- Lecture 1: Introduction, Abstract Rewriting
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Outline

Strategies

Definition

A rewrite strategy $\mathcal{S}$ is mapping that assigns to every reducible term t a nonempty set of finite nonempty rewrite sequences starting from t.

- $\mathcal{S}$ is deterministic if $|\mathcal{S}(t)| = 1$ for every reducible term t
- $\mathcal{S}$ normalizes term t if there are no infinite $\mathcal{S}$ rewrite sequences starting from t
- $\mathcal{S}$ is normalizing if it normalizes every term that has a normal form
- $\mathcal{S}$ is perpetual if every maximal $\mathcal{S}$ rewrite sequence starting from any non-terminating term is infinite

Definition

A rewrite sequence is maximal if it is infinite, or it ends in a normal form.

Lemma

For terminating TRSs every strategy is normalizing and perpetual.

Definition

Let $\rho = t_0 \to t_1 \to \ldots$ be a finite or infinite rewrite sequence.

- Consider a redex occurrence s is some term t_n of ρ.

  Then s is secured if eventually there are no residuals of s left.

  That is, there exits $m > n$ such that t_m contains no residuals of s.

- The reduction ρ is fair if every redex occurring in ρ is secured.

Definition

A strategy $\mathcal{S}$ is fair if every every maximal $\mathcal{S}$ rewrite sequence is fair.

Definition

A one-step strategy maps every reducible term to a set of one-step reductions.

Example

There exists no fair one-step strategy for $\mathcal{R} = \{I(x) \to I(x)\}$.

For the term:

$$t = I(I(x))$$

there are only 3 possible mappings:

- $\mathcal{S}(t) = \{I(I(x)) \to_\varepsilon I(I(x))\}$,
- $\mathcal{S}(t) = \{I(I(x)) \to_1 I(I(x))\}$, or
- $\mathcal{S}(t) = \{I(I(x)) \to_\varepsilon I(I(x)),\ I(I(x)) \to_1 I(I(x))\}$.

None of these is fair as we can always continue to reduce the same occurrence of I.

Example

- rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

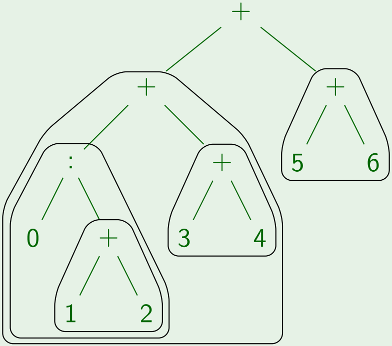
- term

$$((0:(1+2))+(3+4))+(5+6)$$

Example (cont'd)

term $\boxed{\boxed{0 : \boxed{1+2}} + \boxed{3+4}} + \boxed{5+6}$

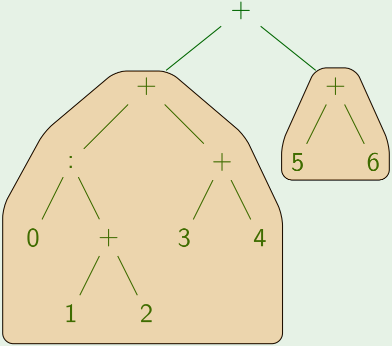
tree representation



Example (cont'd)

term $(0 : (1+2)) + (3+4) + (5+6)$

tree representation

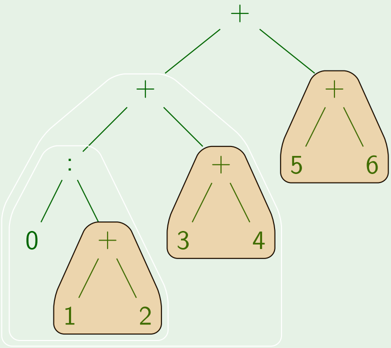


outermost redexes

Example (cont'd)

term $0 : 1 + 2 + 3 + 4 + 5 + 6$

tree representation

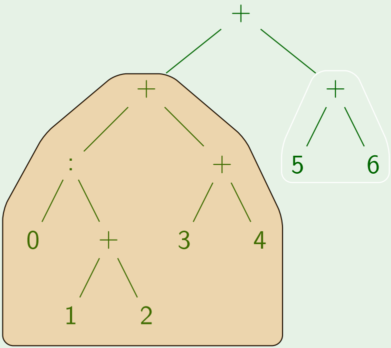


innermost redexes

Example (cont'd)

