# Degrees of Transducibility


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**Abstract.** Our objects of study are infinite sequences and how they can be transformed into each other. As transformational devices, we focus here on Turing Machines, sequential finite state transducers and Mealy Machines. For each of these choices, the resulting transducibility relation $\geq$ is a preorder on the set of infinite sequences. This preorder induces equivalence classes, called *degrees*, and a partial order on the degrees.

For Turing Machines, this structure of degrees is well-studied and known as *degrees of unsolvability*. However, in this hierarchy, all the computable streams are identified in the bottom degree. It is therefore interesting to study transducibility with respect to weaker computational models, giving rise to more fine-grained structures of degrees. In contrast with the degrees of unsolvability, very little is known about the structure of degrees obtained from finite state transducers or Mealy Machines.


## 1 Introduction

In recent times, computer science, logic and mathematics have extended the focus of interest from finite data types to include infinite data types, of which the paradigm notion is that of infinite sequences of symbols, or *streams*. As Democritus in his adagium Panta Rhei already observed, streams are ubiquitous. Indeed they appear in functional programming, formal language theory, in the mathematics of dynamical systems, fractals and number theory, in business (financial data streams) and in physics (signal processing).

The title of this paper is inspired by the well-known 'degrees of unsolvability', described in Shoenfield[12]. Here sets of natural numbers are compared by means of transducibility using Turing Machines (TMs). The ensuing hierarchy of degrees of unsolvability has been widely studied in the 60's and 70's of the last century and later. We use the notion of degrees of unsolvability as a guiding analogy. In our case, we will deal with streams, noting that a set of natural numbers (as the subject of degrees of unsolvability) is also a stream over the alphabet $\{0, 1\}$ via its characteristic function. However, Turing Machines are too strong for our purposes, since typically we are interested in computable streams and they would all be identified by transducibility via Turing Machines.

We are therefore interested in studying transducibility of streams with respect to less powerful devices. A reduction of the computational power results in a finer structure of degrees. For transforming streams, a few choices present themselves: (sequential) finite state transducers (FSTs) or Mealy Machines (MMs). There are other possibilities, for instance: morphisms, many-one-reducibility, 1-reducibility and tt-reducibility (truth-table reducibility). The last three are more interesting in the context of degrees of unsolvability (Turing degrees).

Let us now describe the contents of this paper. In Section 2, we start with the formal definition of the three main notions of degrees, as generated by transducibility using Turing Machines, by sequential finite state transducers, and by Mealy Machines. The latter two machine models will be defined formally, for Turing Machines we will suppose familiarity, making a definition superfluous. At the end of this preliminary section, we briefly mention the possible employment of infinitary rewriting as an alternative way of phrasing the various transductions. The next section (Section 3) can be considered to be the heart of this paper, with a comparison between Turing degrees and Transducer degrees (arising from Turing Machines and finite state transducers, respectively). Here we mention a dozen of the main properties of Turing degrees, all without the jump operator, and compare these with the situation for Transducer degrees. This yields a number of open questions. In Section 4 we zoom in on an interesting area in the partial order of Transducer degrees, namely the area of 'rarefied ones'[4] streams. It turns out that the degree structure of this restricted area is already surprisingly rich. Remarkably the structure of degrees of these streams requires neither insight about finite state transducers, nor about infinite sequences. We conclude with an extensive list of questions about Transducer degrees, see Section 5.

A general word of warning may be in order. While we think that the questions arising from finite state transducibility of streams are fascinating, they seem to be challenging, some might even be intractable with the current state of the art.

## 2 Preliminaries

We briefly introduce the dramatis personae: (sequential) finite state transducers, Mealy Machines and Turing Machines. For a thorough introduction of finite automata and transducers, we refer the reader to [1,11].

Let $\Sigma$ be an alphabet. We use $\varepsilon$ to denote the empty word. We use $\Sigma^*$ to denote the set of finite words over $\Sigma$, and let $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. The set of infinite sequences over $\Sigma$ is $\Sigma^\omega = \{\sigma \mid \sigma : \mathbb{N} \to \Sigma\}$ and we let $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. In this paper, we consider only sequences over finite alphabets. Without loss of generality we assume that the alphabets are of the form $\Sigma_n = \{0, 1, \ldots, n-1\}$ for some $n \in \mathbb{N}$. Then there are countably many alphabets and countably many finite state transducers over these alphabets. We write $\mathbf{S}$ for the set of all streams over these alphabets, that is $\mathbf{S} = \bigcup_{n \in \mathbb{N}} \Sigma_n^\omega$.

### 2.1 Finite State Transducers and Mealy Machines

Sequential finite state transducers, also known as *deterministic generalised sequential machines (DGSMs)*, are finite automata with input letters and output words along the edges.

