- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Modularity

Modularity

Definition

A property of TRSs is modular if it is preserved under union.

Remark

Without further restrictions 'no' property of TRSs is modular:

termination	$a \to b$	$b \to a$
confluence	$a \to b$	$a \to c$

Definition

A property P is preserved under signature extension if:

$$(\Sigma, R) \vDash P \quad \Longrightarrow \quad (\Sigma \cup \mathcal{G}, R) \vDash P$$

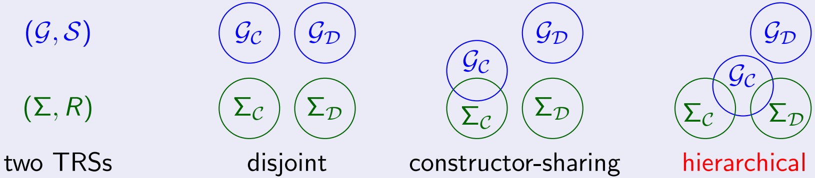
for all TRSs (Σ, R) and signatures $\mathcal{G}$ with $\Sigma \cap \mathcal{G} \neq \varnothing$.

Definition

TRS R over signature Σ

- defined symbols $\quad \Sigma_{\mathcal{D}} = \{\, \mathrm{root}(\ell) \mid \ell \to r \in R \,\}$
- constructors $\quad \Sigma_{\mathcal{C}} = \Sigma \setminus \Sigma_{\mathcal{D}}$

More Interesting Combinations



Example

①	$0 + y \to y$ $s(x) + y \to s(x + y)$	$0 \times y \to 0$ $s(x) \times y \to x \times y + y$	②
③	$0 - y \to 0$ $x - 0 \to x$ $s(x) - s(y) \to x - y$	$fib(0) \to s(0)$ $fib(s(0)) \to s(0)$ $fib(s(s(x))) \to fib(s(x)) + fib(x)$	④
⑤	$nil +\!\!+\ x \to x$ $(x : y) +\!\!+\ z \to x : (y +\!\!+\ z)$	$0 \div s(y) \to 0$ $s(x) \div s(y) \to s((x - y) \div s(y))$	⑥
⑦	$true \wedge false \to false$ $false \wedge true \to false$ $x \wedge x \to x$	$x < 0 \to false$ $0 < s(y) \to true$ $s(x) < s(y) \to x < y$	⑧
⑨	$sum(nil) \to 0$ $sum(x : y) \to x + sum(y)$	$length(nil) \to 0$ $length(x : y) \to s(length(y))$	⑩

① $\overset{}{+}$ ② + ③ + ④ + ⑤ + ⑥ + ⑦ + ⑧ + ⑨ + ⑩

①+②	②+③	③+④	④+⑤	⑤+⑥	⑥+⑦	⑦+⑧	⑧+⑨	⑨+⑩
h	cs	h	d	h	d	cs	h	cs

Outline

- Overview
- Modularity
  - Definitions
  - Results

Theorem (Toyama's Theorem)

Confluence is modular for disjoint TRSs.

Remark

Confluence is not modular for constructor-sharing TRSs.

Example

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{g}(x)) &\to \mathsf{b} \end{aligned} \qquad\qquad \mathsf{c} \to \mathsf{g}(\mathsf{c})$$

$$\mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$

Theorem

Termination is not modular for disjoint TRSs.

Example

$$f(a, b, x) \to f(x, x, x)$$

$$g(x, y) \to x$$
$$g(x, y) \to y$$

duplicating collapsing

$$\begin{aligned} f(a, b, g(a, b)) &\to f(g(a, b), g(a, b), g(a, b)) \\ &\to f(a, g(a, b), g(a, b)) \\ &\to f(a, b, g(a, b)) \end{aligned}$$

Theorem

The disjoint union of terminating TRSs R and $\mathcal{S}$ is terminating if:

- *R and $\mathcal{S}$ lack collapsing rules, or*
- *R and $\mathcal{S}$ lack duplicating rules, or*
- *R or $\mathcal{S}$ lacks both collapsing and duplicating rules.*

Corollary

Termination is preserved under signature extension.

Theorem

Termination is not modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x)$$

$$g(x, y) \rightarrow x$$
$$g(x, y) \rightarrow y$$

duplicating

collapsing & not confluent

$$\begin{aligned} f(a, b, g(a, b)) &\rightarrow f(g(a, b), g(a, b), g(a, b)) \\ &\rightarrow f(a, g(a, b), g(a, b)) \\ &\rightarrow f(a, b, g(a, b)) \end{aligned}$$

Theorem

Termination is not modular for disjoint confluent TRSs.

Example

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c} \qquad\qquad \mathsf{g}(x,y,y) \to x$$

$$\mathsf{f}(x,y,z) \to \mathsf{c} \qquad\qquad \mathsf{b} \to \mathsf{c} \qquad\qquad \mathsf{g}(y,y,x) \to x$$

no constructor system — not left-linear

$$\begin{aligned}
\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b})) &\to \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{b},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c},\mathsf{c},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))\\
&\to \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b},\mathsf{b}))
\end{aligned}$$

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS R over signature Σ is constructor system (CS) if $l_1, \ldots, l_n \in \mathcal{T}(\Sigma_{\mathcal{C}}, \mathcal{X})$ for every left-hand side $f(l_1, \ldots, l_n)$ of rewrite rules in R.

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*
- *semi-completeness is modular for constructor-sharing TRSs*