term $0 : 1+2 + 3+4 + 5+6$

tree representation



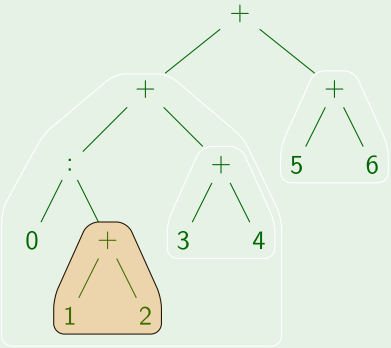
leftmost outermost strategy

Example (cont'd)

term

$$\boxed{\boxed{0 : \boxed{1 + 2}} + \boxed{3 + 4}} + \boxed{5 + 6}$$

tree representation

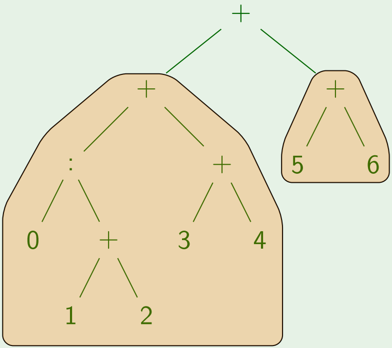


leftmost innermost strategy

Example (cont'd)

term $0 : 1 + 2 + 3 + 4 + 5 + 6$

tree representation

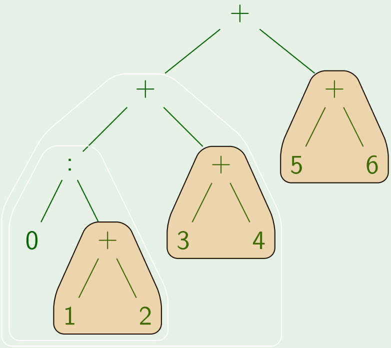


parallel outermost strategy

Example (cont'd)

term $0 : 1 + 2 + 3 + 4 + 5 + 6$

tree representation



parallel innermost strategy

Definition

Leftmost outermost strategy always reduces the leftmost of the outermost redexes.

Example (cont'd)

$$
\begin{aligned}
((0:(1+2))+(3+4))+(5+6) &\to (0:((1+2)+(3+4)))+(5+6)\\
&\to 0:(((1+2)+(3+4))+(5+6))\\
&\to ((1+2)+(3+4))+(5+6)\\
&\to (3+(3+4))+(5+6)\\
&\to (3+7)+(5+6)\\
&\to (1:0)+(5+6)\\
&\to 1:(0+(5+6))\\
&\to 1:(0+(1:1))\\
&\to 1:(1:(0+1))\\
&\to (1+1):(0+1)\\
&\to 2:(0+1)\\
&\to 2:1
\end{aligned}
$$

Definition

Leftmost innermost strategy always reduces the leftmost of the innermost redexes.

Example (cont'd)

$$
\begin{aligned}
(0 : (1+2)) + (3+4) + (5+6) &\to ((0:3) + (3+4)) + (5+6) \\
&\to (3 + (3+4)) + (5+6) \\
&\to (3+7) + (5+6) \\
&\to (1:0) + (5+6) \\
&\to (1:0) + (1:1) \\
&\to 1 : (0 + (1:1)) \\
&\to 1 : (1 : (0+1)) \\
&\to 1 : (1:1) \\
&\to (1+1) : 1 \\
&\to 2 : 1
\end{aligned}
$$

Definition

Parallel outermost strategy always reduces all outermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\mathrel{+\!\!\!\!\!\!\rightarrow} (0 : ((1+2) + (3+4))) + (1 : 1) \\
&\to 0 : (((1+2) + (3+4)) + (1 : 1)) \\
&\to ((1+2) + (3+4)) + (1 : 1) \\
&\to 1 : (((1+2) + (3+4)) + 1) \\
&\mathrel{+\!\!\!\!\!\!\rightarrow} 1 : ((3+7) + 1) \\
&\to 1 : ((1 : 0) + 1) \\
&\to 1 : (1 : (0+1)) \\
&\to (1+1) : (0+1) \\
&\mathrel{+\!\!\!\!\!\!\rightarrow} 2 : 1
\end{aligned}
$$

Definition

Parallel innermost strategy always reduces all innermost redexes in parallel.

Example (cont'd)

$$
\begin{aligned}
((0 : (1+2)) + (3+4)) + (5+6) &\;\;\#\!\!\!\rightarrow ((0:3)+7)+(1:1) \\
&\rightarrow (3+7)+(1:1) \\
&\rightarrow (1:0)+(1:1) \\
&\rightarrow 1:(0+(1:1)) \\
&\rightarrow 1:(1:(0+1)) \\
&\rightarrow 1:(1:1) \\
&\rightarrow (1+1):1 \\
&\rightarrow 2:1
\end{aligned}
$$

Definition

A development of set of redex positions Q in term t is a rewrite sequence starting from t in which all contracted redex positions descend from positions in Q.