[4] The name 'rarefied ones' for the stream $01001000100001\cdots$ occurs in [7, p.208] in the context of dynamical systems.

**Definition 2.1.** A *(sequential) finite state transducer (FST)* $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ consists of

(i) a finite input alphabet $\Sigma$,
(ii) a finite output alphabet $\Gamma$,
(iii) a finite set of states $Q$,
(iv) an initial state $q_0 \in Q$,
(v) a transition function $\delta : Q \times \Sigma \to Q$, and
(vi) an output function $\lambda : Q \times \Sigma \to \Gamma^*$.

Whenever $\Sigma$ and $\Gamma$ are clear from the context we write $A = \langle Q, q_0, \delta, \lambda \rangle$.

A finite state transducer reads an input stream letter by letter and produces a prefix of the output stream in each step.

**Definition 2.2.** Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a finite state transducer. We homomorphically extend the transition function $\delta$ to $Q \times \Sigma^* \to Q$ by

$$\delta(q, \varepsilon) = q \qquad\qquad \delta(q, au) = \delta(\delta(q, a), u)$$

for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^*$, and the output function $\lambda$ to $Q \times \Sigma^\infty \to \Gamma^\infty$ by

$$\lambda(q, \varepsilon) = \varepsilon \qquad\qquad \lambda(q, au) = \lambda(q, a) \cdot \lambda(\delta(q, a), u)$$

for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^\infty$.

A Mealy Machine is an FST that outputs precisely one letter in each step.

**Definition 2.3.** A *Mealy Machine (MM)* is an FST $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ such that $|\lambda(q, a)| = 1$ for every $q \in Q$ and $a \in \Sigma$.

For convenience, we sometimes consider the output function of a Mealy Machine as having type $\lambda : Q \times \Sigma \to \Gamma$.

### 2.2 Degrees of Transducibility

We define the partial orders of degrees of streams arising from Turing Machines, finite state transducers and Mealy Machines. First, we define transducibility relations $\geq_{\mathrm{TM}}$, $\geq_{\mathrm{FST}}$ and $\geq_{\mathrm{MM}}$ on the set of streams.

**Definition 2.4.** Let $\Sigma, \Gamma$ be finite alphabets, and $\sigma \in \Sigma^\omega$, $\tau \in \Gamma^\omega$ streams. For an FST $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$, we write $\sigma \geq_{\mathrm{A}} \tau$ if $\tau = \lambda(q_0, \sigma)$.

(i) We write $\sigma \geq_{\mathrm{FST}} \tau$ if there exists an FST $A$ such that $\sigma \geq_{\mathrm{A}} \tau$.
(ii) We write $\sigma \geq_{\mathrm{MM}} \tau$ if there exists an MM $A$ such that $\sigma \geq_{\mathrm{A}} \tau$.
(iii) We write $\sigma \geq_{\mathrm{TM}} \tau$ if $\tau$ is computable by a TM with oracle $\sigma$.

Note that the relations $\geq_{\mathrm{TM}}$, $\geq_{\mathrm{FST}}$ and $\geq_{\mathrm{MM}}$ are preorders on $\mathbf{S}$. Each of these preorders $\geq$ induces a partial order of 'degrees', the equivalence classes with respect to $\geq \cap \leq$. We denote equivalence using $\equiv$.

**Definition 2.5.** Let $\mathrm{T} \in \{\mathrm{FST}, \mathrm{MM}, \mathrm{TM}\}$. We define $\equiv_{\mathrm{T}}$ as the equivalence relation $\geq_{\mathrm{T}} \cap \leq_{\mathrm{T}}$. The *T-degree* $[\sigma]_{\mathrm{T}}$ of a stream $\sigma \in \mathbf{S}$ is the equivalence class of $\sigma$ with respect to $\equiv_{\mathrm{T}}$, that is, $[\sigma]_{\mathrm{T}} = \{\tau \in \mathbf{S} \mid \sigma \equiv_{\mathrm{T}} \tau\}$. For a set of streams $X \subseteq \mathbf{S}$, we write $[X]_{\mathrm{T}}$ for the set of degrees $\{[\sigma]_{\mathrm{T}} \mid \sigma \in X\}$.

