter removed ($Q(2k) = P(3k)$, $Q(2k+1) = P(3k+1)$), and W is the Mephisto Waltz, the fixed point of $0 \mapsto 001$, $1 \mapsto 110$.

The open problems. The web page [7], `https://joerg.endrullis.de/research/finite-state-transducers/`, lists open problems about this hierarchy, together with their solutions and the formalization status. This note collects the results that are formalized, in the order of that page.

Formalization. The new results were found with the help of Claude (Anthropic) in October 2026, with one exception credited where it appears: $Q < P$ was found by Marcel Horbach with ChatGPT. Known results from the literature that are formalized here (Section 2 and Theorem 4.3) are marked with their reference. All results are formalized in Lean 4 (v4.30.0) with Mathlib. Every main theorem depends only on Lean's standard axioms `propext`, `Classical.choice` and `Quot.sound`; there is no `sorry` and there is no custom axiom. Where a result from the literature is used, it enters the Lean theorem as an explicit, named hypothesis, and we state it as such in the theorem ("Assume .... Then ..."). After each theorem, "Lean:" gives the module and the theorem name. The Lean project has two libraries: `Transducts` (modules `Transducts.*`) and `TransducerDegrees` (modules `TransducerDegrees.*`).

Conventions of the formalization. The modules use several, equivalent, formal definitions of FSTs; all are deterministic sequential transducers with finitely many states and word outputs (empty outputs allowed), and all require the output on the infinite input to be infinite. Some modules work with binary streams $\mathbb{N} \to \mathrm{Bool}$; this is no loss of generality, since every stream over a finite alphabet is equivalent to a binary stream (`TransducerDegrees.exists_binary_equiv` in `Transducts.TDNormalForm`). Statements of the form "$\neg\sigma \geq \tau$" are stated in Lean as the non-existence of an FST; "σ is not an atom" is usually stated with an explicit witness τ with $\sigma \geq \tau$, $\tau \notin \mathbf{0}$ and $\tau \not\geq \sigma$.

2 Transducts of spiralling streams

Many results rest on a description of all transducts of $\langle g \rangle$ for spiralling g [12, 13], which we formalized.

Definition 2.1. A *weighted product* α consists of $K, L \geq 1$, window boundaries $0 = s_0 \leq s_1 \leq \cdots \leq s_K = L$ with $s_1 > 0$, weights $a_t \in \mathbb{Q}_{\geq 0}$ and constants $b_r \in \mathbb{Q}$. It acts on $f : \mathbb{N} \to \mathbb{Q}$ by

$$(\alpha \otimes f)(Kq + r) = b_r + \sum_{s_r \leq t < s_{r+1}} a_t \, f(qL + t) \qquad (r < K).$$

It has *positive mass* if every window $[s_r, s_{r+1})$ contains some t with $a_t > 0$.

Theorem 2.2 (normal form)**.** *Let g be spiralling.*

1. *If $\langle g \rangle \geq \sigma$ (for σ over any finite alphabet), then $\sigma \in \mathbf{0}$, or $\sigma \equiv \langle E \rangle$ where $E = \alpha \otimes g(n_1 + \cdot)$ for a weighted product α of positive mass and some n_1.*
2. *If $f(n) \to \infty$, then $\langle g \rangle \geq \langle f \rangle$ if and only if there are a weighted product α of positive mass and n_1, n_2 with $f(n_2 + n) = (\alpha \otimes g(n_1 + \cdot))(n)$ for all n.*
3. *Every weighted product with nonnegative weights is realized: if $E = \alpha \otimes f$ (with values in $\mathbb{N}$) then $\langle f \rangle \geq \langle E \rangle$, for every f.*

Proof sketch. (3) is an explicit transducer that reads L blocks per period and writes the weighted sums. For (1), the formal proof does not follow the generator/parser argument of the literature. Instead σ is replaced by its equivalent *period-break stream*; on the long 0-runs of $\langle g \rangle$ the transducer enters cycles whose outputs contain no 1, and since g is spiralling the cycle positions are periodic in the block index. Hence the gaps between consecutive 1s of the period-break stream are literally a weighted product of a tail of g; windows with only zero weights are then deleted. (2) combines (1), applied to $\sigma = \langle f \rangle$, with a per-position exactness argument, and (3). □

Lean: `Transducts.TDNormalForm`, `TransducerDegrees.normal_form`, `TransducerDegrees.normal_form_alph`, `TransducerDegrees.geq_iff_wprod`, `TransducerDegrees.exact_form`, `TransducerDegrees.geq_wprod`

3 The structure of the order

3.1 Least upper bounds

Theorem 3.1. *Let x, y be streams over finite alphabets with $x \not\geq y$ and $y \not\geq x$. For every z with $z \geq x$ and $z \geq y$ there is an interleaving w of x and y with $w \geq x$, $w \geq y$ and $w \not\geq z$. Consequently, two degrees have a least upper bound if and only if they are comparable.*

Proof sketch. w is built by a diagonal construction against the countably many FSTs A that could map w to z. The key lemma: if all admissible continuations of the current interleaving produce outputs compatible with z and the output can be made arbitrarily long, then z is below x or y; this needs Ramsey's theorem for pairs (proved in the file) to make the state transformations on the runs of x and y homogeneous, and a shortest-witness argument to bound the output. Since $z \geq x$ and $z \geq y$ with x, y incomparable, z is below neither, so each requirement can be met. □

Lean: `TransducerDegrees.NoSuprema`, `thm15`, `sup_iff_comparable`, `no_sup_of_incomparable`

Corollary 3.2. *A finite lattice embeds into the degrees preserving least upper bounds if and only if it is a chain.*

3.2 Greatest lower bounds and exact pairs

Theorem 3.3. *For every stream σ over a finite alphabet there are $\chi_1, \chi_2 \in \{0,1\}^{\mathbb{N}}$ such that, with $\tau_i = (\sigma(n), \chi_i(n))_n$: $\tau_i > \sigma$, τ_1 and τ_2 are incomparable, and every stream w (over any alphabet) with $\tau_1 \geq w$ and $\tau_2 \geq w$ satisfies $\sigma \geq w$. Hence every degree is the greatest lower bound of two incomparable degrees strictly above it.*

Proof sketch. Forcing with finite conditions on (χ_1, χ_2). Requirements $C(\sigma) \neq \tau_i$ are met by flipping a bit. For a requirement "$A(\tau_1) = B(\tau_2) = w$ implies $\sigma \geq w$": if incompatible outputs can be forced, do so; if A can be made silent, do so; otherwise a *greedy simulation* (a subset construction that follows a longest-output run over all admissible bits) computes w from σ. □

Lean: `TransducerDegrees.InfimumOfPair`, `every_degree_is_inf_of_two_incomparable`

Theorem 3.4 (exact pairs)**.** *Let $g_0, g_1, \ldots$ be streams over a common finite alphabet Σ. There are streams a, b over $\Sigma \cup \{\sharp\}$ with $a \geq g_j$ and $b \geq g_j$ for all j, such that every stream e (over any alphabet) with $a \geq e$ and $b \geq e$ satisfies $g_j \geq e$ for some j. Moreover, a, b have a greatest lower bound (among streams over finite alphabets) if and only if $\{g_j\}$ has a greatest element up to $\equiv$; and if the g_j form an antichain, then the maximal common lower bounds of a, b are exactly the g_j up to $\equiv$.*

Proof sketch. a and b are generic packet sequences: packets $\langle j, c\rangle$ carry letters of g_j on track j; tracks are committed one by one, and free packets of larger tags may be inserted. For each pair of FSTs (A, B) one first makes the set of admissible round trips homogeneous for A (Ramsey), then forces incompatibility or silence; if neither is possible, a key lemma (pure tails, equal lengths of equal configurations, shortest witnesses) shows that the common output lies below one g_j. □

Lean: `Transducts.TDExactPairCountable`, `BoundsEPC.exact_pair_countable`, `BoundsEPC.inf_iff_principal`, `BoundsEPC.antichain_coding`

3.3 Infinite chains

Theorem 3.5 (ascending sequences)**.** *Let $\sigma_0 \leq \sigma_1 \leq \cdots$ be streams over a common finite alphabet. There are upper bounds v_1, v_2 of all σ_k such that every stream w with $v_1 \geq w$ and $v_2 \geq w$ satisfies $\sigma_k \geq w$ for some k (an analogue of Spector's exact pair theorem [27]). Consequently, no strictly ascending sequence of degrees has a least upper bound.*

