- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

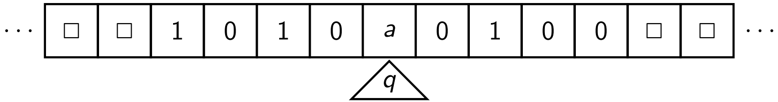
Decidability

Definition

A Turing machine is a tuple $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ where:

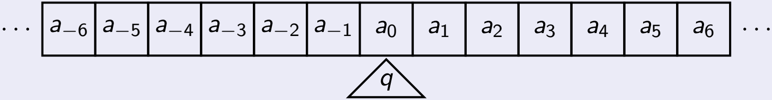
- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $F \subseteq Q$ is the set of final states,
- Σ is a finite alphabet,
- $\square \in \Sigma$ is the blank symbol,
- $\delta : (Q \setminus F) \times \Sigma \to Q \times \Sigma \times \{L, R\}$ is the transition function.

Turing machines work on a two-sided infinite tape with one read/write head:

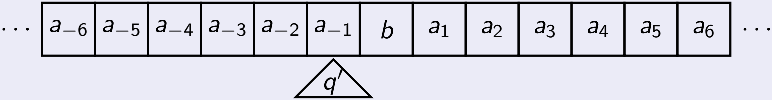


Definition (State Transition)

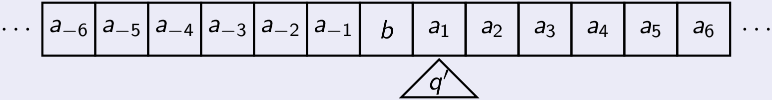
The current state:



If $\delta(q, a_0) = (b, q', L)$, then the following state is:

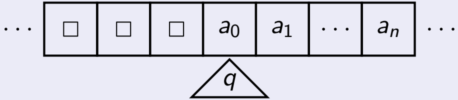


If $\delta(q, a_0) = (b, q', R)$, then the following state is:



Definition

A Turing machine $\langle Q, q_0, F, \Sigma, \Box, \delta \rangle$ is said to halt on $w = a_1 a_2 \dots a_n \in \Sigma^*$ if it reaches a final state when started on the configuration:



The tape content upon reaching the final state is called output.

Definition (Decidable Problems)

A predicate $P \subseteq \mathbb{N}^d$ is decidable if there is a Turing machine M such that:

- M terminates on all inputs $s^{n_1}0 \cdots s^{n_d}0$ for $\langle n_1, \dots, n_d \rangle \in \mathbb{N}^d$, and
- M halts with output 0 if and only if $\langle n_1, \dots, n_d \rangle \in A$.

Problem (Halting Problem)

Input: *Turing machine* M*, word* $w \in \Sigma^*$
Question: *Does* M *halt on* w*?*

Problem (Empty-Tape Halting Problem)

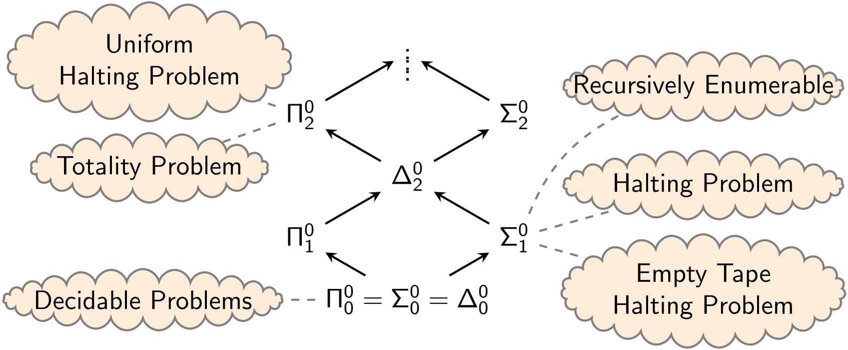
Input: *Turing machine* M
Question: *Does* M *halt on the empty tape* ϵ*?*

Problem (Uniform Halting Problem)

Input: *Turing machine* M
Question: *Does* M *halt on all configurations?*

Problem (Totality Problem)

Input: *Turing machine* M
Question: *Does* M *halt on all natural numbers* $\{s^n 0 \mid n \in \mathbb{N}\}$*?*