The *T-degrees of transducibility* is the partial order $\langle [\mathbf{S}]_{\mathrm{T}}, \geq_{\mathrm{T}} \rangle$ induced by the preorder $\geq_{\mathrm{T}}$ on $\mathbf{S}$, that is, for $\sigma, \tau \in \mathbf{S}$ we have $[\sigma]_{\mathrm{T}} \geq_{\mathrm{T}} [\tau]_{\mathrm{T}} \iff \sigma \geq_{\mathrm{T}} \tau$. We introduce some notation:

- We use $\mathbf{0}_{\mathrm{T}}$ to denote the *bottom degree* of $\langle [\mathbf{S}]_{\mathrm{T}}, \geq_{\mathrm{T}} \rangle$, that is, the unique degree $\mathbf{a} \in [\mathbf{S}]_{\mathrm{T}}$ such that $\mathbf{a} \leq_{\mathrm{T}} \mathbf{b}$ for every $\mathbf{b} \in [\mathbf{S}]_{\mathrm{T}}$.
- A *minimal cover* of a degree $\mathbf{a}$ is a degree $\mathbf{b}$ such that $\mathbf{a} <_{\mathrm{T}} \mathbf{b}$ and there exists no degree strictly between $\mathbf{a}$ and $\mathbf{b}$.
- An *atom* is a minimal cover of the bottom degree $\mathbf{0}_{\mathrm{T}}$.

In the sequel, we will refer to

- TM-degrees $\langle [\mathbf{S}]_{\mathrm{TM}}, \geq_{\mathrm{TM}} \rangle$ as *Turing degrees*,
- FST-degrees $\langle [\mathbf{S}]_{\mathrm{FST}}, \geq_{\mathrm{FST}} \rangle$ as *Transducer degrees* and
- MM-degrees $\langle [\mathbf{S}]_{\mathrm{MM}}, \geq_{\mathrm{MM}} \rangle$ as *Mealy degrees.*

*Machine models via infinitary rewriting.* As we have seen, degrees of transducibility depend on the machine used. It is worth remarking that describing such machine models, including the transduction of streams, can be conveniently phrased in the framework of rewriting, in particular infinitary rewriting [6,5], including infinitary $\lambda$-calculus.

Clearly, Turing Machines are tantamount to finite $\lambda$-terms, as to their expressive power to define computable functions. Interestingly, oracle Turing Machines can also be described in $\lambda$-calculus, this time in infinitary $\lambda$-calculus. For a set $X \subseteq \mathbb{N}$, we use $\underline{X}$ to denote the infinite $\lambda$-term obtained using iterated pairing that describes the characteristic function of $X$. For example, if $X = \{1, 2, 4, 7, \ldots\}$ then the infinite $\lambda$-term $\underline{X}$ is

$$\underline{X} = \langle \underline{0}, \langle \underline{1}, \langle \underline{1}, \cdots \rangle \rangle \rangle = \lambda z.\, z\underline{0}(\lambda z.\, z\underline{1}(\lambda z.\, z\underline{1}(\cdots)))$$

Here $\langle p, q \rangle = \lambda z.\, zpq$ is the usual pairing in $\lambda$-calculus. Turing reducibility is then a matter of infinitary rewriting $\twoheadrightarrow\!\!\!\!\!\rightarrow$: for $X, Y \subseteq \mathbb{N}$, $X$ is Turing reducible to $Y$, $X \leq_{\mathrm{TM}} Y$, if there exists a finite $\lambda$-term $M$ such that $M\underline{Y} \twoheadrightarrow\!\!\!\!\!\rightarrow \underline{X}$. Sequential finite state transducers and Mealy Machines can be described using restricted forms of $\lambda$-terms $M$ or infinitary first-order rewriting.

## 3 Comparison

In this section, we compare the structure of degrees of transducibility arising from Turing Machines with that obtained from sequential finite state transducers. We will also mention a few facts about the degrees obtained from Mealy Machines.

All three partial orders have very different structural properties, to wit:

– In contrast to the Mealy degrees, there exist atoms (minimal non-zero degrees) in the Turing degrees and Transducer degrees.
– The Turing degrees and Mealy degrees form semi-lattices in contrast to the Transducer degrees for which there exist pairs of degrees without supremum.

*Turing degrees.* Our comparison will be guided by questions that have been studied for Turing degrees, and we start by recalling some of the classical results. The bottom degree $\mathbf{0}_{\mathrm{TM}}$ of this hierarchy consists of all computable streams.

In the following theorem, we summarise a few known results about Turing degrees. For each result we indicate on the right using ✔, ✗ and ? whether the property holds, does not hold or is open for Transducer degrees, respectively. For further reading on Turing degrees we refer the reader to [12,13,16,8,15,9].