Proof sketch. The same packet forcing as for Theorem 3.4, with conditions consisting of a stage s, a committed packet prefix and a position in a fixed interleaving J_{s-1} of $\sigma_0, \ldots, \sigma_{s-1}$; in the remaining case, greedy simulation with free insertions shows $\sigma_{s-1} \geq w$. If u were a least upper bound, then $v_1, v_2 \geq u$, so $\sigma_k \geq u \geq \sigma_{k+1}$ for some k. □

Lean: `TransducerDegrees.AscendingChains`, `exact_pair`, `no_least_upper_bound`

Theorem 3.6 (descending sequences)**.** *For every stream σ over a finite alphabet there is a strictly descending sequence $c_0 > c_1 > c_2 > \cdots$ with $c_j > \sigma$ for all j, such that every w with $c_j \geq w$ for all j satisfies $\sigma \geq w$. So every degree is the greatest lower bound of a strictly descending sequence.*

Proof sketch. Take one generic bit stream χ for masked products: $c_j = \sigma \otimes_{M_j} \chi$ shows χ only on a residue class M_j modulo 2^j, where $M_0 \supseteq M_1 \supseteq \cdots$ follow a 2-adic integer b. Masks with disjoint classes have common lower bounds below σ; this gives strictness, and a Cantor diagonal over the countably many FSTs chooses b such that all common lower bounds lie below σ. □

Lean: `TransducerDegrees.DescendingChains`, `descending_chain_glb'`

3.4 A dense chain

Theorem 3.7. *For rational $\rho > 0$ let $d_\rho = \langle 2^{\lfloor \rho n \rfloor} \rangle$. Then for rational $\rho, \rho' > 0$: $d_\rho \geq d_{\rho'}$ if and only if $\rho \leq \rho'$. Hence the degrees d_ρ form a dense chain without endpoints.*

Proof sketch. "$\Leftarrow$" is an explicit transducer that, writing $\rho = a/D$, $\rho' = b/D$, deletes rounds and multiplies block lengths by suitable powers of 2. "$\Rightarrow$": the weighted-product representation of a transduct of d_ρ gives exponentials in the period index whose growth rate per output block is at least ρ; comparing exponents and counting blocks gives $\rho' \geq \rho$. □

Lean: `TransducerDegrees.DenseChain`, `chain_iff`, `chain_dense`, `chain_no_endpoints`

Remark 3.8. The web page also states that the degrees below $\langle b^n \rangle$ are exactly $\mathbf{0}$ and the d_ρ with $\rho \geq 1$ (for $b = 2$), and gives the concrete descending example $d_{1/2} > d_{2/3} > \cdots$ with glb d_1. These are not formalized; Theorem 3.6 is the formalized general statement.

3.5 Embeddings of partial orders

Theorem 3.9. *For non-empty finite $S \subseteq \{1, 2, \dots\}$ let $\sigma_S = \langle fzip_{k \in S}(n+1)^k \rangle$ (exponents in increasing order). Then $\sigma_S \geq \sigma_T$ if and only if $T \subseteq S$. Hence every finite partial order embeds into the degrees. Moreover, for every stream σ and every finite partial order $(X, \leq)$ there are streams $f_x > \sigma$ $(x \in X)$ with $x \leq y$ iff $f_y \geq f_x$.*

Proof sketch. "$\Leftarrow$" deletes the blocks of exponents not in T. "$\Rightarrow$": on each output class a transduct is a nonnegative combination of polynomials in the period index with degrees in S, so no leading terms cancel and each $t \in T$ must occur in S. The embedding above an arbitrary σ uses masked generic streams as in Theorem 3.6: $x \mapsto \sigma \otimes_{M_x} \chi$ with masks indexed by the down-sets of x. □

Lean: `TransducerDegrees.PosetEmbedding`, `set_embedding`, `finite_poset_embedding`

Lean: `TransducerDegrees.DescendingChains`, `finite_poset_embeds`

Remark 3.10. The web page writes $fzip_{k \in S} n^k$; the Lean statement uses $(n+1)^k$, which avoids the degenerate block 0^{0^k}. Embeddings of countable partial orders are not formalized.

3.6 An interval isomorphic to periodic sequences

Let $\mathcal{T}_2$ be the set of periodic $c : \mathbb{Z} \to \{0, 1, 2\}$, preordered by $c' \sqsubseteq c$ iff $\exists d \, \forall t \; c'(t) \leq c(t + d)$.

Theorem 3.11. *Let $g_P = fzip(2^n, 3^n, 3^n)$. There is a map $c \mapsto x_c$ such that $\langle g_P \rangle \geq x_c \geq \langle 2^n \rangle$ for all $c \in \mathcal{T}_2$, every x with $\langle g_P \rangle \geq x \geq \langle 2^n \rangle$ is equivalent to some x_c, and $x_c \geq x_{c'}$ iff $c' \sqsubseteq c$ (for periodic c, c'). Moreover $x_2 = \langle g_P \rangle$ and $x_0 = \langle 2^n \rangle$.*

Proof sketch. By Theorem 2.2 every stream in the interval is a weighted product of g_P; the comparison between such products reduces to the combinatorics of how many 3-blocks each output window takes, modulo shift, which is the order $\sqsubseteq$. □

Lean: `Transducts.TDTheorem17`, `TransducerDegrees.theorem17`, `TransducerDegrees.theorem17_equiv`

3.7 The first-order theory

Theorem 3.12. *The first-order theory of the transducer degrees in the language $(\geq, =)$ is recursively isomorphic to (the theory of) second-order arithmetic: there is a bijection e between the sentences of second-order arithmetic (in the language $0, 1, +, \times, <, \in$) and the first-order sentences over $(\geq, =)$, with e and e^{-1} computable, such that φ is true in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ iff $e(\varphi)$ is true in the degrees. In particular the theory is undecidable and not analytical.*

Proof sketch. Lower bound. Natural numbers are coded as the maximal common lower bounds of a pair of parameters (Theorem 3.4); addition, multiplication and arbitrary sets of numbers are coded by exact pairs, so J. Robinson's definability theorem is not needed. A first-order formula χ axiomatizes the coded structure, and standardness is expressed by induction over all pairs; the countable exact pair theorem shows that standard models exist and that every set is coded. The intended parameters are built inside the interval of Theorem 3.11, via a Chinese remainder lemma in $\mathcal{T}_2$. *Upper bound.* $\sigma \geq \tau$ is expressed in second-order arithmetic by the existence of a consistent run with an output-position function. Both translations are primitive recursive; padding makes them injective, and Myhill's isomorphism theorem [19] (proved in Lean) yields the recursive isomorphism. This is the analogue of Simpson's theorem [25] for the Turing degrees. □

Lean: `Transducts.TDFO_Iso`, `TransducerDegrees.FO.problem11_iso`, `TransducerDegrees.FO.problem11_iso_rel`

Remark 3.13. The Lean statement asserts that e and e^{-1} are `Computable`; the construction makes them primitive recursive, but the theorem statement records only computability. The many-one equivalence is also stated separately (`TransducerDegrees.FO.problem11`).

4 Atoms

4.1 General structure of atoms

Theorem 4.1 (dichotomy)**.** *Let x be a stream over a finite alphabet whose degree is an atom. Then*

1. *$x \equiv y$ for a binary stream y with infinitely many 1s and arbitrarily long runs of 0s (so $y = \langle g \rangle$ with g unbounded), or*
2. *x is* asymptotically uniformly recurrent*: every word that occurs infinitely often in x occurs with bounded gaps.*

Proof sketch. If some factor u occurs infinitely often but with unbounded gaps, a window transducer marks the occurrences of u and produces a non-periodic block stream y with $x \geq y$; as x is an atom, $y \geq x$. □

Lean: `Transducts.AtomsR7`, `Transducts.AtomsR7.atom_dichotomy`

Theorem 4.2. *Every transduct of a uniformly recurrent stream has a uniformly recurrent tail.*

Proof sketch. The run of a DFA on a uniformly recurrent word is eventually uniformly recurrent (as a stream of states and letters); the output is the image of that run under a morphism, and the claim transfers. □

Lean: `Transducts.AtomsR7`, `Transducts.AtomsR7.transduct_evUR`

Lean: `Transducts.URTransducts`, `Transducts.Transducer.transforms_evUR`

4.2 Atoms among quasi-polynomial streams

Theorem 4.3 ([12], formalized)**.** *For every $k \geq 1$, $\langle p_k \rangle$ is an atom; for every polynomial $G \in \mathbb{Q}[X]$ of degree $k \geq 1$ with positive leading coefficient and every $g : \mathbb{N} \to \mathbb{N}$ that eventually agrees with G, $\langle g \rangle \geq \langle p_k \rangle$.*

