Syllogistic Logic with "Most"

Jörg Endrullis[1] and Lawrence S. Moss[2]

[1] Department of Computer Science, VU University Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands. j.endrullis@vu.nl
[2] Department of Mathematics, Indiana University, Bloomington IN 47405 USA. lsm@cs.indiana.edu

Abstract. This paper presents a sound and complete proof system for the logical system whose sentences are of the form *All X are Y*, *Some X are Y* and *Most X are Y*, where we interpret these sentences on finite models, with the meaning of "most" being "strictly more than half." Our proof system is syllogistic; there are no individual variables.

1 Introduction

The classical syllogistic is the logical system whose sentences are of the form All X are Y, Some X are Y, and No X are Y. These sentences are evaluated in a model by assigning a set $[\![X]\!]$ to the variable X and then using the evident truth definition. This logical system lies at the root of the western logical tradition. For this reason, modern logicians have occasionally looked back on it with an eye to its theoretical properties or to extending it in various ways.

This paper presents an extension of the syllogistic which includes sentences of the form Most X are Y. Variables are interpreted by subsets of a given finite set, with the understanding that Most X are Y means that strictly more than half of the X's are Y's. We present a proof system which is *strongly complete* relative to the semantics: for every finite set Γ of sentences and every sentence φ, $\Gamma \vdash \varphi$ in our system if and only if $\Gamma \models \varphi$. (This last assertion means that every model of Γ is a model of φ.)

To get a feeling for the logical issues we present a few valid and non-valid assertions. Note first that

$$\{\, \text{Most } X \text{ are } Y,\ \text{Most } X \text{ are } Z \,\} \quad \models \quad \text{Some } Y \text{ are } Z\,.$$

For if $[\![Y]\!] \cap [\![Z]\!] = \emptyset$ in a particular model, then it cannot be the case that $[\![X]\!] \cap [\![Y]\!]$ and $[\![X]\!] \cap [\![Z]\!]$ each have more than half of the elements of $[\![X]\!]$.
For a second example, we might ask whether

$$\{\, \text{Most } X \text{ are } Y,\ \text{Most } Y \text{ are } Z,\ \text{Most } Z \text{ are } W \,\} \quad \models \quad \text{Some } X \text{ are } W\,.$$

The answer here is negative: take $[\![X]\!] = \{1, 2, 3\}$, $[\![Y]\!] = \{2, 3, 4\}$, $[\![Z]\!] = \{3, 4, 5\}$, and $[\![W]\!] = \{4, 5, 6\}$.

Another positive assertion:

$$\{\, \text{All } Y \text{ are } X,\ \text{All } X \text{ are } Z,\ \text{Most } Z \text{ are } Y \,\} \quad \models \quad \text{Most } X \text{ are } Y\,.$$

This turns out to be a sound rule of inference in our system. Continuing, we may ask whether

$$\{\,\textsf{All } X \textsf{ are } Z,\ \textsf{All } Y \textsf{ are } Z,\ \textsf{Most } Z \textsf{ are } Y,\ \textsf{Most } Y \textsf{ are } X\,\} \quad \models \quad \textsf{Most } X \textsf{ are } Y\ ?$$

Again, the conclusion does not follow. One can take $[\![X]\!] = \{1,2,3,4,5,6,7\}$, $[\![Y]\!] = \{5,6,7,8,9\}$, and $[\![Z]\!] = \{1,\ldots,9\}$. This example is from [4].

For a final point in this direction, here is a challenge for the reader. Let Γ contain the sentences below

$$\begin{array}{lll} \textsf{Most } U \textsf{ are } A^* & \textsf{All } A^* \textsf{ are } W & \textsf{All } D \textsf{ are } B^* \\ \textsf{Most } V \textsf{ are } B^* & \textsf{All } U \textsf{ are } V & \textsf{All } A^* \textsf{ are } E \\ \textsf{Most } W \textsf{ are } A^* & \textsf{All } V \textsf{ are } W & \textsf{All } B^* \textsf{ are } E \\ \textsf{Most } A^* \textsf{ are } U & \textsf{All } D \textsf{ are } A^* & \textsf{Most } E \textsf{ are } U \end{array} \tag{1}$$

We ask: does $\Gamma \models \textsf{Some } A^* \textsf{ are } B^*$, or not?

The main work of the paper presents a sound and complete proof system for this semantics. The proof system is found in Section 3 and the completeness itself is in Section 4. The last section discusses a fine point on our logical system: it has infinitely many rules, and this is unavoidable.

Prior work on this topic. The problem of axiomatizing the syllogistic logic of Most originates with [4]. That paper obtained some very simple results in the area, such as a completeness result for syllogistic reasoning using Some and Most (but not All), and also explicit statements of some of the very simplest of the infinite rule scheme that we employ in this paper, the scheme of ($\triangleright$). The full formulation of these ($\triangleright$) rules in our logic is new, as is the completeness result.

2 Syntax and semantics

For the syntax of our language, we start with a collection of *nouns*. (These are also called *unary atoms* or *variables* in this area, and we shall use these terms interchangeably.) We use upper-case Roman letters like A, B, …, X, Y, Z for nouns. We are only interested in *sentences* of one of the following three forms:

(i) All X are Y,
(ii) Some X are Y, and
(iii) Most X are Y.

We mentioned sentences No X are Y in the Introduction, but we are ignoring No in what follows; it is open to extend what we do to the larger syllogistic fragment with No.

For the semantics, we use *models* $\mathcal{M}$ consisting of a finite set M together with *interpretations* $[\![X]\!] \subseteq M$ of each noun X. We then interpret our sentences in a model as follows

$$\begin{array}{ll} \mathcal{M} \models \textsf{All } X \textsf{ are } Y & \text{iff } [\![X]\!] \subseteq [\![Y]\!] \\ \mathcal{M} \models \textsf{Some } X \textsf{ are } Y & \text{iff } [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ \mathcal{M} \models \textsf{Most } X \textsf{ are } Y & \text{iff } |[\![X]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![X]\!]| \end{array}$$

Observe that if $[\![X]\!]$ is empty, then automatically $\mathcal{M} \not\models \mathsf{Most}\ X\ \mathsf{are}\ Y$.