**Theorem 3.1.** *For Turing degrees we have:*

*(i) (Kleene, Post) Every degree is countably infinite.* ✔
*(ii) (Kleene, Post) There are $2^{\aleph_0}$ distinct degrees.* ✔
*(iii) (Kleene, Post) For every degree* $\mathbf{a}$*,* $\mathbf{a}{\downarrow} = \{\mathbf{b} \mid \mathbf{a} \geq \mathbf{b}\}$ *is countable.* ✔
*(iv) (Kleene, Post) For every degree* $\mathbf{a}$*, the set* $\mathbf{a}{\uparrow} = \{\mathbf{b} \mid \mathbf{b} \geq_{TM} \mathbf{a}\}$ *has cardinality* $2^{\aleph_0}$*.* ✔
*(v) (Spector) There exists an atom.* ✔
*(vi) (Spector) Every degree has a minimal cover.* ?
*(vii) (Kleene, Post) Every finite set of degrees has a least upper bound.* ✗
*(viii) (Kleene, Post, Spector) No infinite ascending sequence of degrees has a least upper bound.* ?
*(ix) (Kleene, Post) There are pairs of degrees without greatest lower bound.* ✔
*(x) (Kleene, Post) For every degree* $\neq \mathbf{0}$ *there exists an incomparable degree.* ✔
*(xi) (Sacks) Every countable partially ordered set can be embedded.* ?
*(xii) (Sacks) The recursively enumerable degrees are dense: whenever* $\mathbf{a} < \mathbf{c}$ *for recursively enumerable degrees* $\mathbf{a}, \mathbf{c}$*, then there exists a recursively enumerable degree* $\mathbf{b}$ *such that* $\mathbf{a} < \mathbf{b} < \mathbf{c}$*.* ✗
*(xiii) (Simpson) The first-order theory of* $[\mathbf{S}]_{TM}$ *in the language* $\langle \geq, = \rangle$ *is recursively isomorphic to that of true second order arithmetic.* ?

The items (viii) and (ix) of Theorem 3.1 are corollaries of the following famous result by Kleene, Post and Spector.

**Theorem 3.2 (Kleene, Post, Spector [12]).** *Let* $\mathbf{a_0} <_{TM} \mathbf{a_1} <_{TM} \cdots$ *be an infinite ascending sequence of Turing degrees. Then there exist Turing degrees* $\mathbf{b}, \mathbf{c}$ *such that* $\mathbf{b}$ *and* $\mathbf{c}$ *are upper bounds for* $\{\mathbf{a_n}\}_{n \in \mathbb{N}}$*, and there is no Turing degree that is both an upper bound for* $\{\mathbf{a_n}\}_{n \in \mathbb{N}}$ *and a lower bound for* $\{\mathbf{b}, \mathbf{c}\}$*.*

The structure of Turing degrees is extremely complicated. Shore [13] discussed some conjectures about this structure due to Sacks, such as:

(C4) A partially ordered set $P$ is embeddable in the Turing degrees if and only if $P$ has at most continuum cardinality, and each downward cone is at most countable.

(C5) If $S$ is a set of independent Turing degrees of cardinality less than continuum, then there exists a degree $\mathbf{d} \notin S$ such that $S \cup \{\mathbf{d}\}$ is an independent set of degrees.

The first conjecture is still open[5], and the second was shown to be independent of the axioms of ZFC set theory! We expect that the structure of Transducer degrees is more tractable, less complicated than the structure of Turing degrees.

*Transducer degrees.* Sequential finite state transducers are less powerful than Turing machines, and consequently, the Transducer degrees are more fine-grained than the Turing degrees. The Transducer degrees provide an interesting complexity measure for streams. On the one hand, transducers are 'weak enough' to exhibit a rich structure within the computable streams (which trivialise in the bottom degree of the Turing degrees). On the other hand, finite state transduction generalises several usual transformations in dealing with streams, such as alphabet renaming, insertion and removal of elements, or applying a morphism that substitutes words for letters.

The structure of the Transducer degrees is largely unexplored territory with a large number of interesting open questions. An initial study of this partial order of degrees has been carried out in [4,3]. The bottom degree $\mathbf{0}$ of the hierarchy is formed by the ultimately periodic streams. There exist infinite ascending and infinite descending sequences, and thus the hierarchy is not well-founded. It is not difficult to see that there exists no maximal degree, and a set of degrees has an upper bound if and only if the set is countable. The cardinality results (i)–(iv) of Theorem 3.1 hold also for the Transducer degrees. In [4] it has been shown that the degree of the stream

$$\Pi = 1101001000100001000001\ldots$$

is an atom (a minimal non-zero degree) and hence Theorem 3.1(v) is valid for the Transducer degrees. We refer to Section 4 for more on atom degrees. However, it is *open* whether every degree has a minimal cover (compare with Theorem 3.1(vi)). Analogously to the degrees of unsolvability, we call a Transducer degree recursively enumerable if it contains a recursively enumerable stream. As a consequence of the degree of $\Pi$ being an atom, it follows that the recursively enumerable Transducer degrees are not dense, and hence Theorem 3.1(xii) fails for the Transducer degrees. However, it is interesting and *open* whether there exist dense substructures (e.g. dense intervals).