Lean: `TransducerDegrees.Atoms`, `TransducerDegrees.pk_atom`

Lean: `TransducerDegrees.Infimum`, `TransducerDegrees.geq_pk`

Theorem 4.4. *Let $q : \mathbb{N} \to \mathbb{N}$ with $q(n) \to \infty$, and suppose there are $P \geq 1$, N and polynomials $Q_r \in \mathbb{Q}[X]$ with positive leading coefficients such that $q(Pn + r) = Q_r(n)$ for all $r < P$ and $n \geq N$. Then $\langle q \rangle$ is an atom if and only if $\langle q \rangle \equiv \langle p_k \rangle$ for some $k \geq 1$.*

Proof sketch. Selecting the class r gives $\langle q \rangle \geq \langle Q_r \rangle \geq \langle p_{\deg Q_r} \rangle$ (Theorem 4.3). If $\langle q \rangle$ is an atom it is equivalent to each of these. The converse holds since $\langle p_k \rangle$ is an atom. □

Lean: `TransducerDegrees.PropC`, `TransducerDegrees.propC_atoms`, `TransducerDegrees.propC`

4.3 Fast-growing spiralling functions

Theorem 4.5. *Let f be spiralling with $f(n+1)/f(n) \to \infty$ (in Lean: for every R, eventually $R\, f(n) \leq f(n+1)$). Then no stream σ with $\langle f \rangle \geq \sigma$ is an atom. For example $\langle n! \rangle$ bounds no atom.*

Proof sketch. By Theorem 2.2 a non-zero σ is $\equiv \langle h \rangle$ with h a positive-mass weighted product of f, and $h(j+1)/h(j) \to \infty$. Then $g(n) = h(2n) + h(2n+1)$ satisfies $\langle h \rangle \geq \langle g \rangle \notin \mathbf{0}$; if $\langle g \rangle \geq \langle h \rangle$, the weighted-product representation would bound $h(j)$ from below by a positive multiple of $h(2j + O(1))$, which contradicts the growth. □

Lean: `Transducts.TDAtomsB`, `TransducerDegrees.AtomsB.theoremB`

4.4 Chacón and Toeplitz words

Theorem 4.6. *Let C be the Chacón word, the fixed point of $0 \mapsto 0010$, $1 \mapsto 1$, and d the ternary Toeplitz word, the fixed point of $0 \mapsto 010$, $1 \mapsto 011$. Then $C \equiv d$, $d \geq e$ for $e(n) = d(2n)$, $e \notin \mathbf{0}$, and $e \not\geq d$. Hence the Chacón word is not an atom.*

Proof sketch. $C \to d$ outputs a letter iff the previous letter was 0, $d \to C$ is the morphism $0 \mapsto 0$, $1 \mapsto 01$, and $d \to e$ keeps even positions. For $e \not\geq d$: a synchronization lemma (output count has bounded discrepancy from a linear function) applies since d is recognizable in base 3; then output positions are pulled back to input positions near an integer coordinate, where e is regular (its singular point is $-1/2$), and the fibre lemma together with recognizability of e yields two occurrences of the same local data whose output positions have different lowest non-zero ternary digits. □

Lean: `Transducts.Chacon`, `Transducts.chacon_toeplitz`, `Transducts.chacon_not_atom`

Theorem 4.7. *Let $r \geq 3$ and $s_1, \ldots, s_{r-1} \in \{0, 1\}$ not all equal, and let x be the fixed point of $a \mapsto s_1 \cdots s_{r-1} a$ ($a \in \{0, 1\}$). Then $y(n) = x((r-1)n)$ satisfies $x \geq y$, $y \notin \mathbf{0}$ and $y \not\geq x$; so x is not an atom. In particular, for the period doubling word, $\mathbf{0} < P(3n) < P$.*

Proof sketch. $x(i) = s_\delta$ for the lowest non-zero base-r digit δ of $i+1$. The argument of Theorem 4.6 generalizes: synchronization in base r, regularity of y at integer coordinates, and multiplicativity of the lowest non-zero digit. □

Lean: `Transducts.ToeplitzS`, `Transducts.toeplitz_not_atom'`, `Transducts.ys_not_to_xs`, `Transducts.P3_not_to_P`, `Transducts.P_not_atom`

4.5 Sturmian words with quadratic slope

For $0 < \alpha < 1$ irrational and $u \in \mathbb{R}$ let $s_{\alpha,u}(n) = \lfloor (n+1)\alpha + u \rfloor - \lfloor n\alpha + u \rfloor$, and $O = \mathbb{Z} + \mathbb{Z}\alpha$.

Theorem 4.8. *Let $\alpha \in (0,1)$ be a quadratic irrational, $A\alpha^2 + B\alpha + C = 0$ with $A, B, C \in \mathbb{Z}$, $A \neq 0$, and $\alpha' = -B/A - \alpha$ its conjugate. For all $u, v \in \mathbb{R}$:*

$$s_{\alpha,u} \geq s_{\alpha,v} \iff \exists b_0, b_1 \in \mathbb{Z} :\ \beta = b_0 + b_1\alpha \text{ satisfies } \beta O \subseteq O,\ b_0 + b_1\alpha' > 0 \text{ and } \beta v - u \in O.$$

Moreover, no $s_{\alpha,u}$ is an atom.

Proof sketch. "⇐": a β-division transducer reads a window of $s_{\alpha,u}$ around a position moving at rate $\beta' > 0$ and decides each output letter by a count (walk lemma, determinacy lemma). "⇒" (Lean route, without unique ergodicity): Gauss steps are transducer equivalences up to tracking of the intercept; by Lagrange's theorem (proved) the slope becomes purely periodic after finitely many steps, where a synchronization argument and a boundary-matching lemma give the multiplier condition; the conditions are transported back step by step. Not an atom: if $s_{\alpha,u}$ were an atom, after reducing to a purely periodic slope both $s_{\beta,(u'+c)/2}$ $(c = 0, 1)$ would transduce back, giving multipliers with incompatible parities. □

Lean: `Transducts.Sturm.QuadIff`, `Transducts.Sturm.quad_geq_iff`

Lean: `Transducts.Sturm.QuadMain`, `Transducts.Sturm.quad_sturmian_not_atom'`

4.6 Thue–Morse and period doubling

Theorem 4.9 (M. Horbach with ChatGPT [18])**.** *$P \geq Q$ and $Q \not\geq P$. Hence $\mathbf{0} < Q < P \equiv M$ and the degree of the Thue–Morse word is not an atom.*

Proof sketch of the formal proof. Synchronization: lift the run of a transducer T on Q to a fixed point of a 2^r-uniform morphism; on a primitive letter class the normalized output lengths contract, and recognizability of P forces the output count to be exactly linear at return points, so $|q\,N(n) - p\,n|$ is bounded. *No synchronized transducer:* output positions $p'2^T - 1$ come from input positions $q2^{T-s} + c^*$ with an integer c^*, where Q is Toeplitz-regular; the fibre lemma gives occurrences with equal state and discrepancy at positions σ with $\nu_2(\sigma - c^*) = M$ for all large M, while P at the corresponding outputs alternates with the parity of M. $M \equiv P$ by explicit transducers, and $Q \notin \mathbf{0}$. □

Lean: `Transducts.QvsP`, `Transducts.Q_lt_P`

Lean: `Transducts.TableR7`, `Transducts.M_gt_Q`

4.7 The Mephisto Waltz and prime generated streams

For a set A of primes let $v_A(n) = \left(\sum_{p\in A} \nu_p(n)\right) \bmod 2$ ($n \geq 1$); in Lean the stream is indexed from 0 as $n \mapsto v_A(n+1)$.