We sometimes use φ and ψ as variables ranging over all sentences in the language, and Γ as a variable denoting arbitrary finite sets of sentences.

We say that $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models \psi$ for all $\psi \in \Gamma$.

The main semantic definition is that $\Gamma \models \varphi$ if for all (finite) models $\mathcal{M}$, if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \models \varphi$. The central point of this paper is to provide a proof system which defines a relation $\Gamma \vdash \varphi$ in terms of proof trees, and to prove the soundness and completeness of the system: $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.

3 Proof system

The logical system is a syllogistic one. See Figure 1 for the rules of the system. The rules of All and Some are familiar from basic logic, and the interesting rules of the system are the ones involving Most.

$$\dfrac{}{\mathsf{All}\ X\ \mathsf{are}\ X} \qquad \dfrac{\mathsf{All}\ X\ \mathsf{are}\ Y \quad \mathsf{All}\ Y\ \mathsf{are}\ Z}{\mathsf{All}\ X\ \mathsf{are}\ Z}$$

$$\dfrac{\mathsf{Some}\ X\ \mathsf{are}\ Y}{\mathsf{Some}\ Y\ \mathsf{are}\ X} \qquad \dfrac{\mathsf{Some}\ X\ \mathsf{are}\ Y}{\mathsf{Some}\ X\ \mathsf{are}\ X}$$

$$\dfrac{\mathsf{Some}\ X\ \mathsf{are}\ Y \quad \mathsf{All}\ Y\ \mathsf{are}\ Z}{\mathsf{Some}\ X\ \mathsf{are}\ Z}$$

$$\dfrac{\mathsf{Most}\ X\ \mathsf{are}\ Y}{\mathsf{Some}\ X\ \mathsf{are}\ Y}\ m_1 \qquad \dfrac{\mathsf{Some}\ X\ \mathsf{are}\ X}{\mathsf{Most}\ X\ \mathsf{are}\ X}\ m_2$$

$$\dfrac{\mathsf{Most}\ X\ \mathsf{are}\ Y \quad \mathsf{All}\ Y\ \mathsf{are}\ Z}{\mathsf{Most}\ X\ \mathsf{are}\ Z}\ m_3$$

$$\dfrac{\mathsf{Most}\ X\ \mathsf{are}\ Z \quad \mathsf{All}\ X\ \mathsf{are}\ Y \quad \mathsf{All}\ Y\ \mathsf{are}\ X}{\mathsf{Most}\ Y\ \mathsf{are}\ Z}\ m_4$$

$$\dfrac{\mathsf{All}\ Y\ \mathsf{are}\ X \quad \mathsf{All}\ X\ \mathsf{are}\ Z \quad \mathsf{Most}\ Z\ \mathsf{are}\ Y}{\mathsf{Most}\ X\ \mathsf{are}\ Y}\ m_5$$

$$\dfrac{X_1 \triangleright_{A,B} Y_1 \quad Y_1 \triangleright_{B,A} X_2 \cdots X_n \triangleright_{A,B} Y_n \quad Y_n \triangleright_{B,A} X_1}{\mathsf{Some}\ A\ \mathsf{are}\ B}\ \triangleright$$

Fig. 1. Rules of the logical system for All, Some, and Most. The last line is an infinite rule scheme, and the syntax is explained in Section 3.

We write $\Gamma \vdash \varphi$ to mean that there is a tree $\mathcal{T}$ labeled with sentences from our language such that (a) all of the leaves of $\mathcal{T}$ are labeled with sentences which belong to Γ (or are *axioms* of the form All X are X); (b) each node which is not a leaf matches one of the rules in the system; (c) the root is labeled φ.

As an example, Some X are X, All X are $Y \vdash$ Most X are Y via the tree below:

$$\dfrac{\dfrac{\text{Some } X \text{ are } X}{\text{Most } X \text{ are } X}\,m_2 \qquad \text{All } X \text{ are } Y}{\text{Most } X \text{ are } Y}\,m_3$$

For more on syllogistic logics in general, see [6].

The system is sound: if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. The proof is a routine induction on proof trees in the system. We comment on the soundness of the rules concerning Most.

For (m_1), if Most X are Y in a model $\mathcal{M}$, then in that model, $|[\![X]\!] \cap [\![Y]\!]| > 0$, and so Some X are Y holds. And for (m_2), if $|[\![X]\!]| > 0$, then $|[\![X]\!] \cap [\![X]\!]| = |[\![X]\!]| > \frac{1}{2}|[\![X]\!]|$.

For (m_3), suppose that $|[\![X]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![X]\!]|$ and that $[\![Y]\!] \subseteq [\![Z]\!]$. Then $|[\![X]\!] \cap [\![Z]\!]| \geq |[\![X]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![X]\!]|$, and so we have Most X are Z.

Turning to (m_4), if $|[\![X]\!] \cap [\![Z]\!]| > \frac{1}{2}|[\![X]\!]|$, and also $[\![X]\!] \subseteq [\![Y]\!] \subseteq [\![X]\!]$, then $[\![X]\!] = [\![Y]\!]$, and so $|[\![Y]\!] \cap [\![Z]\!]| > \frac{1}{2}|[\![Y]\!]|$.

For (m_5), assume that $[\![Y]\!] \subseteq [\![X]\!] \subseteq [\![Z]\!]$ and that $|[\![Z]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![Z]\!]|$. Then $Z \cap Y = Y = X \cap Y$, and so $|[\![X]\!] \cap [\![Y]\!]| > \frac{1}{2}|[\![Z]\!]| \geq \frac{1}{2}|[\![X]\!]|$.

For the infinite scheme of ($\triangleright$) rules, we need a preliminary result. In the figures below, and for later in this paper, we present facts about the interpretations of various variables inside a given model using special diagrams. For the most part, the notation is self-explanatory given our statements in Proposition 1 below. We merely alert the reader to the two types of arrows, one (with an open arrowhead and an "inclusion" tail) for All sentences, and one (with a solid arrowhead) for Most sentences.