Theorem 3.1(ix) holds for the Transducer degrees: there exist pairs of degrees without a greatest lower bound. In contrast to the Turing degrees, there also exist pairs of Transducer degrees without a least upper bound and thus Theorem 3.1(vii) fails. It is *open* whether there exist infinite ascending sequences of

[5] The conjecture was open at the time of Shore [13] and it has remained open to the best knowledge of the authors.

Transducer degrees with a least upper bound (Theorem 3.1(viii)). The validity of Theorem 3.1(x) for Transducer degrees (the existence of incomparable degrees) follows immediately from the fact that finite state transducers are weaker than Turing Machines. Theorem 3.1(xi) is *open* for Transducer degrees. It is even *open* whether every finite distributive lattice can be embedded. Finally, also the complexity of the first-order theory of Transducer degrees in the language $\langle \geq, = \rangle$ is *open* (compare with Theorem 3.1(xiii)).

*Mealy degrees.* The hierarchy of degrees induced by transducibility via Mealy Machines, has been studied by Rayna in [10] and Belov in [2]. We briefly mention a few interesting facts about this hierarchy. The bottom degree $\mathbf{0}$ of consists of the ultimately periodic streams, just as for the Transducer degrees. Except for the common bottom degree, the Mealy degrees and Transducer degrees exhibit very different properties. In the Mealy degrees, every stream $\sigma \notin \mathbf{0}$ admits an infinite descending chain, while there exist atom degrees in the Transducer degrees. In the Mealy degrees, the degree of a stream $\sigma \notin \mathbf{0}$ is always strictly lower than the degree of every strict suffix of $\sigma$. In contrast, the Transducer degree of a stream is invariant under removal and insertion of finitely many elements. In the Mealy degrees, every finite set of degrees has a least upper bound.

## 4 Atoms and Polynomials

In this section, we want to highlight an intriguing connection between finite state transduction and number theory.

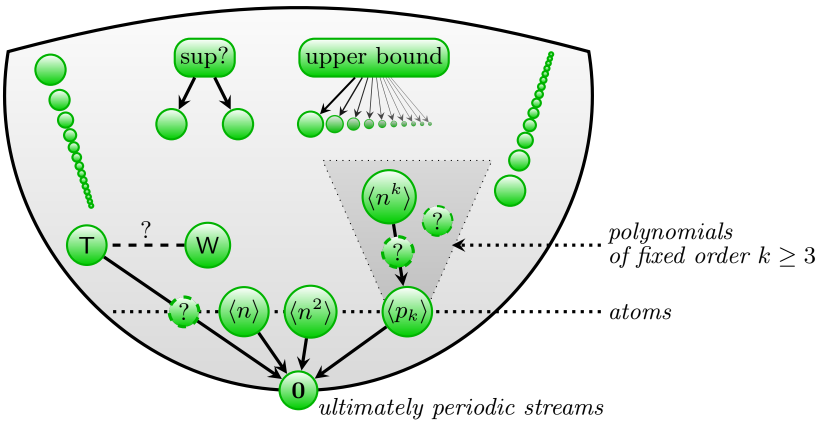



**Fig. 1.** *The partial order of Transducer degrees. Question marks indicate open problems. Here $p_k$ is a polynomial of order $k$, see Section 4 for the form of this polynomial. The degree of $\langle p_k \rangle$ is an atom and all other polynomials of order $k$ can be transduced to $p_k$. Note that $\langle n^k \rangle$ is not an atom for $k \geq 3$.*

We will consider the following ‘rarefied ones’ streams: for $f : \mathbb{N} \to \mathbb{N}$ we use $\langle f \rangle \in \mathbf{2}^\omega$ to denote the sequence

$$\langle f \rangle = \prod_{i=0}^{\infty} 0^{f(i)} 1 = 0^{f(0)}\, 10^{f(1)}\, 10^{f(2)} \cdots ,$$

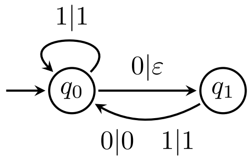
Note that for every stream $\sigma \in \{0,1\}^\omega$ there exists $f : \mathbb{N} \to \mathbb{N}$ such that $\sigma = \langle f \rangle$, and for every stream $\sigma \in \mathbf{S}$ there exists $f : \mathbb{N} \to \mathbb{N}$ such that $\sigma \equiv_{\mathrm{FST}} \langle f \rangle$.

In general, it is difficult to characterise the set of transducts of a sequence $\langle f \rangle$. We will therefore consider the case where the function $f$ is a polynomial. Surprisingly, even for this simple class of functions, there is a rich structure in the degrees, and we reach very soon a large terra incognita.

