

- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: **Modularity**
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Modularity

Modularity

Definition

A property of TRSs is **modular** if it is preserved under union.

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Remark

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<i>termination</i>	$a \rightarrow b$	$b \rightarrow a$
<i>confluence</i>	$a \rightarrow b$	$a \rightarrow c$

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Without further restrictions 'no' property of TRSs is modular:

<i>termination</i>	$a \rightarrow b$	$b \rightarrow a$
<i>confluence</i>	$a \rightarrow b$	$a \rightarrow c$

Definition

A property P is **preserved under signature extension** if:

$$(\Sigma, R) \models P \implies (\Sigma \cup \mathcal{G}, R) \models P$$

for all TRSs (Σ, R) and signatures \mathcal{G} with $\Sigma \cap \mathcal{G} \neq \emptyset$.

Definition

TRS R over signature Σ

- defined symbols $\Sigma_{\mathcal{D}} = \{ \text{root}(l) \mid l \rightarrow r \in R \}$

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More Interesting Combinations

$(\mathcal{G}, \mathcal{S})$



(Σ, R)



two TRSs

disjoint

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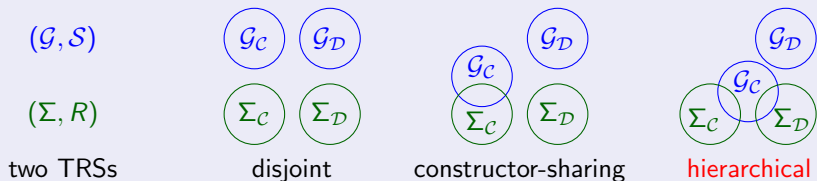
constructor-sharing

Definition

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- constructors $\Sigma_{\mathcal{C}} = \Sigma \setminus \Sigma_{\mathcal{D}}$