Proposition 1. *Let $\mathcal{M}$ be a (finite) model which satisfies all of the sentences*

$$\begin{array}{ll} \text{Most } Y \text{ are } A' & \text{All } A' \text{ are } A \\ \text{Most } X \text{ are } B' & \text{All } B' \text{ are } B\,, \end{array} \tag{2}$$

and either the sentence All X are Y,

$$\begin{tikzcd} & & Y \arrow[dl] & & \\ A & A' \arrow[l, hook] & & B' \arrow[r, hook] & B \\ & & X \arrow[uu, hook] \arrow[ur] & & \end{tikzcd}$$

or the sentence All B' are Y

$$\begin{tikzcd} & & Y \arrow[dl] & & \\ A & A' \arrow[l, hook] & & B' \arrow[r, hook] \arrow[ul, hook] & B \\ & & X \arrow[ur] & & \end{tikzcd}$$

Also, assume that in addition, $[\![A]\!] \cap [\![B]\!] = \emptyset$. *Let*

$$X_A = |[\![X]\!] \cap [\![A]\!]|\,,$$

and similarly for X_B, Y_A, and Y_B. Then

$$\min(Y_A, Y_B) > \min(X_A, X_B)\,.$$

Proof. By (2) we have $Y_A > \frac{1}{2}|Y|$. Since $\llbracket A \rrbracket \cap \llbracket B \rrbracket = \emptyset$, it follows $Y_A > Y_B$. Similarly, $X_B > X_A$.

We have two cases, depending on whether All X are Y is true in $\mathcal{M}$, or All B' are Y. In the first case, $Y_A > Y_B \geq X_B > X_A$. In the second case,

$$\begin{aligned}
X_A = |X \cap A| &\leq |X \setminus B'| && \text{since All } B' \text{ are } B \text{ and } \llbracket A \rrbracket \cap \llbracket B \rrbracket = \emptyset \\
&< |X \cap B'| && \text{since Most } X \text{ are } B' \\
&\leq |B'| = |Y \cap B'| && \text{since All } B' \text{ are } Y \\
&\leq |Y \cap B| = Y_B
\end{aligned}$$

Thus $X_A < Y_B, Y_A$. □

In order to state the rules of the system in a concise way, we need to introduce some notation based on what we saw in Proposition 1. We write $X \triangleright_{A,B} Y$ for either

(i) the assertions (2) and All X are Y, or
(ii) the assertions (2) and All B' are Y.

This notation is found in the last rule in Figure 1; actually, this is a rule scheme with infinitely many instances. When we write $X \triangleright_{A,B} Y$, we fix the variables X, Y, A, B, but the additional variables A' and B' are arbitrary. When we have more than one assertion with a $\triangleright$, we permit different additional variables to be used. To be concrete, the first ($\triangleright$) rule would be

$$\frac{X \triangleright_{A,B} Y \quad Y \triangleright_{B,A} X}{\text{Some } A \text{ are } B} \tag{3}$$

This is shorthand for four rules. One of them is: From

$$\begin{array}{lll}
\text{Most } Y \text{ are } A' & \text{All } A' \text{ are } A & \\
\text{Most } X \text{ are } B' & \text{All } B' \text{ are } B & \text{All } X \text{ are } Y \\
\text{Most } Y \text{ are } A'' & \text{All } A'' \text{ are } A & \\
\text{Most } X \text{ are } B'' & \text{All } B'' \text{ are } B & \text{All } A'' \text{ are } X
\end{array} \tag{4}$$

infer Some A are B. Here is a diagrammatic form of this rule:

$$\begin{tikzcd}
A' \arrow[d, hook, -{Triangle[open]}] & Y \arrow[l, -{Triangle}] \arrow[d, -{Triangle}] & B' \arrow[r, hook, -{Triangle[open]}] & B \\
A & A'' \arrow[l, hook, -{Triangle[open]}] \arrow[r, hook, -{Triangle[open]}] & X \arrow[u, -{Triangle}] \arrow[ul, hook, -{Triangle[open]}] \arrow[r, -{Triangle}] & B'' \arrow[u, hook, -{Triangle[open]}]
\end{tikzcd}$$

We mentioned that the ($\triangleright$) rule shown in (3) is shorthand for *four* rules. The other three differ in the assertions in the third column, corresponding to the differing possibilities in Proposition 1. One of these other rules is:

$$\begin{tikzcd}
A' \arrow[d, hook] & Y \arrow[l] \arrow[d] \arrow[dr, hook, bend left] & B' \arrow[r, hook] & B \\
A & A'' \arrow[l, hook'] & X \arrow[ul, hook, bend left] \arrow[u] \arrow[r] & B'' \arrow[u, hook']
\end{tikzcd}$$

where All A'' are X is replaced by All Y are X.

As with all rules in logic, we may identify variables. For example, taking Y to be X, also A' and A'' to be A, and finally B' and B'' to be B, we get

$$\begin{tikzcd}
A \arrow[d, hook] & X \arrow[l] \arrow[d] \arrow[dr, hook, bend left] & B \arrow[r, hook] & B \\
A & A \arrow[l, hook'] & X \arrow[ul, hook, bend left] \arrow[u] \arrow[r] & B \arrow[u, hook']
\end{tikzcd}$$

Dropping repeated premises and the premises All X are X, All A are A and All B are B, we obtain a simpler form of this rule:

$$\frac{\text{Most } X \text{ are } A \quad \text{Most } X \text{ are } B}{\text{Some } A \text{ are } B} \tag{5}$$

This was the rule that we began with, back in (1).

Lemma 1. *Every* ($\triangleright$) *rule is sound.*

Proof. The soundness follows from Proposition 1. We shall only go into details concerning one instance of the rule scheme, the rule 5 described just before the statement of this lemma. Let $\mathcal{M}$ be a (finite) model satisfying all 10 sentences in (4). Assume towards a contradiction that $[\![A]\!] \cap [\![B]\!] = \emptyset$. One use of Proposition 1 shows that $\min(Y_A, Y_B) > \min(X_A, X_B)$. A second use shows $\min(X_A, X_B) > \min(Y_A, Y_B)$. This is a contradiction. ◻

4 Completeness

The next theorem is the main result in this paper.

Theorem 1. *Let Γ be a finite set of sentences in our fragment. If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

The rest of this section is devoted to the proof.