Let us lead up to the situation as described in Figure 1 by the following stepwise example; afterwards we present the technical key to establish these facts. In the sequel, when speaking about polynomials, we always mean polynomials with non-negative integer coefficients.

(i) *Linear functions.*

All linear functions $\langle an + b \rangle$ are equivalent to $\langle n \rangle$, and the degree of $\langle n \rangle$ is an atom. For example, the following transducer transforms $\langle 2n + 1 \rangle$ to $\langle n \rangle$:




The way back from $\langle n \rangle$ to $\langle 2n + 1 \rangle$ is an easy exercise. The proof that the degree of $\langle n \rangle$ is an atom requires an understanding of the method explained below.

(ii) *Quadratic functions.*

Every quadratic function $\langle an^2 + bn + c \rangle$ transduces to $\langle n^2 \rangle$ and the degree of $\langle n^2 \rangle$ is an atom. This has been shown in [3] using the technical analysis described below. We expect that the same argument yields that $\langle n^2 \rangle$ also transduces to $\langle an^2 + bn + c \rangle$, and hence all quadratic polynomials have the same degree. We wonder whether there is a relation to the well-known geometrical fact that the graphs of quadratic polynomials, parabolas, coincide up to translation and scaling.

Let's work out a typical example which gives a feeling for the capabilities of transducers. We show that $\langle 2n^2 + n + 3 \rangle \geq_{\mathrm{FST}} \langle n^2 \rangle$:

$$\begin{aligned}
\langle 2n^2 + n + 3 \rangle &\equiv \langle 2n^2 + n \rangle && \textit{subtracting a constant} \\
&\equiv \langle 2(n+1)^2 + (n+1) \rangle = \langle 2n^2 + 5n + 3 \rangle && \textit{taking the tail} \\
&\geq \langle 1(2(2n)^2 + 5(2n) + 3) + && \textit{merging even \& odd blocks} \\
&\quad\; 3(2(2n+1)^2 + 5(2n+1) + 3) \rangle && \textit{multiplying odd blocks by 3} \\
&= \langle 32n^2 + 64n \rangle \equiv \langle 32(n+1)^2 \rangle && \textit{adding a constant}
\end{aligned}$$

$$\equiv \langle (n+1)^2 \rangle \qquad \textit{division by 32}$$
$$\equiv \langle n^2 \rangle \qquad \text{prefixing 1}$$

(iii) *Cubic functions and higher order.*

For polynomials of order three and higher the picture becomes much more complicated. For $k \geq 3$, the degree of $\langle n^k \rangle$ is *not* an atom. But nevertheless there are polynomials $p_k(n)$ of order $k$ that do have an atom degree, namely

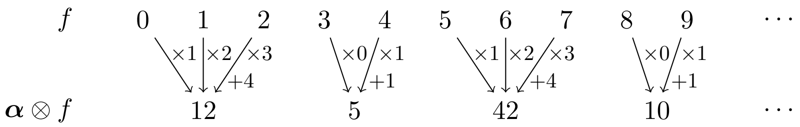
$$p_k(n) = (kn + 0)^k + (kn + 1)^k + \cdots + (kn + (k-1))^k \ .$$

Moreover, all degrees of polynomials of order $k$ are above or equal to this atom degree (and hence the atom degree is unique among them). For example, the unique atom for polynomials of order 3 is the degree of

$$\begin{aligned} p_3(n) &= (3n + 0)^3 + (3n + 1)^3 + (3n + 2)^3 \\ &= 81n^3 + 81n^2 + 45n + 9 \ . \end{aligned}$$

The results on polynomials of order 3 and higher are very recent and still unpublished. We include then as they indicate that there is a rich structure inside this ‘polynomial subhierarchy’.

Let us briefly discuss the technical key observations underlying these results. A *block* is an occurrence of a word $100 \cdots 0$ in a stream. Finite state transducers can multiply (and divide) the length of a block by any non-negative rational number. A transducer can ‘merge’ consecutive blocks by erasing the 1 between the blocks. Moreover, transducers have a finite number of states, so they can multiply and merge in a periodic fashion. This is the essence of what we call ‘weighted products’, denoted by $\boldsymbol{\alpha} \otimes f$. Here $\boldsymbol{\alpha}$ is a tuple of weights and a weight is a tuple of rational numbers. For example, let us consider $f(n) = n$ and $\boldsymbol{\alpha} = \langle \alpha_1, \alpha_2 \rangle$ with $\alpha_1 = \langle 1, 2, 3, 4 \rangle$, $\alpha_2 = \langle 0, 1, 1 \rangle$. Then:




Intuitively, the weight $\alpha_1 = \langle 1, 2, 3, 4 \rangle$ means that three consecutive blocks are merged, where the length of the blocks is multiplied by 1, 2 and 3, respectively, and finally 4 is added to the result. Likewise, the weight $\alpha_2 = \langle 0, 1, 1 \rangle$ means that two consecutive blocks are merged while being multiplied by 0 and 1, respectively, and 1 is added to the result.