More Interesting Combinations



Example

$$0 + y \rightarrow y$$

$$s(x) + y \rightarrow s(x + y)$$

$$0 - y \rightarrow 0$$

$$x - 0 \rightarrow x$$

$$s(x) - s(y) \rightarrow x - y$$

$$\text{nil} ++ x \rightarrow x$$

$$(x : y) ++ z \rightarrow x : (y ++ z)$$

$$\text{true} \wedge \text{false} \rightarrow \text{false}$$

$$\text{false} \wedge \text{true} \rightarrow \text{false}$$

$$x \wedge x \rightarrow x$$

$$\text{sum}(\text{nil}) \rightarrow 0$$

$$\text{sum}(x : y) \rightarrow x + \text{sum}(y)$$

$$0 \times y \rightarrow 0$$

$$s(x) \times y \rightarrow x \times y + y$$

$$\text{fib}(0) \rightarrow s(0)$$

$$\text{fib}(s(0)) \rightarrow s(0)$$

$$\text{fib}(s(s(x))) \rightarrow \text{fib}(s(x)) + \text{fib}(x)$$

$$0 \div s(y) \rightarrow 0$$

$$s(x) \div s(y) \rightarrow s((x - y) \div s(y))$$

$$x < 0 \rightarrow \text{false}$$

$$0 < s(y) \rightarrow \text{true}$$

$$s(x) < s(y) \rightarrow x < y$$

$$\text{length}(\text{nil}) \rightarrow 0$$

$$\text{length}(x : y) \rightarrow s(\text{length}(y))$$

Example

①	$0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$	$0 \times y \rightarrow 0$ $s(x) \times y \rightarrow x \times y + y$	②
③	$0 - y \rightarrow 0$ $x - 0 \rightarrow x$ $s(x) - s(y) \rightarrow x - y$	$\text{fib}(0) \rightarrow s(0)$ $\text{fib}(s(0)) \rightarrow s(0)$ $\text{fib}(s(s(x))) \rightarrow \text{fib}(s(x)) + \text{fib}(x)$	④
⑤	$\text{nil} ++ x \rightarrow x$ $(x : y) ++ z \rightarrow x : (y ++ z)$	$0 \div s(y) \rightarrow 0$ $s(x) \div s(y) \rightarrow s((x - y) \div s(y))$	⑥
⑦	$\text{true} \wedge \text{false} \rightarrow \text{false}$ $\text{false} \wedge \text{true} \rightarrow \text{false}$ $x \wedge x \rightarrow x$	$x < 0 \rightarrow \text{false}$ $0 < s(y) \rightarrow \text{true}$ $s(x) < s(y) \rightarrow x < y$	⑧
⑨	$\text{sum}(\text{nil}) \rightarrow 0$ $\text{sum}(x : y) \rightarrow x + \text{sum}(y)$	$\text{length}(\text{nil}) \rightarrow 0$ $\text{length}(x : y) \rightarrow s(\text{length}(y))$	⑩

①

②

③

④

⑤

⑥

⑦

⑧

⑨

⑩

Example

①	$0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$	$0 \times y \rightarrow 0$ $s(x) \times y \rightarrow x \times y + y$	②
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① **+** ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩
h

Example

①	$0 + y \rightarrow y$ $s(x) + y \rightarrow s(x + y)$	$0 \times y \rightarrow 0$ $s(x) \times y \rightarrow x \times y + y$	②
③	$0 - y \rightarrow 0$ $x - 0 \rightarrow x$ $s(x) - s(y) \rightarrow x - y$	$\text{fib}(0) \rightarrow s(0)$ $\text{fib}(s(0)) \rightarrow s(0)$ $\text{fib}(s(s(x))) \rightarrow \text{fib}(s(x)) + \text{fib}(x)$	④
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① **+** ② **+** ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩
 h **cs**

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③	$0 - y \rightarrow 0$ $x - 0 \rightarrow x$ $s(x) - s(y) \rightarrow x - y$	$\text{fib}(0) \rightarrow s(0)$ $\text{fib}(s(0)) \rightarrow s(0)$ $\text{fib}(s(s(x))) \rightarrow \text{fib}(s(x)) + \text{fib}(x)$	④
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① \oplus h ② \oplus cs ③ \oplus h ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

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① \vdash ② \vdash ③ \vdash ④ \vdash ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

h cs h d

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① \oplus h ② \oplus cs ③ \oplus h ④ \oplus d ⑤ \oplus h ⑥ ⑦ ⑧ ⑨ ⑩

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⑨	$\text{sum}(\text{nil}) \rightarrow 0$ $\text{sum}(x : y) \rightarrow x + \text{sum}(y)$	$\text{length}(\text{nil}) \rightarrow 0$ $\text{length}(x : y) \rightarrow s(\text{length}(y))$	⑩

① \vdash h ② \vdash cs ③ \vdash h ④ \vdash d ⑤ \vdash h ⑥ \vdash d ⑦ \vdash cs ⑧ ⑨ ⑩

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① \oplus h ② \oplus cs ③ \oplus h ④ \oplus d ⑤ \oplus h ⑥ \oplus d ⑦ \oplus cs ⑧ \oplus h ⑨ ⑩

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① \oplus h ② \oplus cs ③ \oplus h ④ \oplus d ⑤ \oplus h ⑥ \oplus d ⑦ \oplus cs ⑧ \oplus h ⑨ \oplus cs ⑩

Outline

- Overview
- Modularity
 - Definitions
 - Results

Theorem (Toyama's Theorem)

Confluence is modular for disjoint TRSs.

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Remark

*Confluence is **not** modular for constructor-sharing TRSs.*

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Example

$$\begin{array}{l} f(x, x) \rightarrow a \\ f(x, g(x)) \rightarrow b \end{array} \quad c \rightarrow g(c)$$

Theorem (Toyama's Theorem)

Confluence is modular for disjoint TRSs.

Remark

*Confluence is **not** modular for constructor-sharing TRSs.*

Example

$$\begin{array}{l} f(x, x) \rightarrow a \\ f(x, g(x)) \rightarrow b \end{array} \qquad c \rightarrow g(c)$$

Theorem (Toyama's Theorem)

Confluence is modular for disjoint TRSs.