Notation 2 *If $\Gamma \vdash$ All X are Y, we write $X \hookrightarrow Y$. If $\Gamma \vdash$ Most X are Y, we write $X \to Y$. If $\Gamma \vdash$ Some X are Y, we write $X \downarrow Y$.*

Notice that our notation suppresses the underlying set Γ of assumptions. It should also be noted that these shortened notations are intended to be used only for the statements having to do with formal proofs in our system. When discussing a particular model $\mathcal{M}$ of Γ, we generally prefer to write, for example, $\mathcal{M} \models$ Most X are Y instead of $X \to Y$ in $\mathcal{M}$.

The proof of Theorem 1 is by cases as to φ. To keep the cases separate, we treat each in its own subsection. The bulk of the work turns out to be for the case φ is of the form Some A^* are B^*.

4.1 The proof when φ is All A^* are B^*.

This is the easiest case. Let Γ_{all} be the set of All sentences in Γ. We claim that in this case $\Gamma_{\mathsf{all}} \models \varphi$. To see this, let $\mathcal{M} \models \Gamma_{\mathsf{all}}$. Add k fresh points to the interpretation $[\![X]\!]$ of every noun, where k is chosen large enough so that the new interpretations now overlap in most elements. Then the expanded model $\mathcal{M}^+$ satisfies (i) all most sentences, (ii) all some sentences, and (iii) the same all sentences as $\mathcal{M}$. Thus $\mathcal{M}^+ \models \Gamma$, and $\mathcal{M}^+ \models \varphi$. Then the original model $\mathcal{M}$ also satisfies φ since φ is an All-sentence. It follows $\Gamma \vdash \varphi$ since the derivation rules for all are complete, see [4].

4.2 The proof when φ is Most A^* are B^*.

We assume in that $\Gamma \nvdash$ Most A^* are B^*. We build a finite model $\mathcal{M} \models \Gamma$ where $|[\![A^*]\!] \cap [\![B^*]\!]| \leq \frac{1}{2}|[\![A^*]\!]|$. We have three subcases.

The first subcase is when $A^* \hookrightarrow B^*$. In this subcase we have $\neg(A^* \downarrow A^*)$, for otherwise by (m_2) and (m_3), $\Gamma \vdash$ Most A^* are B^*. We define a model $\mathcal{M}$ by $M = \{*\}$, and $[\![X]\!] = \{*\}$ if $X \downarrow X$ and $[\![X]\!] = \emptyset$, otherwise. We check that $\mathcal{M} \models \Gamma$. Consider a sentence $X \rightarrow Y$ in Γ. The rule (m_1) tells us that $X \downarrow X$ and $Y \downarrow Y$, and indeed Most X are Y holds in the model. The same holds for sentences $X \downarrow Y$. For the sentences $X \hookrightarrow Y$, note that if $[\![X]\!] \neq \emptyset$, then $X \downarrow X$. So by the logic, $Y \downarrow Y$, and $[\![X]\!] = \{*\} = [\![Y]\!]$. This concludes the verification that $\mathcal{M} \models \Gamma$. Clearly $[\![A^*]\!] = \emptyset$, and so $\mathcal{M} \not\models$ Most A^* are B^*.

The second subcase is when $\neg(A^* \hookrightarrow B^*)$ *and* $B^* \hookrightarrow A^*$. We divide our variables in three classes:

$$\mathcal{A} = \{X : A^* \hookrightarrow X\} \qquad \mathcal{B} = \{X : X \hookrightarrow B^*\} \qquad \mathcal{C} = \text{all others}$$

Define a model $\mathcal{M}$ using $M = \{1, 2, 3, 4\}$ and

$$[\![X]\!] = \begin{cases} \{1,2,3,4\} & \text{if } X \in \mathcal{A} \\ \{1,2\} & \text{if } X \in \mathcal{B} \\ \{1,2,3\} & \text{if } X \in \mathcal{C} \end{cases} \tag{6}$$

Every sentence of the form All X are Y is true in $\mathcal{M}$, except for the ones with $X \in \mathcal{C}$ and $Y \in \mathcal{B}$, and those with $X \in \mathcal{A}$ and $Y \in \mathcal{B} \cup \mathcal{C}$. But if $Y \in \mathcal{B}$, and if Γ contains All X are Y, then $X \in \mathcal{B}$ as well. If $X \in \mathcal{A}$, and if Γ contains All X are Y, then $Y \in \mathcal{A}$ also. So the All sentences in Γ all hold in $\mathcal{M}$.

Every sentence of the form Some X are Y is true in $\mathcal{M}$.

Every sentence of the form Most X are Y is true in $\mathcal{M}$, except for the ones with $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. But if $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, we cannot have $X \rightarrow Y$: if we did have this, then using (m_5) we would have $A^* \rightarrow B^*$, contradicting our assumption in this section that $\neg(A^* \rightarrow B^*)$.

We conclude that $\mathcal{M} \models \Gamma$. Finally, Most A^* are B^* is false in Γ.

The final subcase is when $\neg(A^* \hookrightarrow B^*)$ *and* $\neg(B^* \hookrightarrow A^*)$. We divide the set of variables into six classes and assign interpretations as follows:

class	variables X such that	interpretation
$\mathcal{A}$	$A^* \hookrightarrow X, \neg(X \hookrightarrow A^*)$	$\{0, 1, 2, 3, 4, 5, 6\}$
$\mathcal{B}$	$A^* \hookrightarrow X$ and $X \hookrightarrow A^*$	$\{1, 2, 3, 4, 5, 6\}$
$\mathcal{C}$	$X \hookrightarrow A^*, \neg(A^* \hookrightarrow X)$, $\neg(X \hookrightarrow B^*)$	$\{1, 2, 3, 4\}$
$\mathcal{D}$	$X \hookrightarrow B^*, \neg(X \hookrightarrow A^*)$	$\{0, 1, 2, 3\}$
$\mathcal{E}$	$X \hookrightarrow A^*, X \hookrightarrow B^*$	$\{1, 2, 3\}$
$\mathcal{F}$	all others	$\{0, 1, 2, 3, 4\}$

This defines a model which we call $\mathcal{M}$. $\mathcal{M}$ satisfies all Some sentences. We omit the proof that $\mathcal{M} \models \Gamma$ but $\mathcal{M} \not\models$ Most A^* are B^*.