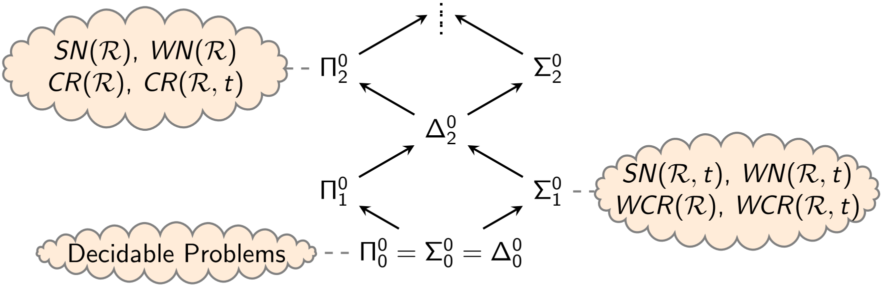


Definition

A problem $A \subseteq \mathbb{N}^d$ is:

- in Σ_1^0 if $\vec{x} \in A \iff \exists y.\ P(\vec{x}, y)$,
- in Π_2^0 if $\vec{x} \in A \iff \forall y.\ \exists z.\ P(\vec{x}, y, z)$,

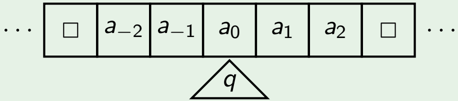
for some decidable predicate P.



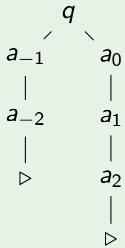
Theorem

- *$SN(\mathcal{R})$ and $WN(\mathcal{R})$ are Π^0_2-complete.*
- *$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ^0_1-complete.*
- *$WCR(\mathcal{R})$ and $WCR(\mathcal{R}, t)$ are Σ^0_1-complete.*
- *$CR(\mathcal{R})$ and $CR(\mathcal{R}, t)$ are Π^0_2-complete.*

Example (From Turing Machine Configurations to Terms)



becomes $q(a_{-1}(a_{-2}(\triangleright)), a_0(a_1(a_2(\triangleright))))$:



$$q(x, f(y)) \to q'(f'(x), y) \qquad \text{for every } \delta(q, f) = \langle q', f', R\rangle$$

$$q(g(x), f(y)) \to q'(x, g(f'(y))) \qquad \text{for every } \delta(q, f) = \langle q', f', L\rangle$$

Definition (From Turing Machines to Term Rewriting Systems)

For a Turing machine $M = \langle Q, q_0, F, \Sigma, \Box, \delta \rangle$ we define a TRS $\mathcal{R}_M$ as follows:

$$q(x, f(y)) \to q'(f'(x), y) \qquad \text{for every } \delta(q, f) = \langle q', f', R \rangle$$
$$q(g(x), f(y)) \to q'(x, g(f'(y))) \qquad \text{for every } \delta(q, f) = \langle q', f', L \rangle$$

together with four rules for 'extending the tape':

$$q(\triangleright, f(y)) \to q'(\triangleright, \Box(f'(y))) \qquad \text{for every } \delta(q, f) = \langle q', f', L \rangle$$
$$q(x, \triangleright) \to q'(f'(x), \triangleright) \qquad \text{for every } \delta(q, \Box) = \langle q', f', R \rangle$$
$$q(g(x), \triangleright) \to q'(x, g(f'(\triangleright))) \qquad \text{for every } \delta(q, \Box) = \langle q', f', L \rangle$$
$$q(\triangleright, \triangleright) \to q'(\triangleright, \Box(f'(\triangleright))) \qquad \text{for every } \delta(q, \Box) = \langle q', f', L \rangle$$

Observations on $\mathcal{R}_M$

Theorem

A Turing machine M halts on w if and only if $q_0(\triangleright, w(\triangleright))$ terminates in $\mathcal{R}_M$.

Theorem

A Turing machine M halts on all configurations if and only if $\mathcal{R}_M$ terminates.

Lemma

For ever Turing machine M, the TRS $\mathcal{R}_M$ is orthogonal.

Complexity of (Weak) Normalization

As a consequence we obtain:

Theorem

$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ^0_1-complete.

Proof.

