

- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: **Decidability**
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Decidability

Decidability

Definition

A **Turing machine** is a tuple $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ where:

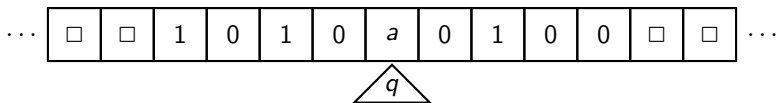
- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $F \subseteq Q$ is the set of final states,
- Σ is a finite alphabet,
- $\square \in \Sigma$ is the blank symbol,
- $\delta : (Q \setminus F) \times \Sigma \rightarrow Q \times \Sigma \times \{L, R\}$ is the **transition function**.

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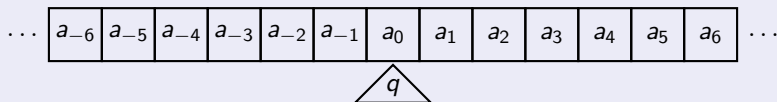
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Turing machines work on a two-sided infinite tape with one read/write head:

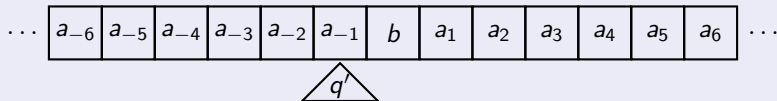


Definition (State Transition)

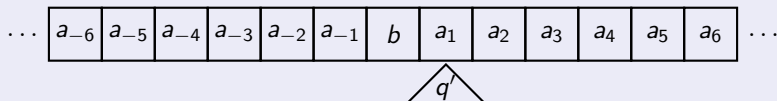
The current state:



If $\delta(q, a_0) = (b, q', L)$, then the following state is:

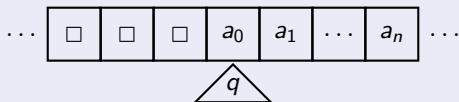


If $\delta(q, a_0) = (b, q', R)$, then the following state is:



Definition

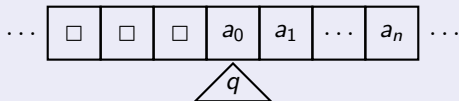
A Turing machine $\langle Q, q_0, F, \Sigma, \square, \delta \rangle$ is said to **halt on** $w = a_1 a_2 \dots a_n \in \Sigma^*$ if it reaches a final state when started on the configuration:



The tape content upon reaching the final state is called **output**.

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Definition (Decidable Problems)

A predicate $P \subseteq \mathbb{N}^d$ is **decidable** if there is a Turing machine M such that:

- M terminates on all inputs $s^{n_1} 0 \dots s^{n_d} 0$ for $\langle n_1, \dots, n_d \rangle \in \mathbb{N}^d$, and
- M halts with output 0 if and only if $\langle n_1, \dots, n_d \rangle \in A$.

Problem (Halting Problem)

Input: Turing machine M , word $w \in \Sigma^$*

Question: Does M halt on w ?

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Problem (Empty-Tape Halting Problem)

Input: Turing machine M

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Problem (Uniform Halting Problem)

Input: Turing machine M

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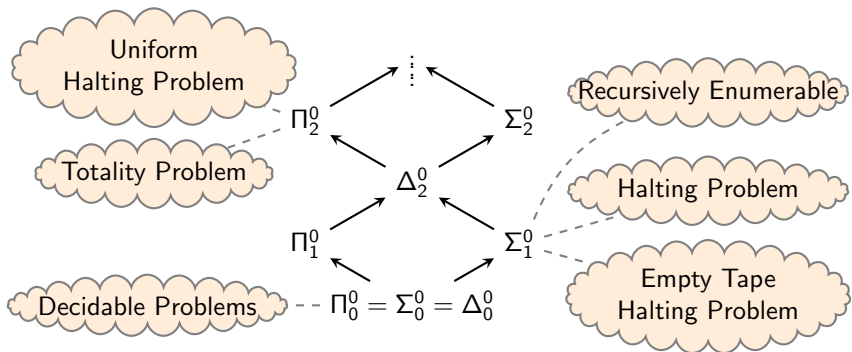
Input: Turing machine M