4.3 Starting the proof when φ is Some A^* are B^*

We are still proving Theorem 1. We are left with the case that $\Gamma \not\vdash$ Some A^* are B^*. We build a finite model $\mathcal{M} \models \Gamma$ where $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$.

We divide the unary atoms (nouns) in $\Gamma \cup \{\varphi\}$ into five classes:

$$\begin{aligned}
\mathcal{A} &= \{X : X \hookrightarrow A^* \text{ but } \neg(X \hookrightarrow B^*)\} \\
\mathcal{B} &= \{X : X \hookrightarrow B^* \text{ but } \neg(X \hookrightarrow A^*)\} \\
\mathcal{D} &= \{X \notin \mathcal{A} \cup \mathcal{B} : X \hookrightarrow A^* \text{ and } X \hookrightarrow B^*\} \\
\mathcal{C} &= \{X \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{D} : \text{for some } Y, \\
&\qquad X \hookrightarrow Y \to A^* \text{ or } X \hookrightarrow Y \to B^*\} \\
\mathcal{E} &= \text{all other nouns}
\end{aligned}$$

Notation 3 *Henceforth, we use A as a variable for the elements of $\mathcal{A}$, and similarly for B, C, D, and E. Also, we continue to use X as an arbitrary noun, one which might belong to any of the collections $\mathcal{A}, \ldots, \mathcal{E}$.*

Proposition 2. *The following hold:*

1. *$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are pairwise disjoint.*
2. *If $D \in \mathcal{D}$, then for all X, $\neg(D \downarrow X)$.*
3. *If $E \in \mathcal{E}$, then $\neg(E \hookrightarrow X)$ for all $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.*
4. *For $A \in \mathcal{A}, B \in \mathcal{B}$, and $C \in \mathcal{C}$, $\neg(C \hookrightarrow A)$ and $\neg(C \hookrightarrow B)$.*
5. *If $E \in \mathcal{E}$, then $E \not\to X$ for all $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$.*

Proof. Part (1) is an easy consequence of the definitions.

In part (2), from if $D \downarrow X$, and $X \in \mathcal{D}$, then using our logic, we get $A^* \downarrow B^*$. This contradicts our overall assumption that $\Gamma \not\vdash$ Some A^* are B^*.

Part (3) comes down to two similar facts: if $X \hookrightarrow A^*$ or $X \hookrightarrow B^*$, then $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$, and if $C \in \mathcal{C}$ and $X \hookrightarrow C$, then $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. This fact is also behind part (4).

For the last part, assume that $E \in \mathcal{E}$, $E \to X$ and $X \in \mathcal{A} \cup \mathcal{D}$. Then $E \to X \hookrightarrow A^*$, and so $E \hookrightarrow A^*$ by our logic. Thus $E \in \mathcal{A} \cup \mathcal{D}$, contradicting the definition of $\mathcal{E}$.

This completes the proof. □

Dispensing with a trivial case. Since $A^* \hookrightarrow A^*$, we have $A^* \in \mathcal{A} \cup \mathcal{D}$. In case $A^* \in \mathcal{D}$, the model construction is very easy indeed. We let

$$[\![X]\!] = \begin{cases} \emptyset & \text{if } X \hookrightarrow A^* \\ \{*\} & \text{otherwise} \end{cases} \tag{7}$$

It is easy to check that this gives a model of Γ, and clearly $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$. We omit these details.

All the points in our last paragraph apply as well to the case $B^* \in \mathcal{D}$. So from this time forward we avoid these trivial cases and instead make the following assumption.

$$A^* \in \mathcal{A} \text{ and } B^* \in \mathcal{B} \tag{8}$$

This assumption will only be used once, at the very end of our proof.

We now return to the model construction. In the model that we eventually build, we'll have $[\![D]\!] = \emptyset$ for $D \in \mathcal{D}$. And for $E, E' \in \mathcal{E}$, $[\![E]\!] = [\![E']\!]$. We'll also make sure that Most X are E holds for $X \in \mathcal{C}$.

The idea, part I. The high-level description of our semantics is that each interpretation $[\![X]\!]$ will be a disjoint union of four sets:

(i) A variable set that represents $[\![X]\!] \cap [\![A^*]\!]$.
(ii) A constant set that represents additional material added to $[\![A]\!]$ for all $A \in \mathcal{A}$, and also to $[\![Y]\!]$ for $Y \in \mathcal{C} \cup \mathcal{E}$.
(iii) A variable set that represents $[\![X]\!] \cap [\![B^*]\!]$.
(iv) A constant set that represents additional material added to $[\![B]\!]$ for all $B \in \mathcal{B}$, and also to $[\![Y]\!]$ for $Y \in \mathcal{C} \cup \mathcal{E}$.

Each of these sets will be of the form $\{1, \ldots, n\}$ for some n depending on which collection X belongs to, and also some other factors that will be explained in due course. To be a bit more concrete, let us write (i) – (iv) as

$$\{1, \ldots, n_X^1\} + \{1, \ldots, n_{A^*}\} + \{1, \ldots, n_X^2\} + \{1, \ldots, n_{B^*}\},$$

where + denotes disjoint union. (See Section 4.6.) Note that in order that a sentence of the form All U are V be true in the model, we need only arrange that $n_U^1 \leq n_V^1$ and $n_U^2 \leq n_V^2$. To arrange that a sentence of the form Most U are V is true, we shall employ two different ideas. First, in many cases we can simply arrange that the constant sets, the ones in (ii) and (iv) above, are large. This will indeed insure that many Most sentences hold in our model. But more delicately, if $C \in \mathcal{C}$ and $A \in \mathcal{A}$ and we wish to insure that Most C are A holds, then we need to arrange that $[\![A]\!] \subseteq [\![A^*]\!]$, $[\![B]\!] \subseteq [\![B^*]\!]$, $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$, and

$$|[\![A]\!]| > \tfrac{1}{2}\Big(|[\![C]\!] \cap [\![A^*]\!]| + |[\![C]\!] \cap [\![B^*]\!]|\Big). \tag{9}$$

For this purpose, the variables $C \in \mathcal{C}$ lead us to two numerical parameters, C_a and C_b. These are intended to be $|[\![C]\!] \cap [\![A^*]\!]|$ and $|[\![C]\!] \cap [\![B^*]\!]|$. And to get our hands on the values of these parameters, we introduce a set and a well-founded relation of it, and then take the heights of various elements in this relation.

