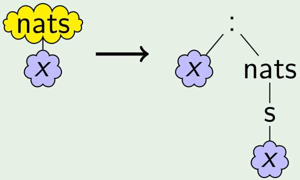
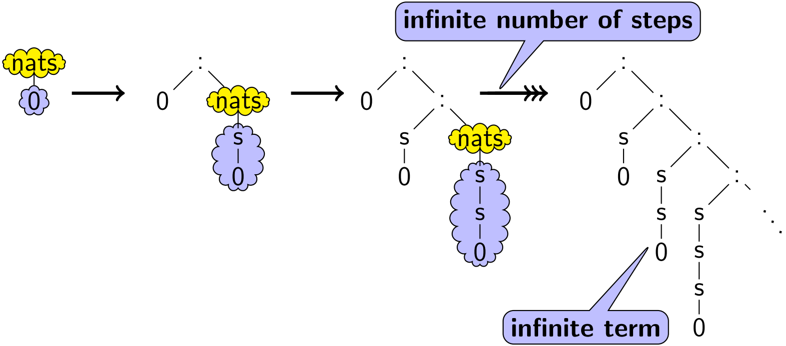
- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

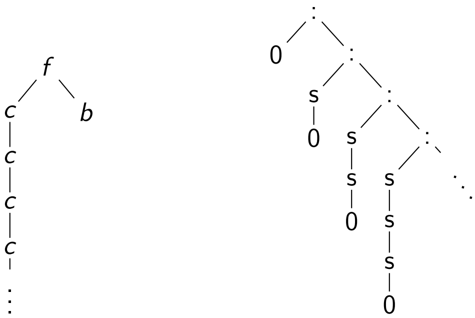
Infinitary Rewriting

Example (The Stream of Natural Numbers)





Infinite Terms



Definition

An infinite term is a partial map $t : \mathbb{N}^* \rightharpoonup \Sigma$ from positions to symbols such that:

- $t(\epsilon) \in \Sigma$, and
- $t(ip) \in \Sigma \iff 1 \leq i \leq \#(t(p))$

The set of finite and infinite terms is denoted by $\mathcal{T}^\infty(\Sigma, \mathcal{X})$.

Infinite Terms as Metric Space

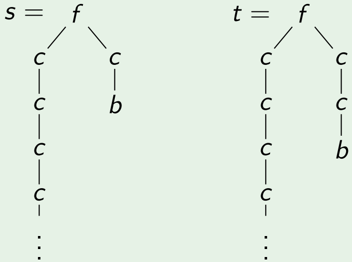
Definition

We define a metric d on $\mathcal{T}^\infty(\Sigma, \mathcal{X})$ by:

$$d(s,t) = 2^{-|p|} \text{ where } p \text{ is the highest position such that } s(p) \neq t(p)$$

Note that $d(s,t) = 0 \iff s = t$.

Example



The first difference is at depth 2, hence $d(s,t) = 2^{-2} = 0.25$.

Infinitary Rewriting

Example

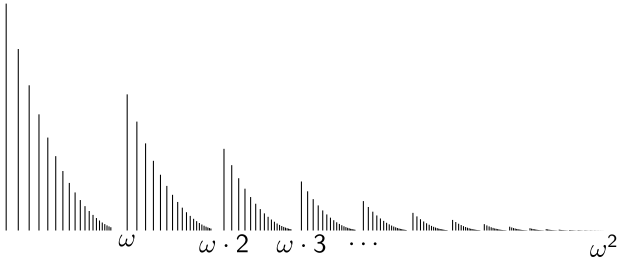
$$\begin{aligned} f(x,x) &\to f(a,b) \\ a &\to c(a) \\ b &\to c(b) \end{aligned}$$

$$\begin{aligned} f(a,b) &\to f(c(a),b) \\ &\to f(c(c(a)),b) \\ &\twoheadrightarrow\!\!\!\!\rightarrow f(c(c(c(\ldots))),b) \\ &\to f(c(c(c(\ldots))),c(b)) \\ &\to f(c(c(c(\ldots))),c(c(b))) \\ &\twoheadrightarrow\!\!\!\!\rightarrow f(c(c(c(\ldots))),c(c(c(\ldots)))) \\ &\to f(a,b) \\ &\to \ldots \end{aligned}$$

We need transfinite reductions...