The term $t = q_0(\triangleright, \triangleright)$ is normalizing in $\mathcal{R}_M \iff M$ halts on the blank tape.
Moreover $SN(t) \iff WN(t)$ since every reduct contains at most one redex. ■

Theorem

$SN(\mathcal{R})$ and $WN(\mathcal{R})$ are Π^0_2-complete.

Complexity of Local Confluence

Definition

For a Turing machine M we define H_M to be $\mathcal{R}_M$ extended with:

$$q(x, f(y)) \to T \qquad \text{for every } f \in \Sigma, q \in Q \text{ with } \delta(q, f) \text{ is undefined}$$

$$q(x, \triangleright) \to T \qquad \text{for every } q \in Q \text{ with } \delta(q, \Box) \text{ is undefined}$$

Theorem

$WCR(\mathcal{R})$ and $WCR(\mathcal{R}, t)$ are Σ^0_1-complete.

Proof.

We extend H_M with:

$$S \to T \qquad\qquad S \to q_0(\triangleright, \triangleright)$$

Then $\langle T, q_0(\triangleright, \triangleright)\rangle$ and $\langle q_0(\triangleright, \triangleright), T\rangle$ are the only critical pairs.

Hence $WCR(\mathcal{R})$ and $WCR(\mathcal{R}, S)$ hold

$\iff q_0(\triangleright, \triangleright) \to^* T$

$\iff M$ halts on the empty tape. ∎

Complexity of Confluence

Definition

For a Turing machine M we define H_M to be $\mathcal{R}_M$ extended with:

$$q(x, f(y)) \to T \qquad \text{for every } f \in \Sigma, q \in Q \text{ with } \delta(q, f) \text{ is undefined}$$
$$q(x, \triangleright) \to T \qquad \text{for every } q \in Q \text{ with } \delta(q, \Box) \text{ is undefined}$$

Theorem

$CR(\mathcal{R}, t)$ for single terms is Π^0_2-complete.

Proof.

We extend H_M with:

$$S(x) \to T \qquad S(x) \to S(s(x)) \qquad S(x) \to q_0(\triangleright, x)$$

Then $T \leftarrow S(0(\triangleright)) \to^* S(s^n(0(\triangleright))) \to q_0(\triangleright, s^n(0(\triangleright)))$ for all $n \in \mathbb{N}$.

Hence $CR(\mathcal{R}, S(0(\triangleright)))$ holds

$\iff q_0(\triangleright, s^n(0(\triangleright))) \to^* T$ for all $n \in \mathbb{N}$

$\iff M$ is total. ∎

Complexity of Confluence

Theorem

$CR(\mathcal{R})$ is Π^0_2-complete.

Proof.

We extend H_M with:

$$S(x, \triangleright) \to T$$
$$S(\triangleright, y) \to q_0(\triangleright, y)$$
$$S(x, s(y)) \to S(s(x), y)$$
$$S(s(x), y) \to S(x, s(y))$$

Then

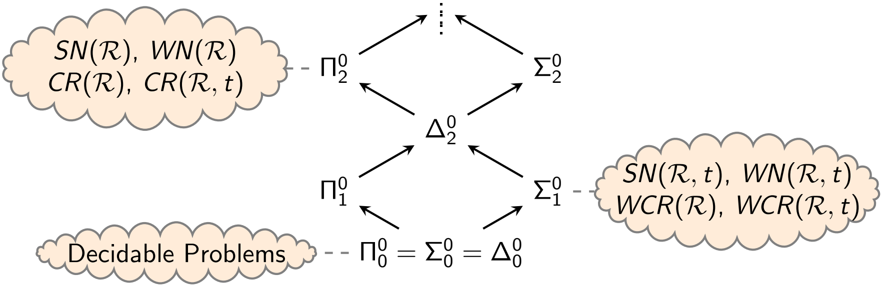
$$T \leftarrow S(u, \triangleright) \leftrightarrow^* S(\triangleright, v) \to q_0(\triangleright, v)$$

if and only if $u = v = s^n(\triangleright)$ for some $n \in \mathbb{N}$.

Hence $CR(\mathcal{R})$ holds

$\iff q_0(\triangleright, s^n(\triangleright)) \to^* T$ for all $n \in \mathbb{N}$

$\iff M$ is total.



Theorem

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