Definition 1. *As a step towards the semantics, we consider a set $\mathcal{G}$ and two relations $\blacktriangleleft$ and $\leq$.*

> *$\mathcal{G} = \mathcal{A} \cup \mathcal{B} \cup \{C_a : C \in \mathcal{C}\} \cup \{C_b : C \in \mathcal{C}\}$. When we need to refer to arbitrary elements of $\mathcal{G}$, we use the letter g.*
> *If $C \to A$, then $C_b \blacktriangleleft C_a, A$.*
> *If $C \to B$, then $C_a \blacktriangleleft C_b, B$.*
> *If $C \hookrightarrow C'$, then $C_a \leq C'_a$ and $C_b \leq C'_b$.*
> *If $A \hookrightarrow A'$, then $A \leq A'$.*
> *If $B \hookrightarrow B'$, then $B \leq B'$.*
> *If $A \hookrightarrow C$, then $A \leq C_a$.*
> *If $B \hookrightarrow C$, then $B \leq C_b$.*

Notice that $\leq$ is a preorder on $\mathcal{G}$.

The idea, part II. The most interesting parts of Definition 1 are the parts having to do with the $\blacktriangleleft$ relation. There is still a ways to go to see how (9) will be arranged, but before we get to this we need to see how $\leq$ and $\blacktriangleleft$ give us a well-founded relation. This is the content of Lemma 3 below, and as a preliminary to this we have Lemma 2.

We write $<$ for the strict part of this preorder, so $g < g'$ means $g \leq g'$ but $\neg(g' \leq g)$.

Here is the only use of the ($\triangleright$) rules of the logic:

Lemma 2. *There are no cycles in $\blacktriangleleft \cup \leq$ which involve a $\blacktriangleleft$ relation. That is, if S is the relation $\blacktriangleleft \cup \leq$, then there is no sequence of length ≥ 2 of the form*

$$g_1 \; S \; g_2 \; \cdots g_i \blacktriangleleft g_{i+1} \; S \; g_k = g_1.$$

Proof. Assume towards a contradiction that we had a sequence as above. By rotating the sequence around, we might as well assume that $i = 1$. Moreover, since $\leq$ is transitive, we may assume that no two successive instances of S are $\leq$. Without loss of generality, g_1 is of the form C^1_a. Thus g_2 is either C^1_b (for the same C) or else g_2 is some B.

We first claim that there must be another instance of $\blacktriangleleft$ on our sequence. For if not, then $g_2 = g_1$. However, we have just seen that g_2 cannot equal g_1. The next instance of $\blacktriangleleft$ must be of one of the two forms $C^2_b \blacktriangleleft C^2_a$, or else $C^2_b \blacktriangleleft A$. Our cycle thus begins

$$C^1_a \; \blacktriangleleft \; g_2 \; \leq C^2_b \; \blacktriangleleft \; g_4 \; \cdots$$

The points before the $\blacktriangleleft$'s alternate between those of the form C_a and those of the form C_b. And so the chain overall may be written

$$C^1_a \; \blacktriangleleft \; g_2 \; \leq C^2_b \; \blacktriangleleft \; g_4 \; \leq C^3_a \; \blacktriangleleft \; \cdots \qquad \cdots \leq C^r_b \; \blacktriangleleft \; g_{2r} \; \leq C^{r+1}_a = C^1_a \; .$$

Now in the first section of the cycle we have $C^1_a \; \blacktriangleleft \; g_2 \; \leq C^2_b \blacktriangleleft g_4$. If g_2 is C^1_b, then we have some $B \in \mathcal{B}$ and $A \in \mathcal{A}$ so that all of the following are provable from Γ:

$$\text{Most } C^1 \text{ are } B, \text{All } C^1 \text{ are } C^2, \text{Most } C^2 \text{ are } A,$$

and also All B are B^* and All A are A^*. (These last two are from the definitions of $\mathcal{A}$ and $\mathcal{B}$.) In short, we get $C^1 \triangleright_{A^*,B^*} C^2$. If g_2 is of the form B, then we again get $C^1 \triangleright_{A^*,B^*} C^2$.

In a similar way, the section of the cycle $C^2_b \blacktriangleleft g_4 \leq C^4_b \blacktriangleleft g_6$, tells us that $C^2 \triangleright_{B,^*,A^*} C^3$. Continuing, we get $C^3 \triangleright_{A^*,B^*} C^4, \ldots, C^r \triangleright_{B,^*,A^*} C^{r+1} = C^1$.

Then by one of the ($\triangleright$)-rules, we see that $\Gamma \vdash$ Some A^* are B^*. But this contradicts the overall assumption made at the very beginning of this section that $\Gamma \nvdash$ Some A^* are B^*. □

We write R for the union $\blacktriangleleft \cup <$. (Note that S used the relation $\leq$ while R uses its strict form $<$.)

Lemma 3. *The relation R is well-founded on $\mathcal{G}$.*

Proof. Suppose $g_0, \ldots, g_n, \ldots$ is an infinite sequence with the property that $g_{n+1}\ R\ g_n$ for all n. Since $\mathcal{G}$ is a finite set, there are $j < k$ such that $g_k = g_j$. Thus

$$g_j = g_k\ R\ g_{k-1}\ R \cdots R\ g_{j+1}\ R\ g_j\ .$$

In this cycle, the instances of R cannot all be from $<$; otherwise, we would have $g_j < g_k$ and $g_j = g_k$, a contradiction. So at least one pair must be related by $\blacktriangleleft$. But then we have a cycle which contradicts Lemma 2. □

The well-foundedness of R implies that there is a unique *rank function* $|\cdot| : \mathcal{G} \to N$ such that for all $g \in \mathcal{G}$, $|g| = \max\{1 + |h| : h\ R\ g\}$.