Remark

*Confluence is **not** modular for constructor-sharing TRSs.*

Example

$$\begin{array}{ll} f(x, x) \rightarrow a & c \rightarrow g(c) \\ f(x, g(x)) \rightarrow b & \end{array}$$

$$a \leftarrow f(c, c) \rightarrow f(c, g(c)) \rightarrow b$$

Theorem

*Termination is **not** modular for disjoint TRSs.*

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x)$$

$$g(x, y) \rightarrow x$$

$$g(x, y) \rightarrow y$$

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x)$$

$$g(x, y) \rightarrow x$$

$$g(x, y) \rightarrow y$$

$$f(a, b, g(a, b))$$

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x)$$

$$g(x, y) \rightarrow x$$

$$g(x, y) \rightarrow y$$

$$f(a, b, g(a, b)) \rightarrow f(g(a, b), g(a, b), g(a, b))$$

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x) \qquad g(x, y) \rightarrow x$$

$$g(x, y) \rightarrow y$$

$$f(a, b, g(a, b)) \rightarrow f(g(a, b), g(a, b), g(a, b)) \\ \rightarrow f(a, g(a, b), g(a, b))$$

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x) \qquad \begin{array}{l} g(x, y) \rightarrow x \\ g(x, y) \rightarrow y \end{array}$$

$$\begin{aligned} f(a, b, g(a, b)) &\rightarrow f(g(a, b), g(a, b), g(a, b)) \\ &\rightarrow f(a, g(a, b), g(a, b)) \\ &\rightarrow f(a, b, g(a, b)) \end{aligned}$$

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x)$$

duplicating

$$g(x, y) \rightarrow x$$

$$g(x, y) \rightarrow y$$

collapsing

$$\begin{aligned} f(a, b, g(a, b)) &\rightarrow f(g(a, b), g(a, b), g(a, b)) \\ &\rightarrow f(a, g(a, b), g(a, b)) \\ &\rightarrow f(a, b, g(a, b)) \end{aligned}$$

Theorem

The disjoint union of terminating TRSs R and S is terminating if:

- *R and S lack collapsing rules, or*
- *R and S lack duplicating rules, or*
- *R or S lacks both collapsing and duplicating rules.*

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The disjoint union of terminating TRSs R and S is terminating if:

- *R and S lack collapsing rules, or*
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Corollary

Termination is preserved under signature extension.

Theorem

Termination is *not* modular for disjoint TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x)$$

duplicating

$$g(x, y) \rightarrow x$$

$$g(x, y) \rightarrow y$$

collapsing & not confluent

$$f(a, b, g(a, b)) \rightarrow f(g(a, b), g(a, b), g(a, b))$$

$$\rightarrow f(a, g(a, b), g(a, b))$$

$$\rightarrow f(a, b, g(a, b))$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x) \quad a \rightarrow c$$

$$f(x, y, z) \rightarrow c \quad b \rightarrow c$$

$$g(x, y, y) \rightarrow x$$

$$g(y, y, x) \rightarrow x$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x) \quad a \rightarrow c$$

$$f(x, y, z) \rightarrow c \quad b \rightarrow c$$

$$g(x, y, y) \rightarrow x$$

$$g(y, y, x) \rightarrow x$$

$$f(a, b, g(a, b, b))$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$\begin{array}{lll} f(a, b, x) \rightarrow f(x, x, x) & a \rightarrow c & g(x, y, y) \rightarrow x \\ f(x, y, z) \rightarrow c & b \rightarrow c & g(y, y, x) \rightarrow x \end{array}$$

$$f(a, b, g(a, b, b)) \rightarrow f(g(a, b, b), g(a, b, b), g(a, b, b))$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$\begin{array}{lll} f(a, b, x) \rightarrow f(x, x, x) & a \rightarrow c & g(x, y, y) \rightarrow x \\ f(x, y, z) \rightarrow c & b \rightarrow c & g(y, y, x) \rightarrow x \end{array}$$