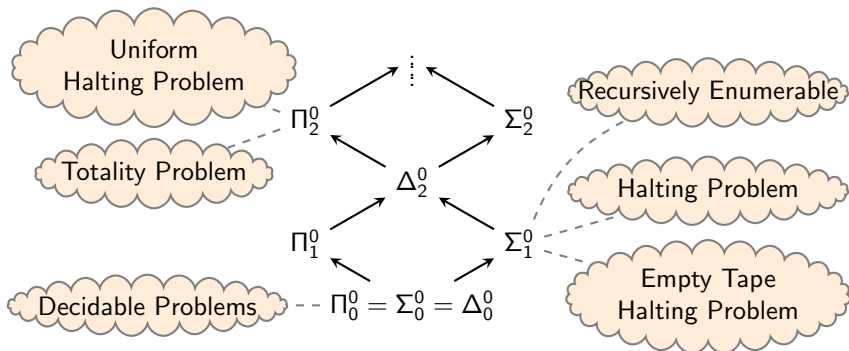
Question: Does M halt on all configurations?

Problem (Totality Problem)

Input: Turing machine M

Question: Does M halt on all natural numbers $\{s^n0 \mid n \in \mathbb{N}\}$?



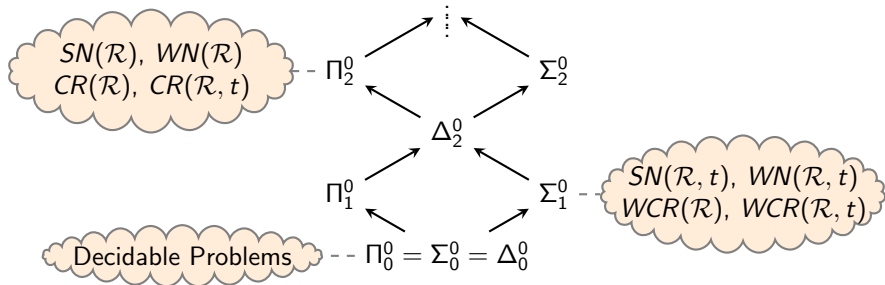


Definition

A problem $A \subseteq \mathbb{N}^d$ is:

- in Σ_1^0 if $\vec{x} \in A \iff \exists y. P(\vec{x}, y)$,
- in Π_2^0 if $\vec{x} \in A \iff \forall y. \exists z. P(\vec{x}, y, z)$,

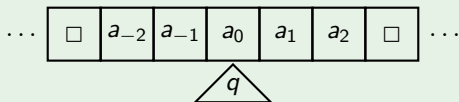
for some decidable predicate P .



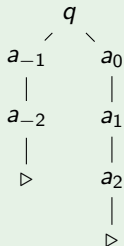
Theorem

- $SN(\mathcal{R})$ and $WN(\mathcal{R})$ are Π_2^0 -complete.
- $SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ_1^0 -complete.
- $WCR(\mathcal{R})$ and $WCR(\mathcal{R}, t)$ are Σ_1^0 -complete.
- $CR(\mathcal{R})$ and $CR(\mathcal{R}, t)$ are Π_2^0 -complete.

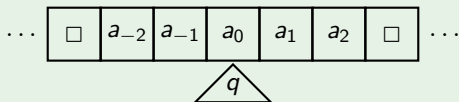
Example (From Turing Machine Configurations to Terms)



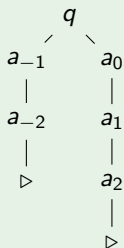
becomes $q(a_{-1}(a_{-2}(\triangleright)), a_0(a_1(a_2(\triangleright))))$:



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$$q(x, f(y)) \rightarrow q'(f'(x), y)$$

$$q(g(x), f(y)) \rightarrow q'(x, g(f'(y)))$$

for every $\delta(q, f) = \langle q', f', R \rangle$

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Definition (From Turing Machines to Term Rewriting Systems)