Example

- rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- rewrite sequences

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0)$$ ☺
$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to \underline{(0 \times (0 \times 0))} + (0 \times 0) \to 0 + (0 \times 0)$$ ☹
$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0$$ ☺

Definition (Overlining)

For a TRS $\mathcal{R} = \langle \Sigma, R \rangle$ we define the overlined TRS $\overline{\mathcal{R}} = \langle \overline{\Sigma}, \overline{R} \rangle$:

- $\overline{\Sigma} = \Sigma \cup \{\overline{f} \mid f \in \Sigma\}$,
- $\overline{R} = \{\overline{\rho} \mid \rho \in R\}$

where $\overline{\rho}$ is obtained from ρ by overlining the head symbol of the left-hand side.

$$\rho : f(s_1, \ldots, s_n) \to r \quad \text{yields} \quad \overline{\rho} : \overline{f}(s_1, \ldots, s_n) \to r$$

Example

The overlined version of Combinatory Logic:

$$\begin{array}{lcl} \overline{Ap}(Ap(Ap(S, x), y), z) & \to & Ap(Ap(x, z), Ap(y, z)) \\ \overline{Ap}(Ap(K, x), y) & \to & x \\ \overline{Ap}(I, x) & \to & x \end{array}$$

We write $t \geq s$ if t can be obtained from s by overlining some redex positions.

Definition (Lifting)

A $\mathcal{R}$ rewrite sequence $A : s_1 \to s_2 \to \ldots \to s_n$ can be lifted if:

$$\begin{array}{ccccccc} t_1 & \xrightarrow{\langle \overline{\rho}_1,\ p_1 \rangle} & t_2 & \xrightarrow{\langle \overline{\rho}_2,\ p_2 \rangle} & \cdots & \xrightarrow{\langle \overline{\rho}_{n-1},\ p_{n-1} \rangle} & t_n \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & & & \text{\rotatebox{90}{$\leq$}} \\ s_1 & \xrightarrow{\langle \rho_1,\ p_1 \rangle} & s_2 & \xrightarrow{\langle \rho_2,\ p_2 \rangle} & \cdots & \xrightarrow{\langle \rho_{n-1},\ p_{n-1} \rangle} & s_n \end{array}$$

for some $\overline{\mathcal{R}}$ rewrite sequence $B : t_1 \to \ldots t_n$.

Lemma

For orthogonal TRSs: a reduction is a development $\iff$ *it can be lifted.*

Theorem

Properties of $\overline{\mathcal{R}}$ for orthogonal TRSs $\mathcal{R}$:

- *$\overline{\mathcal{R}}$ is orthogonal.*
- *$\overline{\mathcal{R}}$ is SN.*
- *$\overline{\mathcal{R}}$ is CR.*

Proof.

The orthogonality of $\overline{\mathcal{R}}$ is immediate. Hence $\overline{\mathcal{R}}$ is CR.

For SN we show that $\overline{\mathcal{R}}$ is ILPO terminating where $\overline{f} > g$ for every $f, g \in \Sigma$.
Let $\ell = \overline{f}(\ell_1, \ldots, \ell_n)$ and $r \in \mathcal{T}(\Sigma, \mathcal{X})$ with $\mathcal{V}\text{ar}(r) \subseteq \mathcal{V}\text{ar}\ell$.
Then $\ell^* \to_{ilpo}^* r$ by induction on r:

- If $r \in \mathcal{V}\text{ar}(\ell)$, then we use $\to_{put}$ and $\to_{select}$.
- If $r = g(r_1, \ldots, r_m)$, then we use $\ell^* \to_{copy} g(\ell^*, \ldots, \ell^*)$.
  Moreover by the induction hypothesis $\ell^* \to_{ilpo}^* r_i$ for every i. ∎

Theorem

Developments are finite.

Definition

A development $A \colon s \to^* t$ of $Q \subseteq \mathcal{P}\text{os}(s)$ is complete if $Q/A = \varnothing$.

We write $s \nrightarrow\!\!\!\!\rightarrow t$ (called multi-step) if there is a complete development $s \to^* t$.

Example

$$\underline{\mathsf{s}(0) \times (0 \times 0)} \to (0 \times \underline{(0 \times 0)}) + (0 \times 0) \to (0 \times 0) + (0 \times 0) \qquad ☹$$

$$\mathsf{s}(0) \times \underline{(0 \times 0)} \to \underline{\mathsf{s}(0) \times 0} \to (0 \times 0) + 0 \qquad ☺$$

Theorem

All complete developments of Q are permutation equivalent.