Ordinals

$0,\ 1,\ 2,\ \ldots,\ \omega,\ \omega+1,\ \omega+2,\ldots,\omega+\omega=\omega\cdot 2,\ \ldots,\ \omega\cdot 3,\ldots,\omega^2,\ldots,\omega^\omega,\ldots$



Note that ω is the smallest infinite ordinal.

Ordinals

Definition

A set S is transitive if $x \in S$ implies $x \subseteq S$.

An ordinal is a transitive set whose elements are transitive sets.

Example $(0, 1, 2, 3, \ldots)$

$\varnothing, \ \{\varnothing\}, \ \{\varnothing, \{\varnothing\}\}, \ \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ \ldots$

Definition

We define $\alpha < \beta \iff \alpha \in \beta$.

Lemma

The relation $<$ is a total order on ordinals.

Lemma

For every ordinal β, we have $\beta = \{\alpha \mid \alpha < \beta\}$.

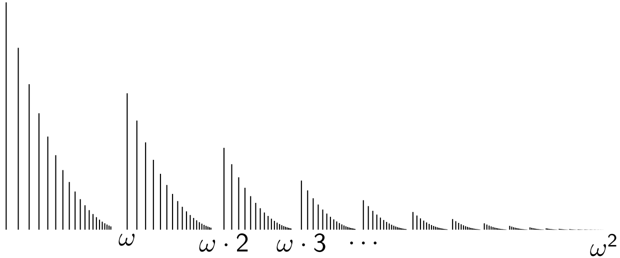
Ordinals

Definition

For ordinals α, we define $\alpha^+ = \alpha \cup \{\alpha\}$, the successor of α.

An ordinal α is a successor ordinal if $\alpha = \beta^+$ for some ordinal β.

If $\alpha \neq 0$ and α is not a successor ordinal, then α is called limit ordinal.



Example

Successor ordinals: $1, 2, \omega + 1, \omega \cdot 3 + 2, \ldots$ Limit ordinals: $\omega, \omega \cdot 2, \omega \cdot 3, \omega^2, \ldots$

Transfinite Reductions

Example

$$f(x,x) \to f(a,b)$$
$$a \to c(a)$$
$$b \to c(b)$$

A reduction of length $\omega \cdot 2 + 1$:

$$f(a,b) \to^{\omega} f(c^{\omega}, b) \to^{\omega} f(c^{\omega}, c^{\omega}) \to f(a,b)$$

A reduction of length $\omega + 1$:

$$f(a,b) \to^{\omega} f(c^{\omega}, c^{\omega}) \to f(a,b)$$

by alternating $f(a,b) \to f(c(a),b) \to f(c(a),c(b)) \to \ldots$.

Transfinite Reductions

Definition

Let α be an ordinal, and $\tau : (t_\beta \to t_{\beta+1})_{\beta<\alpha}$ a sequence of reduction steps.

We use d_β to denote the depth of the rewrite step $t_\beta \to t_{\beta+1}$.

Then τ is an infinite reduction of length α if for every limit ordinal $\lambda < \alpha$:

1. the distance $d(t_\beta, t_\lambda)$ tends to 0, $\forall \epsilon > 0.\ \exists \beta < \lambda.\ \forall \beta < \gamma < \lambda.\ d(t_\gamma, t_\lambda) \le \epsilon$
2. the depth d_β tends to infinity $\forall n.\ \exists \beta < \lambda.\ \forall \beta < \gamma < \lambda.\ d_\gamma \ge n$

as β approaches λ from below.

Example

Let $R = \{a \to a,\ b \to b\}$. Condition (1) excludes jumps in the limit:

$$a \to a \to a \to \ldots \underbrace{b}_{t_\omega} \to b \to \ldots$$

Transfinite Reductions

Definition

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as β approaches λ from below.