Theorem 4.10. *Let $E(n) = (W(2n) + W(2n+1)) \bmod 2$. Then $W \geq E$, $E \notin \mathbf{0}$ and $E \not\geq W$; so W is not an atom. For every prime p, $v_{\{p\}}$ is not an atom; for odd p a witness is $y_p(n) = \nu_p(2n+1) \bmod 2$.*

Proof sketch. A transducer from E to W (or from the witness to $v_{\{p\}}$) is synchronized (it writes at a steady rate); then a readout lemma for one-hole Toeplitz words, recognizability and the fibre lemma produce two input positions with the same local data whose required outputs differ. □

Lean: `Transducts.MorphicAtoms`, `Morphic.mephisto_waltz_not_atom`, `Morphic.vp_not_atom`

Theorem 4.11. *$M \not\geq W$ and $W \not\geq M$.*

Proof sketch. By the synchronization lemma (Theorem 4.9), which works in any base, any transducer between M and W has bounded discrepancy between input and output positions. A discrepancy automaton and the residue structure of the digit-counting sequences in bases 2 and 3 then give a contradiction by pigeonhole. The proof uses no result from the literature (in particular, not Durand's Cobham theorem for substitutions). □

Lean: `Transducts.MWSync`, `Transducts.M_W_incomparable_sync`

4.8 Paperfolding, Rudin–Shapiro, squares and powers of two

Let PF be the regular paperfolding word, RS the Rudin–Shapiro word, Sq and Pow_2 the characteristic sequences of the squares and of the powers of 2, BS the Baum–Sweet sequence, Fib the Fibonacci word.

Theorem 4.12.
1. *M, PF, RS are pairwise incomparable; $RS \equiv PF \oplus P$ (letterwise sum mod 2); the stream of pairs $(P(n), PF(n))_n$ is strictly above each of M, PF, RS.*
2. *$M > Q$, and Q is incomparable with PF and with RS.*
3. *Sq and Pow_2 are incomparable, and for every $x \in \{M, P, PF, RS, Q, W, Fib, C\}$: $x \not\geq Sq$, $x \not\geq Pow_2$, $x \not\geq BS$, $Sq \not\geq x$ and $Pow_2 \not\geq x$.*

Proof sketch. (1),(2): synchronization for 4-letter primitive 2-uniform codings, and a germ lemma: the target letter at a synchronized output position depends only on a "germ" of the input window; collisions of germs with different target values ($u \bmod 4$ for PF, level parity for P) give the incomparabilities; $RS \equiv PF \oplus P$ by explicit transducers. For $PF, RS \not\geq Q$ a germ lemma at rational 2-adic points (the singular point $-2/3$ of Q) is used. (3): transducts of uniformly recurrent streams have uniformly recurrent tails (Theorem 4.2), uniformly recurrent transducts of sparse streams such as Sq, Pow_2 are ultimately periodic, the eight streams are uniformly recurrent and not ultimately periodic, and Sq, Pow_2, BS are not eventually uniformly recurrent; a block normal form handles Sq versus Pow_2. □

Lean: `Transducts.RudinShapiro`, `Transducts.PFRS.M_PF_RS_pairwise_incomparable`, `Transducts.PFRS.RS_equiv_UPF`, `Transducts.PFRS.PPF_above`, `Transducts.PFRS.PPF_strict`

Lean: `Transducts.QPFRS`, `Transducts.PFRS.M_Q_PF_RS`

Lean: `Transducts.TableR7`, `Transducts.table_round7`

Lean: `Transducts.SparseS2`, `Transducts.sq_pow2_incomparable`

5 Polynomial degrees

Throughout, a function $f : \mathbb{N} \to \mathbb{N}$ "is" a polynomial $F \in \mathbb{Q}[X]$ if $f(n) = F(n)$ for all n; the order of f is $\deg F$, and the leading coefficient is positive. For orders $k \geq 3$ we use the cumulants: writing $F(x) = c\big((x+\kappa_1)^k + \binom{k}{2}\kappa_2(x+\kappa_1)^{k-2} + \dots\big)$, κ_1 and κ_2 are rational functions of the top three coefficients. For a cubic $c_3n^3+c_2n^2+c_1n+c_0$ we write $F(x) = c\big((x+\kappa)^3+3s(x+\kappa)\big)+e$, so $\kappa = c_2/(3c_3)$ and $3s = D := c_1/c_3 - c_2^2/(3c_3^2)$.

Theorem 5.1. *1. All polynomials of order 1 are equivalent to $\langle n\rangle$; all polynomials of order 2 are equivalent to $\langle n^2\rangle$.*

2. For $k \geq 3$ there are countably infinitely many degrees of polynomials of order k: the polynomials of order k form a countable set, and the $\langle (2^i n+1)^k\rangle$, $i \in \mathbb{N}$, are pairwise inequivalent.

3. Polynomials of different orders are incomparable.

Proof sketch. (1) For order 2 ("completing the square"): keeping the blocks $0^{m^2}1$ of $\langle n^2\rangle$ with $m \equiv b \pmod{2a}$, $m \geq b$ gives $\langle (2an+b)^2\rangle$, and $(2an+b)^2 = 4a(an^2+bn)+b^2$; removing b^2 zeros, dividing by $4a$ and adding c zeros gives $\langle an^2+bn+c\rangle$. The other direction is Theorem 2.2(3). (2),(3) By Theorem 2.2(2), $\langle g\rangle \geq \langle f\rangle$ makes f on each residue class a nonnegative combination of shifted copies of g, which preserves the degree and is controlled by Theorem 5.2. □

Lean: `Transducts.TDPolyPure`, `TransducerDegrees.deg1_one_degree`, `TransducerDegrees.deg2_one_degree`, `TransducerDegrees.sq_geq_quad`, `TransducerDegrees.countably_many_degrees`

Lean: `Transducts.TDPolyCubic`, `TransducerDegrees.poly_different_orders_incomparable`

Theorem 5.2 (pure powers)**.** *Let $k \geq 3$, $f(n) = c_f(n+\kappa_f)^k + e_f$ and $g(n) = c_g(n+\kappa_g)^k + e_g$ with $c_f, c_g > 0$. Then $\langle g\rangle \geq \langle f\rangle$ iff $\kappa_g - \rho\kappa_f \in \mathbb{Z}$ for some integer $\rho \geq 1$. In particular $\langle (an+1)^k\rangle \geq \langle (bn+1)^k\rangle$ iff $a \mid b$, and*

$$\langle n^k\rangle > \langle (2n+1)^k\rangle > \langle (4n+1)^k\rangle > \langle (8n+1)^k\rangle > \cdots > \langle p_k\rangle.$$

Proof sketch. In a weighted product of a k-th power, comparing the coefficients of q^k, q^{k-1}, q^{k-2} forces each window to be a single point (the variance term must vanish), at an arithmetic progression of positions; the shift condition is then integrality. Conversely the selection is an explicit transducer. The bottom $\langle p_k\rangle$ comes from Theorem 4.3. □

Lean: `Transducts.TDPolyPure`, `TransducerDegrees.pure_geq_iff`, `TransducerDegrees.pow_dvd_iff`, `TransducerDegrees.interpolants`

Theorem 5.3 (rigidity, every order $k \geq 3$)**.** *Let F, G be polynomials of order $k \geq 3$ with $\kappa_2(F) < 0$ or $\kappa_2(F) \geq 1/4$. Then $\langle F\rangle \equiv \langle G\rangle$ iff $F(x) = w\,G(x+d) + b$ for some rational $w > 0$, $d \in \mathbb{Z}$ and $b \in \mathbb{Q}$. Moreover, if $\kappa_2(F) \leq 0$ and $\kappa_2(G) \geq 1/4$, then $\langle F\rangle$ and $\langle G\rangle$ are incomparable.*

Proof sketch. If $\langle g\rangle \geq \langle f\rangle$ with scale $a = L/K$, every window has variance $a^2\kappa_2(f) - \kappa_2(g) \in [0, (a-1)^2/4]$. This yields two branches ($\kappa_2 \leq 0$ decreasing, $\kappa_2 \geq 1/4$ increasing) that cannot meet; for equivalence outside the band $[0, 1/4)$ it forces $a = 1$ and point windows, i.e. an integer shift. □

Lean: `Transducts.TDPolyRigid`, `TransducerDegrees.poly_equiv_iff`, `TransducerDegrees.poly_N_U_incomparable`

Theorem 5.4 (cubic classification)**.** *Let f, g be cubics with data (κ_f, s_f), (κ_g, s_g) as above. Then $\langle f\rangle \equiv \langle g\rangle$ iff*

- *$0 < s_f < 1/4$ and $0 < s_g < 1/4$; or*
- *$s_f = s_g = 0$ and κ_f, κ_g have the same denominator; or*
- *$s_f = s_g \notin [0, 1/4)$ and $\kappa_f - \kappa_g \in \mathbb{Z}$.*

This condition is decided by an explicit Boolean function of $(\kappa_f, s_f, \kappa_g, s_g)$. Moreover: $\langle f\rangle$ is below every cubic iff $0 < s_f < 1/4$; $\langle n^3\rangle \geq \langle f\rangle$ iff $0 < s_f < 1/4$ or $s_f = 0$; for $s_f = s_g = 0$, $\langle g\rangle \geq \langle f\rangle$ iff $\operatorname{den}\kappa_g \mid \operatorname{den}\kappa_f$; and cubics with $s \leq 0$ and with $s \geq 1/4$ are incomparable.