Definition

For orthogonal TRSs the full substitution strategy performs complete development of all redexes.

Example

- rewrite rules

$$0 + y \to y \qquad 0 \times y \to 0$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

- full substitution strategy

$$\begin{aligned}
&\mathsf{s}(\mathsf{s}(0)) \times (\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))) \\
&\quad \twoheadrightarrow (\mathsf{s}(0) \times \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \\
&\quad \twoheadrightarrow ((0 \times \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\quad \to (0 + \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\quad \to \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \\
&\quad \to \ldots \to \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))))
\end{aligned}$$

Outline

- Overview
- Strategies
  - Definitions
  - Results

Theorem

For orthogonal TRSs

- *full substitution* and *parallel outermost* strategies are *normalizing*
- *innermost* strategies are *perpetual*
- *leftmost outermost* strategy is *not normalizing*
- *full substitution* is *fair*

Example

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{c} \qquad \mathsf{f}(x, \mathsf{b}) \to \mathsf{b}$$

- leftmost outermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- leftmost innermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \mathsf{f}(\mathsf{c}, \mathsf{a}) \to \cdots$
- parallel outermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{+\!\!\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \mathsf{b}$
- parallel innermost $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{+\!\!\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \mathsf{f}(\mathsf{c}, \mathsf{b}) \to \cdots$
- full substitution $\quad \mathsf{f}(\mathsf{c}, \mathsf{a}) \mathrel{\circ\!\!\!\!\!\to} \mathsf{f}(\mathsf{c}, \mathsf{b}) \mathrel{\circ\!\!\!\!\!\to} \mathsf{b}$

Definition

A reduction $\rho = t_0 \to t_1 \to \ldots$ is cofinal if for every $t_0 \to^* s$ there exists t_n in ρ such that $s \to^* t_n$.

Definition

A strategy $\mathcal{S}$ is cofinal if every every maximal $\mathcal{S}$ rewrite sequence is cofinal.

Theorem

Cofinal strategies are normalizing.

Proof.

Let $\mathcal{S}$ be a cofinal strategy. Let t_0 be a term that has a normal form u ($t_0 \to^* u$).

Consider a maximal $\mathcal{S}$ rewrite sequence $\rho : t_0 \to t_1 \to \ldots$ starting from t_0.

By cofinality there must be t_n in ρ such that $u \to^* t_n$.

Hence $t_n = u$ since u is a normal form. ∎

Theorem

For orthogonal TRSs, every fair strategy is cofinal.

Proof.

Let $\rho : t_0 \to t_1 \to \ldots$ be a fair rewrite sequence.

Let $\tau : t_0 \to u_0$. We show that $u_0 \to^* t_n$ for some t_n in ρ.

$$\begin{array}{ccccccccccc}
t_0 & \longrightarrow & t_1 & \longrightarrow & t_2 & \longrightarrow & \cdots & \longrightarrow & t_n & \longrightarrow & \cdots \\
\downarrow \tau & & \downarrow\!\!\downarrow \tau/\rho_1 & & \downarrow\!\!\downarrow \tau/\rho_2 & & & & \mid \varnothing & & \\
u_0 & \twoheadrightarrow & u_1 & \twoheadrightarrow & u_2 & \twoheadrightarrow & \cdots & \twoheadrightarrow & u_n & &
\end{array}$$

Here ρ_i consists of the first i steps of ρ.

By fairness of ρ there exists n such that $\tau/\rho_n = \varnothing$. Hence $u_n = t_n$ and $u_0 \to^* t_n$.

The reduction $u_0 \to u_1 \to \ldots \to u_n \to t_{n+1} \to t_{n+2}$ is fair again.
(every redex occurrence in t_n is eventually secured)

By induction over the length of $t_0 \to^* u_0$ we get $u_0 \to^* t_n$ for some t_n in ρ. ∎

Definition

A TRS is left-normal if variables do not precede function symbols in left-hand sides (where the left-hand sides are written in prefix notation).

Example

- $\mathsf{f}(x,\ \mathsf{g}(y,z)) \to \mathsf{g}(y,\ \mathsf{f}(x,z))$ ☹
- $\mathsf{f}(\mathsf{g}(x,y),\ z) \to \mathsf{g}(x,\ \mathsf{g}(y,z))$ ☺

Theorem

Leftmost outermost strategy is normalizing for orthogonal left-normal TRSs.

Remark

Combinatory Logic is left-normal

$$\mathsf{I}\,x \to x \qquad \mathsf{K}\,x\,y \to x \qquad \mathsf{S}\,x\,y\,z \to x\,z\,(y\,z)$$