$$\begin{aligned} f(a, b, g(a, b, b)) &\rightarrow f(g(a, b, b), g(a, b, b), g(a, b, b)) \\ &\rightarrow f(a, g(a, b, b), g(a, b, b)) \end{aligned}$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$\begin{array}{ll}
 f(a, b, x) \rightarrow f(x, x, x) & a \rightarrow c \\
 f(x, y, z) \rightarrow c & b \rightarrow c
 \end{array}
 \qquad
 \begin{array}{l}
 g(x, y, y) \rightarrow x \\
 g(y, y, x) \rightarrow x
 \end{array}$$

$$\begin{aligned}
 f(a, b, g(a, b, b)) &\rightarrow f(g(a, b, b), g(a, b, b), g(a, b, b)) \\
 &\rightarrow f(a, g(a, b, b), g(a, b, b)) \\
 &\rightarrow f(a, g(c, b, b), g(a, b, b))
 \end{aligned}$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$\begin{array}{ll}
 f(a, b, x) \rightarrow f(x, x, x) & a \rightarrow c \\
 f(x, y, z) \rightarrow c & b \rightarrow c
 \end{array}
 \qquad
 \begin{array}{l}
 g(x, y, y) \rightarrow x \\
 g(y, y, x) \rightarrow x
 \end{array}$$

$$\begin{aligned}
 f(a, b, g(a, b, b)) &\rightarrow f(g(a, b, b), g(a, b, b), g(a, b, b)) \\
 &\rightarrow f(a, g(a, b, b), g(a, b, b)) \\
 &\rightarrow f(a, g(c, b, b), g(a, b, b)) \\
 &\rightarrow f(a, g(c, c, b), g(a, b, b))
 \end{aligned}$$

Theorem

Termination is *not* modular for disjoint *confluent* TRSs.

Example

$$f(a, b, x) \rightarrow f(x, x, x) \quad a \rightarrow c$$

$$f(x, y, z) \rightarrow c \quad b \rightarrow c$$

no **constructor system**

$$g(x, y, y) \rightarrow x$$

$$g(y, y, x) \rightarrow x$$

not **left-linear**

$$f(a, b, g(a, b, b)) \rightarrow f(g(a, b, b), g(a, b, b), g(a, b, b))$$

$$\rightarrow f(a, g(a, b, b), g(a, b, b))$$

$$\rightarrow f(a, g(c, b, b), g(a, b, b))$$

$$\rightarrow f(a, g(c, c, b), g(a, b, b))$$

$$\rightarrow f(a, b, g(a, b, b))$$

Theorem

- *termination is modular for disjoint **left-linear** confluent TRSs*

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS R over signature Σ is **constructor system (CS)** if $l_1, \dots, l_n \in \mathcal{T}(\Sigma_C, \mathcal{X})$ for every left-hand side $f(l_1, \dots, l_n)$ of rewrite rules in R .

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS R over signature Σ is constructor system (CS) if $l_1, \dots, l_n \in \mathcal{T}(\Sigma_C, \mathcal{X})$ for every left-hand side $f(l_1, \dots, l_n)$ of rewrite rules in R .

Theorem

- *weak normalization is modular for constructor-sharing TRSs*

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS R over signature Σ is constructor system (CS) if $l_1, \dots, l_n \in \mathcal{T}(\Sigma_C, \mathcal{X})$ for every left-hand side $f(l_1, \dots, l_n)$ of rewrite rules in R .

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*

Theorem

- *termination is modular for disjoint left-linear confluent TRSs*
- *termination is modular for constructor-sharing confluent CSs*

Definition

TRS R over signature Σ is constructor system (CS) if $l_1, \dots, l_n \in \mathcal{T}(\Sigma_C, \mathcal{X})$ for every left-hand side $f(l_1, \dots, l_n)$ of rewrite rules in R .

Theorem

- *weak normalization is modular for constructor-sharing TRSs*
- *local confluence is modular for constructor-sharing TRSs*
- *semi-completeness is modular for constructor-sharing TRSs*