Proof sketch. Rigidity (Theorem 5.3) handles $s \notin [0, 1/4)$; the case $s = 0$ is Theorem 5.2. For $0 < s_f < 1/4$, an explicit weighted product with windows $\{0, h, h+1, 2h+1\}$ at scale $2h+2$ realizes $\langle f\rangle$ from every cubic. □

Lean: `Transducts.TDPolyCubic`, `TransducerDegrees.cubic_equiv_iff`, `TransducerDegrees.cubic_equiv_decide`, `TransducerDegrees.cubic_bottom_iff`, `TransducerDegrees.cubic_below_n3_iff`, `TransducerDegrees.cubic_pure_geq_iff`, `TransducerDegrees.cubic_branches_incomparable`

Remark 5.5. Since $p_3(n) = (3n)^3 + (3n+1)^3 + (3n+2)^3$ has $\kappa = 1/3$ and $s = 2/27 \in (0, 1/4)$, Theorem 5.4 shows that a cubic lies in the degree of $\langle p_3\rangle$ iff $0 < D < 3/4$; this corollary is not a separate Lean theorem. Examples: $\langle n^3 - n\rangle > \langle n^3\rangle$ ($s = -1/3$; the transducer is `cube_sub_geq_cube` in `TransducerDegrees.Polynomials`, strictness by `cubic_below_n3_iff`), and $\langle n^3 + n\rangle$ ($s = 1/3$) is incomparable to $\langle n^3\rangle$ ($s = 0$).

Theorem 5.6 (order between cubics)**.** *For cubics f, g: $\langle g\rangle \geq \langle f\rangle$ iff there is a* chain*: $K, L \geq 1$, consecutive windows $0 = s_0 < s_1 < \cdots < s_K = L$, offsets n_1, n_2 and probability distributions x_r on the windows such that each window has mean $\big((n_2 + r) + \kappa_f\big)L/K$ (measured in $(n_1 + t) + \kappa_g$) and variance $(L^2 s_f - K^2 s_g)/K^2$. Moreover the order is decidable: there is an executable Boolean function `decide_cubic_geq` on the rational coefficient vectors F, G (leading coefficients positive) with `decide_cubic_geq`$(F, G) =$ true iff $\langle g\rangle \geq \langle f\rangle$.*

Proof sketch. The chain criterion is the moment expansion of Theorem 2.2(2) for cubics together with a realization lemma for distributions with prescribed mean and variance. A chain exists iff some coprime scale p/q satisfies an exact closed form (decidable). Scales are bounded ($a \leq s_g/s_f$ in the decreasing branch, $a \leq 2s_g + 1/2$ in the increasing branch). Substituting $t = a^2$ makes all regime boundaries rational; an exact test with sign computations in $\mathbb{Q}(\sqrt{t})$ decides whether a *robust* scale exists (one that works for every representation p/q); otherwise every witness has denominator q below an explicit bound, and a finite search decides. □

Lean: `Transducts.TDPolyCubicOrder`, `TransducerDegrees.cubic_geq_iff_chain`

Lean: `Transducts.TDPolyCubicDecide2`, `TransducerDegrees.cubic_geq_decide`, `TransducerDegrees.cubic_geq_decide_cum`

Remark 5.7. `decide_cubic_geq` is an executable Lean function; a `ComputablePred` statement is not given, since Mathlib has no `Primcodable` instance for $\mathbb{Q}$.

Theorem 5.8 (the cone below $\langle n^3 \rangle$)**.** *A* pattern *is a strictly increasing $e : \mathbb{N} \to \mathbb{N}$ with $e(i+m) = e(i) + L$ for some $m, L \geq 1$; let $E_e(i) = e(i)^3$. If $\langle n^3 \rangle \geq \sigma$, then $\sigma \in \mathbf{0}$, or $\sigma \equiv \langle p_3 \rangle$, or $\sigma \equiv \langle E_e \rangle$ for a pattern e. Conversely $\langle n^3 \rangle \geq \langle E_e \rangle > \langle p_3 \rangle$ for every pattern, and $\langle E_e \rangle \geq \langle E_{e'} \rangle$ iff $\lambda\, e'(\mathbb{N}) + \lambda L'\mathbb{Z} \subseteq e(\mathbb{N}) + L\mathbb{Z}$ for some rational $\lambda > 0$.*

Proof sketch. By Theorem 2.2 a transduct is a weighted product of a tail of n^3. If all windows have positive variance, it lies below $\langle p_3 \rangle$ (hence $\equiv \langle p_3 \rangle$ or $\mathbf{0}$); otherwise the point windows define a pattern, and spread windows are realized by two-sided distributions from the pattern. □

Lean: `Transducts.TDPolyCone2`, `TransducerDegrees.cone_below_n3`

Lean: `Transducts.TDPolyCone`, `TransducerDegrees.pat_geq_iff`, `TransducerDegrees.pat_below_n3`, `TransducerDegrees.pat_above_p3`

Remark 5.9. Removing the blocks $0^{n^3}1$ for all multiples n of 4 from $\langle n^3 \rangle$ is the pattern stream for $e(\mathbb{N}) = \mathbb{N} \setminus 4\mathbb{N}$; by the pattern criterion it lies strictly between $\langle (2n+1)^3 \rangle$ and $\langle n^3 \rangle$.

5.1 Other spiralling families

Theorem 5.10. *1. $\langle n^2 \rangle < \langle \mathit{fzip}(n^2, (n+1)^2) \rangle < \langle \mathit{fzip}(n^2, (n+2)^2) \rangle$.*

2. Let $f(m) = \lambda(m)\big(\pi(m)^2 + v(m)\big) \to \infty$ with $\lambda > 0$ and v periodic of period p and $\pi(m+p) = \pi(m) + c$, $c > 0$. Then $\langle n^2 \rangle \geq \langle f \rangle$ iff π is strictly increasing.

3. For positive integers A, B: $\langle 2^n + 3^n \rangle \geq \langle A2^n + B3^n \rangle$ iff $B/A = (3/2)^e$ for some $e \in \mathbb{Z}$.

4. $\langle n2^n \rangle$ and $\langle (2n+1)2^n \rangle$ are incomparable.

Proof sketch. All four follow from Theorem 2.2(2) by comparing the moments of the windows (for quasi-polynomials, a convex-hull condition on the positions $\pi(m)$) or the coefficients of independent exponentials (for (3), (4)). □

Lean: `Transducts.TDPolyQuasi`, `TransducerDegrees.chain`, `TransducerDegrees.sq_geq_iff'`

Lean: `Transducts.TDPolyTwoBase`, `TransducerDegrees.twobase_iff`

Lean: `Transducts.TDPolyPure`, `TransducerDegrees.n2n_incomparable`

6 Minimal covers and degrees with few degrees below

Theorem 6.1. *Let $Q_f(n) = 16n^3 + 12n = (2n-1)^3 + (2n+1)^3$. The degrees below $\langle Q_f \rangle$ are exactly $\mathbf{0}$, $\langle p_3 \rangle$ and $\langle Q_f \rangle$, and $\langle Q_f \rangle > \langle p_3 \rangle > \mathbf{0}$. So $\langle Q_f \rangle$ has precisely two degrees below itself.*

Proof sketch. By Theorem 2.2 a transduct is a weighted product of a tail of Q_f. If every window is a single point it is equivalent to $\langle Q_f \rangle$ (rigidity); if some window has two or more points, its variance is positive and an explicit realization shows $\langle p_3 \rangle$ above it; conversely every positive-mass product is above $\langle p_3 \rangle$ by moment matching (mean and variance of per-window distributions). □

Lean: `Transducts.TDTheoremA`, `TransducerDegrees.theorem_A`

Theorem 6.2. *For a cubic $R(n) = c_3n^3+c_2n^2+c_1n+c_0$ ($c_3 > 0$): $\langle R\rangle$ is a minimal cover of $\langle p_3\rangle$ (that is, $\langle R\rangle > \langle p_3\rangle$ and nothing lies strictly between) iff $D = c_1/c_3-c_2^2/(3c_3^2) = 3/4$. In that case the degrees below $\langle R\rangle$ are exactly $\mathbf{0}, \langle p_3\rangle, \langle R\rangle$. The cubics $4((i+2)n+1)^3+3(i+2)^2((i+2)n+1)$, $i \in \mathbb{N}$, have $D = 3/4$ and are pairwise incomparable; so $\langle p_3\rangle$ has infinitely many pairwise incomparable minimal covers.*