Such transformations can always be realised by finite state transducers, that is, for every $f : \mathbb{N} \to \mathbb{N}$ and every tuple of weights $\boldsymbol{\alpha}$ we have $\langle f \rangle \geq_{\text{FST}} \langle \boldsymbol{\alpha} \otimes f \rangle$. However, the crucial observation is the following: for a certain class of functions $f$ this is ‘all’ that finite state transducers can do. This is the class of ‘spiralling’ functions; polynomials fall in this class.

**Definition 4.1.** A function $f : \mathbb{N} \to \mathbb{N}$ is called *spiralling* if

(i) $\lim_{n\to\infty} f(n) = \infty$, and
(ii) for every $m \geq 1$, the function $n \mapsto f(n) \bmod m$ is ultimately periodic.

Functions with the property (ii) are called 'ultimately periodic reducible' in [14]. Note that polynomials with non-negative integer coefficients are spiralling.

For a tuple $\boldsymbol{\alpha} = \langle \alpha_0, \ldots, \alpha_m \rangle$ we define its *rotation* by $\boldsymbol{\alpha}' = \langle \alpha_1, \ldots, \alpha_m, \alpha_0 \rangle$.

**Definition 4.2.** A *weight* is a tuple $\langle a_0, \ldots, a_{k-1}, b \rangle \in \mathbb{Q}^{k+1}$ of rational numbers such that $a_0, \ldots, a_{k-1} \geq 0$. Given a weight $\alpha = \langle a_0, \ldots, a_{k-1}, b \rangle$ and a function $f : \mathbb{N} \to \mathbb{N}$ we define $\alpha \cdot f \in \mathbb{Q}$ by

$$\alpha \cdot f \;=\; a_0 f(0) + a_1 f(1) + \cdots + a_{k-1} f(k-1) + b \,.$$

The weight $\alpha$ is said to be *constant* whenever $a_j = 0$ for all $j \in \mathbb{N}_{<k}$.

**Definition 4.3.** For functions $f : \mathbb{N} \to \mathbb{N}$, and tuples $\boldsymbol{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{m-1} \rangle$ of weights, the *weighted product* of $\boldsymbol{\alpha}$ and $f$ is a function $\boldsymbol{\alpha} \otimes f : \mathbb{N} \to \mathbb{Q}$ that is defined by induction on $n$ through the following scheme of equations:

$$\begin{aligned} (\boldsymbol{\alpha} \otimes f)(0) &= \alpha_0 \cdot f \\ (\boldsymbol{\alpha} \otimes f)(n+1) &= (\boldsymbol{\alpha}' \otimes \mathcal{S}^{|\alpha_0|-1}(f))(n) \qquad\qquad (n \in \mathbb{N}) \end{aligned}$$

where $|\alpha_i|$ is the length of the tuple $\alpha_i$, and $\mathcal{S}^k(f)$ is the $k$-th shift of $f$.

The following theorem from [3] characterises up to equivalence the transducts of spiralling sequences in terms of weighted products.

**Theorem 4.4 ([3]).** *Let $f : \mathbb{N} \to \mathbb{N}$ be spiralling, and $\sigma \in \mathbf{2}^\omega$. Then $\langle f \rangle \geq_{FST} \sigma$ if and only if $\sigma \equiv_{FST} \langle \boldsymbol{\alpha} \otimes \mathcal{S}^{n_0}(f) \rangle$ for some $n_0 \in \mathbb{N}$, and a tuple of weights $\boldsymbol{\alpha}$.*

As an immediate consequence of this theorem we obtain that polynomials of degree $k$ are closed under transduction in the following sense.

**Proposition 4.5 ([3]).** *Let $p(n)$ be a polynomial of degree $k$ with non-negative integer coefficients, and let $\sigma$ be a transduct of $\langle p(n) \rangle$ with $\sigma \notin \mathbf{0}$. Then $\sigma \geq_{FST} \langle q(n) \rangle$ for some polynomial $q(n)$ of degree $k$ with non-negative integer coefficients.*

In [3], Proposition 4.5 and Theorem 4.4 are used to show that the degree of $\langle n^2 \rangle$ is an atom. The following theorem characterises transduction between spiralling sequences (without the 'up to equivalence' of Theorem 4.4).