Lemma 4. *Concerning the well-founded relation R:*

1. *If $A \hookrightarrow A'$ and $A' \hookrightarrow A$, then $|A| = |A'|$. (Similar results hold for $B_1, B_2 \in \mathcal{B}$.)*
2. *If $A \hookrightarrow A'$, then $|A| \leq |A'|$.*
3. *If $C \hookrightarrow C'$ and $C' \hookrightarrow C$, then $|C_a| = |C'_a|$ and $|C_b| = |C'_b|$.*
4. *If $C \hookrightarrow C'$, then $|C_a| \leq |C'_a|$ and $|C_b| \leq |C'_b|$.*
5. *If $A \hookrightarrow C$, then $|A| < |C_a|$.*
6. *If $B \hookrightarrow C$, then $|B| < |C_b|$.*

Proof. In part (1), we show that A and A' have the same immediate predecessors under R, hence the same rank. This is an easy consequence of the following rules of the system: the transitivity of All, (*Barbara*); and also the monotonicity rule for Most, (m_3).

The same point works for part (3), except that we need to show that if $C \hookrightarrow C' \hookrightarrow C$, and $C_a \blacktriangleleft C_b$, then also $C'_a \blacktriangleleft C'_b$. This uses the rule ($m_4$).

Part (2) follows from part (1). Assuming that $A \hookrightarrow A'$, then either $A' \hookrightarrow A$ (and then $|A| = |A'|$), or else $\neg(A' \hookrightarrow A)$ (and then the definition of R and the rank function tell us that $|A| < |A'|$).

Here is the argument for (5); part (6) is similar. If $A \hookrightarrow C$, then $A \leq C_a$ by the definition of $\leq$. We cannot have $C_a \hookrightarrow A$, by the definition of $\mathcal{C}$. We also do not have $C_a \blacktriangleleft A$ (see Definition 1). So $A < C_a$, and thus $|A| < |C_a|$. □

4.4 A lemma on falling sums

Let K be any number. We define a function f_K with domain $\{0, \ldots, K\}$ by

$$f_K(i) = \sum_{l=0}^{i} 2^{K-l}. \tag{10}$$

Note that $1 \le f_K(i) \le 2^{K+1} - 1$.

We use these functions f_K in order to insure that the Most sentences in Γ are true in the model which we build. The key point in the verification hinges on the following result.

Lemma 5. *For all $0 \le i < j, k \le K$, $f_K(k) > \frac{1}{2}(f_K(i) + f_K(j))$.*

Proof. It is easy to check that f is strictly increasing. Fix $0 \le i < j, k \le K$. Note that $i < K$ so that $K - i \ge 1$. We drop the subscript K on f, and then:

$$\begin{aligned} f(i) + f(j) \le f(i) + f(K) &= (\textstyle\sum_{l=0}^{i} 2^{K-l}) + \sum_{l=0}^{K} 2^{l} \\ &< (\textstyle\sum_{l=0}^{i} 2^{K-l}) + 2^{K+1} \\ &= 2((\textstyle\sum_{l=0}^{i} 2^{K-l-1}) + 2^{K}) \\ &= 2\textstyle\sum_{l=0}^{i+1} 2^{K-l} = 2f(i+1) \le 2f(k) \end{aligned}$$

□

4.5 Notation for sets in our model construction

We need some notation for sets. Given numbers a, b, c, d, we let

$$\begin{aligned} a + b + c + d = (&\{1, \ldots, a\} \times \{1\}) \cup (\{1, \ldots, b\} \times \{2\}) \\ &\cup (\{1, \ldots, c\} \times \{3\}) \cup (\{1, \ldots, d\} \times \{4\}) \end{aligned}$$

For example, 1+3+0+2 is a shorthand for the set $\{ (1, 1), (1, 2), (2, 2), (3, 2), (1, 4), (2, 4) \}$.

Observe that if $a \le a'$, $b \le b'$, $c \le c'$, and $d \le d'$, then $a+b+c+d \subseteq a'+b'+c'+d'$.

4.6 The model and the verification

At this point we return to the proof of Theorem 1. We have a set Γ and we want to build a model of it where Some A^* are B^* is false. Definition 1 gives a set $\mathcal{G}$ and Lemma 3 a well-founded relation R on it. For $g \in \mathcal{G}$, let $|g|$ be the rank of g in R. We also remind the reader of the functions f_K defined in (10) above. In what follows, we take

$$K = \max_{g \in \mathcal{G}} |g| \qquad \text{and define} \qquad n_g = f_K(|g|). \tag{11}$$

So $n_g = \sum_{l=0}^{|g|} 2^{K-l}$. We also let $N = 1 + \sum_{l=0}^{K} 2^{K-l} = 2^{K+1}$. Then for all g,

$$n_g < 1 + \textstyle\sum_{l=0}^{|g|} 2^{K-l} = N.$$

We now present our model, using all of the notation above. The universe M is $N+N+N+N$; this is a set with $4N$ elements. The rest of the structure is given as follows:

$$\begin{array}{lcllll}
\text{For } A \in \mathcal{A}, [\![A]\!] & = & n_A & + \; 0 & + \; N & + \; 0 \\
\text{For } B \in \mathcal{B}, [\![B]\!] & = & 0 & + \; n_B & + \; 0 & + \; N \\
\text{For } C \in \mathcal{C}, [\![C]\!] & = & n_{C_a} & + \; n_{C_b} & + \; N & + \; N \\
\text{For } D \in \mathcal{D}, [\![D]\!] & = & 0 & + \; 0 & + \; 0 & + \; 0 \\
\text{For } E \in \mathcal{E}, [\![E]\!] & = & N & + \; N & + \; N & + \; N
\end{array}$$

Note that $[\![D]\!] = \emptyset$, while $[\![E]\!] = M$. This defines our model $\mathcal{M}$.

We turn to the verification that it has the properties needed for our theorem: $\mathcal{M} \models \Gamma$, but $\mathcal{M} \not\models$ Some A^* are B^*.

Lemma 6. *If $X \hookrightarrow Y$ then $[\![X]\!] \subseteq [\![Y]\!]$.*

Proof. For $X, Y \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, this result comes from Lemma 4 and the definitions of the model. We also use the fact that no $C \in \mathcal{C}$ is related by $\hookrightarrow$ to any $A \in \mathcal{A}$ or to any $B \in \mathcal{B}$.

If $X \in \mathcal{D}$, then $[\![X]\!] = \emptyset$. If $Y \in \mathcal{D}$, then $X \in \mathcal{D}$ also, by the definition of $\mathcal{D}$.