Proof sketch. For each window of an exact form, $D_FL^2 = K^2(D_G+3\mathrm{Var})$, whence the transfer inequality $K^2D_G \le D_FL^2 \le K^2D_G + \frac{3}{4}(L-K)^2$. For $D = 3/4$ this generalizes Theorem 6.1. For $0 < D < 3/4$, $\langle p_3\rangle \ge \langle R\rangle$. For $D < 0$, $D = 0$ and $D > 3/4$ an explicit intermediate cubic or pattern stream T with $\langle R\rangle > \langle T\rangle > \langle p_3\rangle$ is built (one window; every second value; dropping every $(P+1)$-th value). The family has centres $1/(i+2)$, and a rigidity consequence of the transfer inequality separates them. □

Lean: `Transducts.TDMinCovers`, `TransducerDegrees.problem6`, `TransducerDegrees.theorem_B`, `TransducerDegrees.below_cub`, `TransducerDegrees.mincover_of_defect`, `TransducerDegrees.Qfam_incomparable`

7 Continuum many degrees with almost disjoint cones

For $X \subseteq \mathbb{N}$ (identified with $X : \mathbb{N} \to \{0,1\}$) let $w_X = \langle f_X\rangle$ with $f_X(n) = \big((2n + X(\kappa(n)))!\big)^2$, where $\kappa(n)$ is the first component of the Cantor unpairing of $\lfloor \log_2 n \rfloor$.

Theorem 7.1. *1. For $X \ne Y$, every σ with $w_X \ge \sigma$ and $w_Y \ge \sigma$ is ultimately periodic; $w_X \notin \mathbf{0}$ and $w_X \not\ge w_Y$. No stream below any w_X is an atom.*

2. Every stream has only countably many streams below it.

3. Let C be a countable set of binary streams that are pairwise almost disjoint (the only streams below two of them are ultimately periodic). Then there is $D \supseteq C$, pairwise almost disjoint, whose elements represent continuum many degrees; in fact D contains continuum many non-zero streams w that are inequivalent to every other element of D.

This answers the question after Corollary 36 of [12]: "uncountable" can be replaced by "continuum".

Proof sketch. (1) A common non-zero transduct is a positive-mass weighted product of both f_X and f_Y (Theorem 2.2); a dominance lemma for $F(m) = (m!)^2$ forces the same argument $2\tau+X(\kappa(\tau)) = 2\tau'+Y(\kappa(\tau'))$ along an arithmetic progression, and every value k of κ occurs on every arithmetic progression, so $X = Y$. Atomlessness is Theorem 4.5. (2) There are countably many FSTs, and each determines its output. (3) Each non-zero stream below a member of C lies below at most one w_X, so only countably many X are excluded. □

Lean: `Transducts.TDAtomsB`, `TransducerDegrees.AtomsB.theoremA`, `TransducerDegrees.AtomsB.cor36_continuum`

Lean: `Transducts.TDCor36`, `TransducerDegrees.Cor36.cone_countable`, `TransducerDegrees.Cor36.cor36_extension`

8 Morphic streams

Let N_p be the set of primes that are quadratic non-residues modulo p. The family $\mathcal{F}$ consists of $\{2\}$ and, for every odd prime p, the sets $\{p\}$, N_p and $\{p\}\cup N_p$. (The Mephisto Waltz corresponds to $\{3\} \cup N_3$, and $M \equiv v_{\{2\}}$.)

Theorem 8.1 (prime generated streams are not atoms)**.** *v_A is not an atom for every $A \in \mathcal{F}$, and for every finite set A of primes with $|A| \geq 2$ (the case $|A| = 1$ is Theorem 4.10).*

Proof sketch. As for Theorem 4.10: decimations $n \mapsto v_A(Dn+1)$ are one-hole Toeplitz words in base p^M; synchronization and a readout argument show that the decimation does not transduce back. □

Lean: `Transducts.MorphicFamily`, `Morphic.famF_not_atom`

Lean: `Transducts.MorphicFinite`, `Morphic.finite_not_atom`

Theorem 8.2 (decimations of $v_{\{p\}}$)**.** *Let p be prime and $m, m' \geq 1$ not divisible by p. Then $(v_{\{p\}}(mn+1))_n \geq (v_{\{p\}}(m'n+1))_n$ iff $m \mid m'$. For distinct primes $q, q' \neq p$, the decimations with $m = q$ and $m = q'$ are non-zero, below $v_{\{p\}}$ and incomparable; for a prime $q \neq p$ the decimations with $m = q^j$ form a strictly descending chain; and every finite partial order embeds into the non-zero degrees below $v_{\{p\}}$.*

Proof sketch. "$\Leftarrow$" is an explicit decimation transducer. "$\Rightarrow$": synchronization, a general readout step for locally constant readouts, and a choice of the output residue avoiding singular inputs via p-adic valuations. The chain and the poset embedding follow (subsets of primes $\leftrightarrow$ divisibility of products). □

Lean: `Transducts.MorphicDivisibility`, `Morphic.vp_decim_ge_iff`, `Morphic.vp_antichain`

Lean: `Transducts.MorphicChains`, `Morphic.vp_descending_chain`, `Morphic.poset_embeds_below_vp`

Theorem 8.3 (Conjecture 51 of [10] on a family)**.** *For $A, B \in \mathcal{F}$ with $A \neq B$: $v_A \not\geq v_B$. Moreover, $v_A \not\geq v_B$ for every $A \in \mathcal{F}$ and every finite non-empty set $B \neq A$ of primes. In particular $v_{\{p\}}$ and $v_{\{q\}}$ are incomparable for distinct primes p, q.*

Proof sketch. For the same base prime, a readout lemma separates the three sets. For different primes the paper proof uses Cobham's theorem; the formal proof instead uses the synchronization argument of Theorem 4.11 and needs no result from the literature. □

Lean: `Transducts.MorphicFamily`, `Morphic.conj51_family`

Lean: `Transducts.MorphicFinite`, `Morphic.conj51_F_vs_finite`

Theorem 8.4 (erasing transducers and eigenvalues)**.** *Let x_2 be the fixed point of $0 \mapsto 01$, $1 \mapsto 0a$, $a \mapsto aa$ and x_3 that of $0 \mapsto 01a$, $1 \mapsto 0aa$, $a \mapsto aaa$. Then x_2 is 2-substitutive, x_3 is 3-substitutive, and erasing a from either yields the Fibonacci word. Hence the result of [28] on non-erasing transducers does not extend verbatim to erasing ones.*

Proof sketch. Induction on the iterates: erasing a from the n-th iterate gives the n-th Fibonacci prefix. □

Lean: `Transducts.MorphicErasure`, `Morphic.Erasure.fibonacci_by_erasure`

9 Decidability

A computable binary stream is given by a code c of a partial recursive function that is total with values in $\{0,1\}$.

Theorem 9.1. *For computable streams, transducibility is neither recursively enumerable nor co-recursively enumerable: no r.e. predicate R on pairs of codes and no co-r.e. predicate satisfies $R(c,d) \iff s \geq t$ whenever c,d compute s,t. Already the question whether a computable stream can be transformed into the Thue–Morse word M is undecidable (and neither r.e. nor co-r.e.).*

Proof sketch. Reductions from the halting problem: s_e is M until program e halts and then 0^ω, so $s_e \geq M$ iff e does not halt (transducts of ultimately periodic streams are ultimately periodic); s'_e is 0 until e halts at step h and then M, so $s'_e \geq M$ iff e halts. □

Lean: `Transducts.Undecidable`, `Transducts.transducible_not_re_not_co_re`, `Transducts.transducible_to_M_undecidable`, `Transducts.transducible_to_M_not_re`, `Transducts.transducible_to_M_not_co_re`

pair	relation	Lean
$Q - P$, $Q - M$	$Q < P \equiv M$	`Transducts.Q_lt_P`, `Transducts.M_gt_Q`
$M - W$	incomparable	`Transducts.M_W_incomparable_sync`
M, PF, RS	pairwise incomparable	`Transducts.PFRS.M_PF_RS_pairwise_incomparable`
$RS - PF \oplus P$	equivalent	`Transducts.PFRS.RS_equiv_UPF`
$(P, PF) - M, PF, RS$	strictly above	`Transducts.PFRS.PPF_above`, `PPF_strict`
$Q - PF$, $Q - RS$	incomparable	`Transducts.PFRS.M_Q_PF_RS`
$C - d$	equivalent	`Transducts.chacon_toeplitz`
$Sq - Pow_2$	incomparable	`Transducts.sq_pow2_incomparable`
$x - Sq, Pow_2$ (x one of 8 streams)	incomparable	`Transducts.table_round7`
$x - BS$	$x \not\geq BS$	`Transducts.table_round7`
$v_A - v_B$ ($A \neq B \in \mathcal{F}$)	$v_A \not\geq v_B$	`Morphic.conj51_family`

Table 1: Formalized comparisons of well-known streams; the eight streams are $M, P, PF, RS, Q, W, Fib, C$.