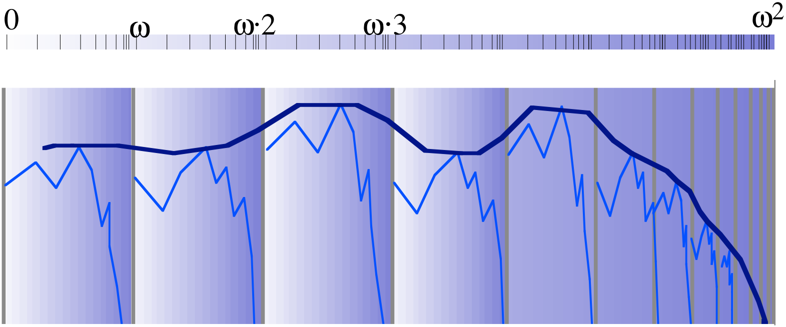
Example (We want more than Cauchy-convergence...)

Let $R = \{f(x) \to f(c(x))\}$. Condition (2) excludes sequences like:

$$f(a) \to_\epsilon f(c(a)) \to_\epsilon f(c(c(a))) \to_\epsilon \ldots \to_\epsilon^\omega f(c^\omega) \to \ldots$$

where the activity does not move downwards.

Transfinite Reductions Visualized



The blue lines indicate the depth of the activity/rewrite steps.

The activity tends to infinity when approaching limit ordinals.

Why more than Cauchy-convergence?

We consider the TRS:

$$f(x,y) \to f(y,x)$$
$$a \to b$$

We start from $f(a,a)$ and trace the left occurrence of a:

$$f(\overline{a},a) \to f(a,\overline{a}) \to f(\overline{a},a) \to f(a,\overline{a}) \to^{\omega} \textcolor{red}{?}$$

The rewrite sequence without overlining is Cauchy-convergent.

However, what are the residuals of the left a after ω-many steps?

Although it appears as if the term has a limit, this is only a syntactic accident. The subterms get swapped all the time...

Definition

A reduction of length α is strongly convergent if for every limit ordinal $\lambda \leq \alpha$ the depth d_β tends to infinity as β approaches λ from below, and divergent, otherwise.

Example

1. $R = \{\ a \to b,\ b \to a\ \}$

   $$a \to b \to a \to b \to \ldots$$

   …is a divergent rewrite sequence of length ω.

2. $R = \{\ f(x,x) \to f(a,b),\ a \to c(a),\ b \to c(b\ \}$

   $$f(a,b) \to^{\omega} f(c^{\omega}, b) \to^{\omega} f(c^{\omega}, c^{\omega}) \to f(a,b)$$

   …is a strongly convergent rewrite sequence of length $\omega \cdot 2 + 1$.

Lemma

A reduction τ is strongly convergent

$\iff$ *for every $n \in \mathbb{N}$ there are only finitely many steps at depth n in τ.*

Definition

We write $s \twoheadrightarrow\!\!\!\!\!\!\rightarrow t$ if the rewrite sequence is strongly convergent and with limit t.

Example

$R = \{\ a \to c(a)\ \}$. Then $a \twoheadrightarrow\!\!\!\!\!\!\rightarrow c^\omega$.

Lemma

Every proper prefix of a (even divergent) rewrite sequence is strongly convergent.

Example

$R = \{\ f(x,x) \to f(a,b),\ a \to c(a),\ b \to c(b\ \}$

$$f(a,b) \to^{\omega \cdot 2+1} f(a,b) \to^{\omega \cdot 2+1} f(a,b) \to^{\omega \cdot 2+1} \ldots$$

…is a divergent rewrite sequence of length ω^2. But every prefix is convergent!

Comparison finitary vs. infinitary rewriting

finitary rewriting	**infinitary rewriting**
finite reduction	strongly convergent reduction
infinite reduction	divergent reduction

Definition

Let $\mathcal{R}$ be a TRS and s a term. Then the term s is

- infinitary strongly normalizing (SN^∞) if s admits no divergent reductions,
- infinitary weakly normalizing (WN^∞) if s admits a reduction to normal form,
- infinitary confluent (CR^∞) if $\forall t_1 \twoheadleftarrow\!\!\!\!\leftarrow s \twoheadrightarrow\!\!\!\!\rightarrow t_2.\ t_1 \twoheadrightarrow\!\!\!\!\rightarrow \cdot \twoheadleftarrow\!\!\!\!\leftarrow t_2$.