If $X \in \mathcal{E}$, then Y cannot belong to $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ by Proposition 2, part (3). For $Y \in \mathcal{E}$, our result comes from the fact that for all X, $[\![X]\!] \subseteq [\![E]\!]$. □

Lemma 7. *If $X \to Y$, then $|[\![X \cap Y]\!]| > \frac{1}{2}|[\![X]\!]|$.*

Proof. Every model satisfies the sentences Most X are X, provided $[\![X]\!] \neq \emptyset$. Our model has $[\![X]\!] \neq \emptyset$ for $X \notin \mathcal{D}$. For all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$, and $E \in \mathcal{E}$, our $\mathcal{M}$ satisfies all sentences of all of the forms Most A are C, Most A are E, Most B are C, Most A are E, and Most E are C. The reason for all of these has to do with the choice of N in the interpretations of the variables, and the fact that $n_g \geq 1$. In other words, we have $|[\![X \cap Y]\!]| > \frac{1}{2}|[\![X]\!]|$ in many cases, even without the assumption that $X \to Y$.

From Proposition 2, we have $\neg(D \to X)$ for all X and also $\neg(E \to X)$ for $X \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$. We also easily have $\neg(A \to D)$ and $\neg(B \to D)$. Finally, we cannot have $A \to B$, since this would easily entail $A^* \downarrow B^*$.

Much of the work in our construction was devoted to insuring that $C \to A$ and $C \to B$. The details on these are similar, so we only discuss $C \to A$. In this case $C_b \blacktriangleright C_a, A$. By Lemma 5, $n_A > \frac{1}{2}(n_{C_a} + n_{C_b})$. Our construction has arranged that $[\![C]\!] \cap [\![X]\!] = \min(n_A, n_{C_a}) + 0 + N + 0$. Recall that n_A is strictly larger than the average of $n(C_a)$ and $n(C_b)$. Since $n(C_a) > n(C_b)$, $n(C_a)$ is also strictly larger than that average. Thus

$$\begin{aligned}
|[\![C \cap A]\!]| &= \min(n_A, n_{C_a}) + N \\
&> \tfrac{1}{2}(n_{C_a} + n_{C_b}) + N \;=\; \tfrac{1}{2}|[\![C]\!]|
\end{aligned}$$

This completes the proof. □

Lemma 8. *If $X \downarrow Y$ then $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$.*

Lemma 9. $[\![A^*]\!] \cap [\![B^*]\!] = \emptyset$.

Proof. Our construction has arranged that $[\![A]\!] \cap [\![B]\!] = \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Recall that we are assuming that $A^* \in \mathcal{A}$ and $B^* \in \mathcal{B}$; see (8) and the discussion preceding it. Our result follows. □

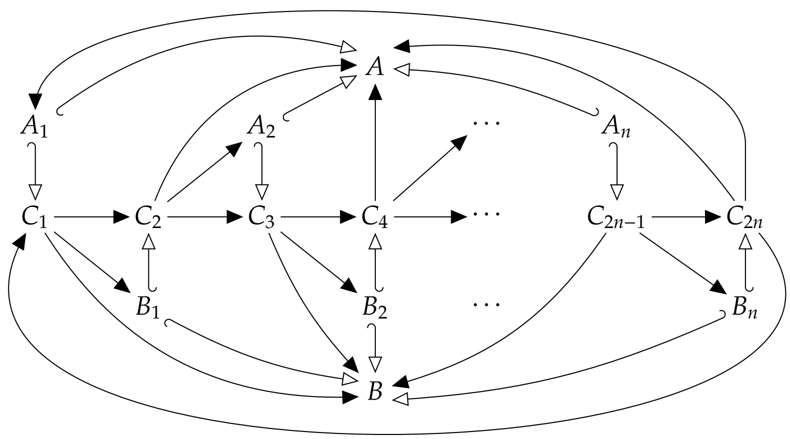
This concludes the proof of Theorem 1.

5 No finite axiomatization

Our set of rules in Figure 1 is infinite: what we write as ($\triangleright$) is an infinite set of axioms. It is natural to ask whether we can obtain a finite axiomatization.

Theorem 4. *There is no finite axiomatization of our fragment of syllogistic logic.*

Proof. We sketch the argument; see [5,6] for proofs of analogous results for other logics; these have some similarity with what we do, and some differences. Let $n > 0$ be arbitrary, and let Γ be given diagrammatically by:



Here we leave a few arrows implicit; every arrow is also a bidirectional some arrow, the graph is reflexive for all and some arrows, and we have all some arrows $A_i \leftrightarrow C_j$ and $B_i \leftrightarrow C_j$.

The diagram is an instance of ($\triangleright$), and so $\Gamma \vdash$ Some A are B. However, if we drop any of the arrows $A_i \hookrightarrow C_{2i-1}$, then Γ is closed under the rules in Figure 1, and hence under non-trivial consequences in the logic by Theorem 1 (completeness). Thus to conclude Some A are B we need a rule that includes all n assumptions $A_i \hookrightarrow C_{2i-1}$. As n was arbitrary, it follows that every complete axiomatization has to be infinite. □

6 Conclusion and Future Work

This paper has presented a sound and complete axiomatization of the logical system whose sentences are of the form All X are Y, Some X are Y, and

Most X are Y. The semantics is the natural one, restricting attention to finite models and using strict majority in the semantics of Most X are Y. We provided a sound and complete proof system.

We have shown that the complexity of the logic is low, and we also have a proof search algorithm. The details on this are suppressed in this publication for lack of space. But the algorithm follows the completeness proof fairly closely.

The next steps would be to add more features to the logic. One would like to add No X are Y, and also sentences like There are at least as many X as Y. In addition, one could hope to add boolean connectives over sentences. A related result appears in Lai et al. [2]; the logic there is propositional logic on top of the sentences Most X are Y (but not containing Some or All sentences). The completeness argument there is rather different, and it is open to merge the approach there with what is done here.

Overall, we would like to explore stronger logics, focusing on the borderline between decidable and undecidable logics, on complexity results and algorithms. This area should be of interest in finite model theory and in branches of logic close to combinatorics.

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