10 Mix-automatic sequences

A *base determiner* D assigns to each state q a base $\beta(q) \geq 2$ and a transition on digits; a sequence is generated with D by a DFAO whose base structure covers D, reading the mixed-radix representation (least significant digit first) [8].

Theorem 10.1. *If the product of the bases along every cycle of D reachable from the initial state is a power of $k \geq 2$, then the sequences generated with D are exactly the k-automatic sequences. In particular, alternating the bases $2, 3$ yields exactly the 6-automatic sequences.*

Proof sketch. Cutting closed sub-walks shows that every walk modulus has the form $c\,k^e$ with c from a finite set. A descent argument then shows that every element of the k-kernel is an arithmetic subsequence $X_q(i + sn)$ of a state sequence with s bounded, so the k-kernel is finite; conversely the mixed kernel of a k-automatic sequence is finite, which gives a covering DFAO. □

Lean: `Transducts.MixAuto29`, `MixAuto.bAutomatic_iff_kAutomatic`, `MixAuto.alt23_iff`

Theorem 10.2. *Assume Cobham's theorem [4, 1]: for multiplicatively independent $k, \ell \geq 2$, every sequence that is k-automatic and ℓ-automatic is ultimately periodic. Let D_1, D_2 be base determiners such that within each strongly connected component the cycle products are powers of a single number r (resp. s), and every such r of D_1 is multiplicatively independent of every such s of D_2. Then every sequence generated with both D_1 and D_2 is ultimately periodic. Without the per-component condition this fails: a base determiner with cycle products 2 and 6 shares non-periodic sequences with base 2 and with base 6.*

Proof sketch. By strong induction on the set of reachable states: sequences at exits of a strongly connected component are ultimately periodic by induction, so they can be replaced by kernel automata in the component's root r, and Theorem 10.1 makes the state sequence r-automatic. The same state sequence is an arithmetic subsequence of a sequence generated with D_2; running the same induction on that side gives automaticity in an independent base, and Cobham's theorem applies. The counterexample uses the characteristic sequences of the powers of 2 and of the numbers $(6^b - 1)/5$. □

Lean: `Transducts.MixAuto29`, `MixAuto.cobham_mix`, `MixAuto.D26_shares`

Theorem 10.3. *Assume Durand's theorem [6]: every uniformly recurrent morphic sequence is primitive substitutive. Let $\gamma_p(n) = \nu_p(n+1) \bmod 2$. Then $zip(\gamma_2, \gamma_3)$ is mix-automatic but not morphic.*

Proof sketch. The zip is generated by a 7-state mix-DFAO (root of base 2, then a 2-DFAO for γ_2 and a 3-DFAO for γ_3). It is uniformly recurrent. A gap lemma shows that the prefix of length $2 \cdot 3^c$ recurs only at positions $\geq 2 \cdot 6^c$. If the zip were morphic, the hypothesis would make it primitive substitutive, and then (Durand 1998 [5], proved in Lean for the needed direction) the prefix of length n would recur within distance Mn; with $n = 2 \cdot 3^c$ this gives $2^c \leq M$ for all c. □

Lean: `Transducts.LitDurand1998`, `LitDurand1998.zgam_mix_not_morphic_H1`

Remark 10.4. In `MixAuto29` morphic sequences are defined with non-erasing morphisms; the class is the same as with erasing ones (a standard fact, not formalized).

11 Number representations

For $r : \mathbb{N} \to \Sigma^*$, a sequence X is r-automatic if $X(n) = A(r(n))$ for a DFA with output A [9].

Theorem 11.1. *Assume Durand's theorem [6] (every uniformly recurrent morphic sequence is primitive substitutive) and the transcendence of $\log 2 / \log 6$ (Gelfond–Schneider [15, 23]). Then for no $r : \mathbb{N} \to \Sigma^*$ (computable or not, injective or not) are the r-automatic sequences over finite alphabets exactly the morphic sequences.*

Proof sketch. r-automatic sequences are closed under pairing. The characteristic sequences of the powers of 2 and of 3 are morphic, but their pairing is not: a transducer turns it into the Sturmian word recording the order of the powers of 2 and 3, which has letter frequency $\log 2 / \log 6$. Morphic images of morphic sequences are morphic, and letters in primitive substitutive sequences have algebraic frequencies (via a contraction argument for primitive matrices); both are proved in Lean. □

Lean: `Transducts.LitNumRep`, `LitNumRep.no_morphic_representation_DT`

12 Finite automata

Theorem 12.1 (decomposition). *Every FST is the composition of a Mealy machine that outputs the transition it takes and a one-state transducer that replaces each transition by its output word. Every one-state transducer (morphism) is a coding after an erasing morphism after a letter-splitting morphism. Compositions of non-erasing FSTs are non-erasing, and compositions of FSTs with outputs of length ≤ 1 keep that property, while erasing and doubling one-state transducers exist; so neither can be avoided. A transducer computing the running products in a group G (or in the flip-flop monoid) has G (resp. the flip-flop) as a quotient of the transition monoid of its reachable states.*

Lean: `Transducts.Aut29_E13_PrimeDecomposition`, `Automata29E13.decomp_transduces`, `Automata29E13.morphism_decomp_word`, `Automata29E13.comp_nonErasing`, `Automata29E13.erasing_prime`, `Automata29E13.comp_nonExpanding`, `Automata29E13.expanding_prime`, `Automata29E13.group_prime`, `Automata29E13.flipflop_prime`

Theorem 12.2 (quotients, single loops). *Let $k \geq 2$ and let u_i, v_i, w_i be words of base-k digits ($i = 1, 2$) with $|v_1| = |v_2|$ and $v_2 \neq \varepsilon$; write $[x]_k$ for the value (most significant digit first). Then some m satisfies $0 \neq [u_2 v_2^m w_2]_k \mid [u_1 v_1^m w_1]_k$ iff some $m \leq M$ does, for an explicit bound M computable from the words.*

Proof sketch. Both tracks satisfy $D[\cdot]_k = A_i K + B_i$ with $K = k^{|v|m}$ and $D = k^{|v|} - 1$. If $A_2 = 0$ the denominator is constant and K modulo it is periodic; if $\Delta = A_2 B_1 - A_1 B_2 \neq 0$, divisibility forces $(A_2 K + B_2) \mid \Delta$, which bounds K; if $\Delta = 0$ the quotient is constant. □

Lean: `Transducts.Aut29_E14_SlenderInteger`, `Automata29E14.slender_integer_iff`

Remark 12.3. Together with the decomposition of slender regular languages into finite unions of uv^*w [24] (not formalized), this decides the problem of Rowland and Shallit [22] for slender automata.

Theorem 12.4 (squares, single loops). *Assume Baker's effective bound for Mordell's equation [2]: a function B with $|x| \leq B(k)$ for every integer solution of $y^2 = x^3 + k$, $k \neq 0$. Then for binary words u, v, w with $v \neq \varepsilon$: some $[uv^m w]_2$ is a square iff some $[uv^m w]_2$ with $m \leq 3\big(B(k_0) + B(k_1) + B(k_2)\big) + 2$ is a square, for explicit k_0, k_1, k_2.*

Proof sketch. With $p = |v|$ and $D = 2^p - 1$ one has $D\,[uv^m w]_2 = A\,2^{pm} + B$. If $A = 0$ or $B = 0$ the question is periodic in m. Otherwise write $pm = 3t + r$ and $c = A 2^r$: from $Dy^2 = c(2^t)^3 + B$ one gets $(D^2 cy)^2 = (Dc2^t)^3 + D^3 c^2 B$, an integer point on a Mordell curve with $k_r = D^3 c^2 B$, so Baker's bound gives $2^t \leq B(k_r)$. □

Lean: `Transducts.Aut29_E16_Slender`, `Automata29E16S.slender_square_iff`

Theorem 12.5. *Let L be the set of words $10^i10^j10^k1$ with $i+j+k \geq 11$. Then L is regular, and L contains the binary representation of no square iff the conjecture of Bennett, Bugeaud and Mignotte [3] holds: $13^2, 15^2, 47^2, 111^2$ are the only squares $2^a + 2^b + 2^c + 1$ with $a > b > c > 0$.*

Lean: `Transducts.Aut29_E16_BBM`, `Automata29E16B.LBBM_no_square_iff_BBM`

Lean: `Transducts.Aut29_R2_LBBMRegular`, `Automata29R2LBBM.LBBM_isRegular`

For a bijection f let $\mathcal{L}_f = \{L : f[L] \text{ regular}\}$ [9].