For a Turing machine $M = \langle Q, q_0, F, \Sigma, \square, \delta \rangle$ we define a TRS \mathcal{R}_M as follows:

$$\begin{array}{ll}
 q(x, f(y)) \rightarrow q'(f'(x), y) & \text{for every } \delta(q, f) = \langle q', f', R \rangle \\
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together with four rules for 'extending the tape':

$$\begin{array}{ll}
 q(\triangleright, f(y)) \rightarrow q'(\triangleright, \square(f'(y))) & \text{for every } \delta(q, f) = \langle q', f', L \rangle \\
 q(x, \triangleright) \rightarrow q'(f'(x), \triangleright) & \text{for every } \delta(q, \square) = \langle q', f', R \rangle \\
 q(g(x), \triangleright) \rightarrow q'(x, g(f'(\triangleright))) & \text{for every } \delta(q, \square) = \langle q', f', L \rangle \\
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Observations on \mathcal{R}_M

Theorem

A Turing machine M halts on w if and only if $q_0(\triangleright, w(\triangleright))$ terminates in \mathcal{R}_M .

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Lemma

For every Turing machine M , the TRS \mathcal{R}_M is orthogonal.

Complexity of (Weak) Normalization

As a consequence we obtain:

Theorem

$SN(\mathcal{R}, t)$ and $WN(\mathcal{R}, t)$ of single terms are Σ_1^0 -complete.

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Proof.

The term $t = q_0(\triangleright, \triangleright)$ is normalizing in $\mathcal{R}_M \iff M$ halts on the blank tape. Moreover $SN(t) \iff WN(t)$ since every reduct contains at most one redex. ■

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$SN(\mathcal{R})$ and $WN(\mathcal{R})$ are Π_2^0 -complete.

Complexity of Local Confluence

Definition

For a Turing machine M we define H_M to be \mathcal{R}_M extended with:

$$\begin{array}{ll}
 q(x, f(y)) \rightarrow T & \text{for every } f \in \Sigma, q \in Q \text{ with } \delta(q, f) \text{ is undefined} \\
 q(x, \triangleright) \rightarrow T & \text{for every } q \in Q \text{ with } \delta(q, \square) \text{ is undefined}
 \end{array}$$

Then $\langle T, q_0(\triangleright, \triangleright) \rangle$ and $\langle q_0(\triangleright, \triangleright), T \rangle$ are the only critical pairs.

Hence $WCR(\mathcal{R})$ and $WCR(\mathcal{R}, S)$ hold

$$\iff q_0(\triangleright, \triangleright) \rightarrow^* T$$

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We extend H_M with:

$$S \rightarrow T$$

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Then $T \leftarrow S(0(\triangleright)) \rightarrow^* S(s^n(0(\triangleright))) \rightarrow q_0(\triangleright, s^n(0(\triangleright)))$ for all $n \in \mathbb{N}$.

Hence $CR(\mathcal{R}, S(0(\triangleright)))$ holds

$$\iff q_0(\triangleright, s^n(0(\triangleright))) \rightarrow^* T \text{ for all } n \in \mathbb{N}$$

$$\iff M \text{ is total.}$$

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$$T \leftarrow S(u, \triangleright) \leftrightarrow^* S(\triangleright, v) \rightarrow q_0(\triangleright, v)$$

if and only if $u = v = s^n(\triangleright)$ for some $n \in \mathbb{N}$.

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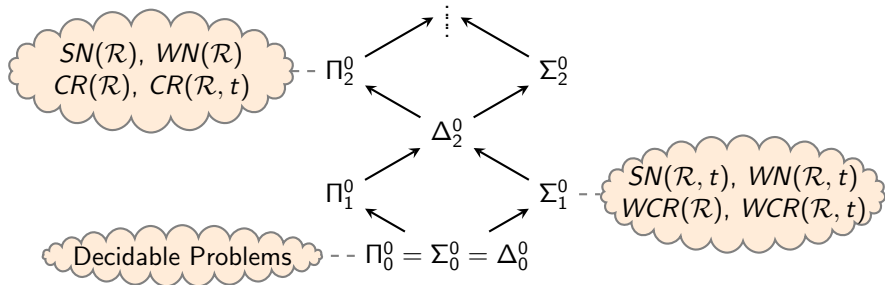
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