Likewise $\mathcal{R}$ has the respective property if all terms from $\mathcal{T}^\infty(\Sigma, \mathcal{X})$ have.

Example

- Let $R = \{\ a \to c(a)\ \}$. Then R is WN^∞, SN^∞ and CR^∞.
- Let $R = \{\ a \to a,\ a \to c(a)\ \}$. Then R is WN^∞ and CR^∞, but not SN^∞.

Remark

- $SN^\infty \not\Rightarrow SN \vee WN$ — $a \to c(a)$
  Here, $a \twoheadrightarrow\!\!\!\!\!\to c^\omega$ which is a normal form.
- $SN \not\Rightarrow SN^\infty \vee WN^\infty$ — $I(x) \to x$
  Here, $I(I(I(\dots)))$ rewrites only to itself.
- $CR^\infty \not\Rightarrow CR$ — $a \to b,\ a \to c,\ b \to d(b),\ c \to d(c)$
  Here, $\neg(b \downarrow c)$, but $b \twoheadrightarrow\!\!\!\!\!\to d^\omega \leftarrow\!\!\!\!\!\twoheadleftarrow c$.
- $CR \not\Rightarrow CR^\infty$ — $A(x) \to x,\ B(x) \to x$
  Here, $A^\omega \leftarrow\!\!\!\!\!\twoheadleftarrow (AB)^\omega \twoheadrightarrow\!\!\!\!\!\to B^\omega$.

Remark

The example $A(x) \to x,\ B(x) \to x$ shows: not every orthogonal TRSs is CR^∞.

Even one collapsing rule is sufficient to violate CR^∞.
Take $R = \{\ f(x,y) \to y\ \}$. Then

$$f(x, f(x, f(x, \dots))) \leftarrow\!\!\!\!\!\twoheadleftarrow f(x, f(y, f(x, f(y, \dots)))) \twoheadrightarrow\!\!\!\!\!\to f(y, f(y, f(y, \dots)))$$

Remark (The failure of Newmann's Lemma for infinitary rewriting)

$$WCR \wedge SN^{\infty} \not\Rightarrow CR^{\infty}$$

For example:

$$\begin{aligned} R = \{\ & a \to b(a), \\ & a \to c(a), \\ & c(b(x)) \to b(b(x))\ \} \end{aligned}$$

is WCR and SN^{∞}, but not CR^{∞}.

Results for (Weakly) Orthogonal TRSs

Theorem

Every weakly orthogonal TRS without collapsing rules is CR^∞.

Definition

A TRS $\mathcal{R}$ is UN^∞ if $s \twoheadleftarrow\!\!\!\!\leftarrow \cdot \twoheadrightarrow\!\!\!\!\rightarrow t \Rightarrow s = t$ for all normal forms $s, t \in \mathcal{T}^\infty(\Sigma, \mathcal{X})$.

Theorem

Every orthogonal TRS is UN^∞.

Example

Weakly orthogonal TRSs are not necessarily UN^∞:

$$S(P(x)) \to x \qquad\qquad P(S(x)) \to x$$

Then

$$S^\omega \twoheadleftarrow\!\!\!\!\leftarrow S^1(P^2(S^3(P^4(\ldots)))) \twoheadrightarrow\!\!\!\!\rightarrow P^\omega$$

Compression and Parallel Moves

Theorem (Compression)

Let $\mathcal{R}$ be an left-linear TRS. Then $s \twoheadrightarrow t$ implies $s \to^{\leq\omega} t$.

That is, every strongly convergent reduction can be compressed to length $\leq \omega$.

Theorem (Parallel Moves)

Let $\mathcal{R}$ be an orthogonal TRS. Then $t_1 \mathrel{\twoheadleftarrow\!\!\!\!\!\!\!\shortparallel} s \twoheadrightarrow t_2 \;\Rightarrow\; t_1 \twoheadrightarrow \cdot \mathrel{\twoheadleftarrow\!\!\!\!\!\!\!\shortparallel} t_2$.