**Theorem 4.6.** *Let $f, g : \mathbb{N} \to \mathbb{N}$ be spiralling functions. Then $\langle g \rangle \geq_{FST} \langle f \rangle$ if and only if there exist $n_0, m_0 \in \mathbb{N}$ and a tuple of weights $\boldsymbol{\alpha}$ such that*

$$\mathcal{S}^{n_0}(f) = \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g) \,.$$

We expect that several questions about the structure of Transducer degrees could be answered if we understood what preorder $\mathcal{S}^{n_0}(f) = \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g)$ induces on spiralling functions and, in particular, on polynomials.

Even among the polynomials there seems to be a rich structure. Theorem 4.6 can be used to obtain the following two results.

**Theorem 4.7.** *For $k \geq 3$, the degree of $\langle n^k \rangle$ is not an atom.*

Nevertheless, it turns out that for every $k \geq 1$ there exists a unique atom among the degrees of sequences $\langle p \rangle$ where $p$ is a polynomial of order $k$.

**Theorem 4.8.** *Let $k \geq 1$. Let $a_0, \ldots, a_{k-1} \geq 1$ and define*

$$p(n) = a_0(kn+0)^k + a_1(kn+1)^k + \cdots + a_k(kn+(k-1)^k$$

*Then for every polynomial $q$ of order $k$ with non-negative integer coefficients it holds that $\langle q \rangle \geq_{FST} \langle p \rangle$. Hence the degree of $\langle p \rangle$ is an atom (the unique atom among polynomials of order $k$).*

## 5 A Plethora of Questions

We mention a few interesting open questions about the Transducer degrees:

(1) How many atom degrees exist? Are there continuum many?

(2) Does every degree have a minimal cover?

(3) Is every degree $\mathbf{a}$ the greatest lower bound of a pair of degrees ($\neq \mathbf{a}$)?

(4) Are there dense intervals? That is degrees $\mathbf{a}$ and $\mathbf{e}$ with $\mathbf{a} < \mathbf{e}$ such that for all degrees $\mathbf{b}, \mathbf{d}$ with $\mathbf{a} \leq \mathbf{b} < \mathbf{d} \leq \mathbf{e}$ there exists $\mathbf{c}$ with $\mathbf{b} < \mathbf{c} < \mathbf{d}$.

(5) Can every finite partial order be embedded in the hierarchy?

(6) Can every finite distributive lattice be embedded in the hierarchy?

(7) When does a pair of degrees have a supremum?

(8) When does a pair of degrees have an infimum?

(9) Are there infinite ascending sequences of degrees with least upper bound?

(10) Are there infinite descending sequences of degrees with greatest lower bound?

(11) What is the structure of degrees of polynomials of order $k$ (for fixed $k \geq 1$) with non-negative integer coefficients. Is the number of degrees finite for every $k \geq 1$?

(12) Is there a degree that has precisely *two* degrees below itself? This is displayed in Figure 2 on the right.

(13) Is there a degree that has precisely *three* degrees below itself: two incomparable degrees and the bottom degree? This is displayed in Figure 2 on the left.

(14) How complex is the first-order theory in the language $\langle \geq, = \rangle$?
(Compare with Theorem 3.1 item (xiii).)

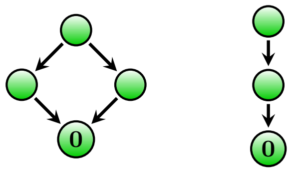



**Fig. 2.** Possible structures in the hierarchy: a diamond, and a line. The arrows $\rightarrow$ mean transducibility $\geq_{\text{FST}}$.

We expect that some of these questions can be answered by better understanding what preorder Theorem 4.6 induces on spiralling functions and polynomials.

There are also intriguing decidability questions, for example:

(15) Is transducibility ($\geq_{\mathrm{FST}}$) decidable for automatic (or morphic) sequences?

(16) Is equivalence ($\equiv_{\mathrm{FST}}$) decidable for automatic (or morphic) sequences?

Moreover, there are challenging questions concerning concrete streams:

(17) Is the degree of Thue-Morse an atom?

(18) Consider the period doubling sequence $\sigma = 1011\ 1010\ 1011\ 1011\ 1011 \cdots$ and drop every third element $\tau = 10\textvisiblespace 1\ 1\textvisiblespace 10\ \textvisiblespace 01\textvisiblespace\ 10\textvisiblespace 1\ 1\textvisiblespace 11 \cdots$. Do we have $\tau \geq_{\mathrm{FST}} \sigma$? If not, then Thue-Morse is not an atom.

(19) Are the degrees of Thue-Morse and Mephisto Waltz incomparable?

(20) Is it decidable whether an automatic sequence can be transduced to ($\geq_{\mathrm{FST}}$) or is equivalent to ($\equiv_{\mathrm{FST}}$) the Thue-Morse sequence?

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