Theorem 12.6 (regularity-reflecting encodings). *1. For every injective f, $\mathcal{L}_f$ is closed under intersection; hence no injective encoding makes exactly the context-free languages regular.*

2. Assume that there is a uniformly decidable sequence of context-free languages over Σ whose cofiniteness is not limit computable (a consequence of the Σ_2-completeness of cofiniteness of CFLs [17, 26]). Then no computable bijection $f : \Sigma^* \to \Gamma^*$, Γ finite, maps every context-free language to a regular language.

3. For a countably infinite effectively presented set X, a class C of subsets of X is contained in $\mathcal{L}_f$ for some computable bijection $f : X \to \{0,1\}^*$ iff C is contained in a family with uniformly computable membership for which it is decidable whether a finite Boolean combination of members is infinite.

4. Assume the classical decidability results for semilinear sets [20, 16, 21] (Parikh, Ginsburg–Spanier, Presburger), stated as: the intersections of regular languages over k letters with Parikh preimages of semilinear sets lie in such a family. Then a single computable bijection makes all these languages regular, for example the language

$$\{w \in \{0,1,2\}^* : |w|_0 = |w|_1 = |w|_2\}.$$

Proof sketch. (1) $f[L \cap L'] = f[L] \cap f[L']$. (2) From f one would compute in the limit whether a CFL is cofinite, by enumerating DFAs (an effective enumeration of regular languages with decidable cofiniteness is proved in Lean). (3) "$\Leftarrow$": enumerate X by atoms of the Boolean algebra and assign codewords so that each member becomes prefix-determined; "$\Rightarrow$": pull back DFAs. (4) is an instance of (3). □

Lean: `Transducts.Aut29_E17_NoEncoding`, `Automata29E17.Lf_inter`

Lean: `Transducts.LitRegEnum`, `LitRegEnum.no_computable_encoding_CFL_fin`, `LitRegEnum.characterization_unconditional`

Lean: `Transducts.Aut29_R2_E17Char`, `Automata29R2E17.PP_encoding`, `Automata29R2E17.equalCounts_mem_PP`

Not formalized

- Decidability of equivalence of polynomials of order $k \geq 4$ inside the band $0 \leq \kappa_2 < 1/4$ near $\langle p_k \rangle$.
- That the polynomial atoms $\langle p_k \rangle$ are morphic (follows from Rigo and Maes).
- Embeddings of countable (infinite) partial orders; the cone below $\langle b^n \rangle$; the concrete descending example $d_{1/2} > d_{2/3} > \cdots$; greatest lower bounds of $\langle \mathit{fzip}(p, 2^n) \rangle$, $\langle \mathit{fzip}(p, 3^n) \rangle$ and of $\langle (an+1)^3 \rangle, \langle (bn+1)^3 \rangle$; glb-preserving embeddings of finite lattices.
- The decomposition of slender regular languages into uv^*w (used to turn Theorems 12.2 and 12.4 into decision procedures).

References

[1] Jean-Paul Allouche and Jeffrey Shallit. *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge University Press, 2003.

[2] Alan Baker. Contributions to the theory of Diophantine equations. II. The Diophantine equation $y^2 = x^3 + k$. *Philosophical Transactions of the Royal Society of London. Series A*, 263:193–208, 1968.

[3] Michael A. Bennett, Yann Bugeaud, and Maurice Mignotte. Perfect powers with few binary digits and related Diophantine problems, II. *Mathematical Proceedings of the Cambridge Philosophical Society*, 153:525–540, 2012.

[4] Alan Cobham. Uniform tag sequences. *Mathematical Systems Theory*, 6:164–192, 1972.

[5] Fabien Durand. A characterization of substitutive sequences using return words. *Discrete Mathematics*, 179:89–101, 1998.

[6] Fabien Durand. Decidability of uniform recurrence of morphic sequences. *International Journal of Foundations of Computer Science*, 24(1):123–146, 2013.

[7] Jörg Endrullis. Research: finite state transducers (open problems). https://joerg.endrullis.de/research/finite-state-transducers/, 2026.

[8] Jörg Endrullis, Clemens Grabmayer, and Dimitri Hendriks. Mix-Automatic Sequences. In *Proc. Conf. on Language and Automata Theory and Applications (LATA 2013)*, volume 7810 of *LNCS*, pages 262–274. Springer, 2013.

[9] Jörg Endrullis, Clemens Grabmayer, and Dimitri Hendriks. Regularity Preserving but Not Reflecting Encodings. In *Proc. Symp. on Logic in Computer Science (LICS 2015)*, pages 535–546. IEEE Computer Society, 2015.

[10] Jörg Endrullis, Dimitri Hendriks, and Jan Willem Klop. Degrees of Streams. *Journal of Integers*, 11B(A6):1–40, 2011. Proceedings of the Leiden Numeration Conference 2010.

[11] Jörg Endrullis, Juhani Karhumäki, Jan Willem Klop, and Aleksi Saarela. Degrees of Infinite Words, Polynomials and Atoms. In *Proc. Conf. on Developments in Language Theory (DLT 2016)*, volume 9840 of *LNCS*, pages 164–176. Springer, 2016.

[12] Jörg Endrullis, Juhani Karhumäki, Jan Willem Klop, and Aleksi Saarela. Degrees of Infinite Words, Polynomials and Atoms. *International Journal of Foundations of Computer Science*, 29(5):825–843, 2018.

[13] Jörg Endrullis, Jan Willem Klop, and Rena Bakhshi. Transducer Degrees: Atoms, Infima and Suprema. *Acta Informatica*, 57(3-5):727–758, 2020.

[14] Jörg Endrullis, Jan Willem Klop, Aleksi Saarela, and Markus A. Whiteland. Degrees of Transducibility. In *Proc. Conf. on Combinatorics on Words (WORDS 2015)*, volume 9304 of *LNCS*, pages 1–13. Springer, 2015.

[15] Alexander O. Gelfond. Sur le septième problème de Hilbert. *Izvestiya Akademii Nauk SSSR*, 7:623–634, 1934.

[16] Seymour Ginsburg and Edwin H. Spanier. Semigroups, Presburger formulas, and languages. *Pacific Journal of Mathematics*, 16(2):285–296, 1966.

[17] John E. Hopcroft and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley, 1979.

[18] Marcel Horbach. A strict finite-state transducer image of the Thue–Morse word. Unpublished manuscript, found with the help of ChatGPT, 2026.

[19] John Myhill. Creative sets. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 1:97–108, 1955.

[20] Rohit J. Parikh. On context-free languages. *Journal of the ACM*, 13(4):570–581, 1966.

[21] Mojżesz Presburger. Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. In *Comptes Rendus du I congrès de Mathématiciens des Pays Slaves*, pages 92–101, 1929.

[22] Eric Rowland and Jeffrey Shallit. Automatic sets of rational numbers. *International Journal of Foundations of Computer Science*, 26(3):343–365, 2015.

[23] Theodor Schneider. Transzendenzuntersuchungen periodischer Funktionen. I. *Journal für die reine und angewandte Mathematik*, 172:65–69, 1935.

[24] Jeffrey Shallit. Numeration systems, linear recurrences, and regular sets. *Information and Computation*, 113(2):331–347, 1994.

[25] Stephen G. Simpson. First-order theory of the degrees of recursive unsolvability. *Annals of Mathematics*, 105(1):121–139, 1977.

[26] Robert I. Soare. *Recursively Enumerable Sets and Degrees.* Springer, 1987.

[27] Clifford Spector. On degrees of recursive unsolvability. *Annals of Mathematics*, 64(3):581–592, 1956.

[28] David Sprunger, William Tune, Jörg Endrullis, and Lawrence S. Moss. Eigenvalues and Transduction of Morphic Sequences. In *Proc. Conf. on Developments in Language Theory (DLT 2014)*, volume 8633 of *LNCS*, pages 239–251. Springer, 2014.