Outline

Confluence

Local Confluence

Critical Pair Lemma

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

Theorem

WCR and CR are decidable properties for terminating, finite TRSs.

Proof.

For terminating R we have CR $\Leftrightarrow$ WCR. We decide WCR as follows:

For every $s \leftarrow \rtimes \rightarrow t \in \mathrm{CP}(\mathcal{R})$ we check $s \downarrow t$ by computing:

- the set of reducts of s: $\rightarrow(s) = \{s' \mid s \rightarrow^* s'\}$, and
- the set of reducts of t: $\rightarrow(t) = \{t' \mid t \rightarrow^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Then $s \downarrow t$ if and only if $\rightarrow(s) \cap \rightarrow(t) \neq \emptyset$. ∎

Proof of the Critical Pair Lemma

Critical Pair Lemma

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

Proof of the Direction $\Rightarrow$.

Let $\mathcal{R}$ be WCR.

Assume that there exists a non-convergent critical pair $s \leftarrow \rtimes \rightarrow t$.

Then $s \leftarrow \cdot \rightarrow t$, and not $s \downarrow t$. This would contradict WCR. ∎

Proof of the Critical Pair Lemma

Critical Pair Lemma

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

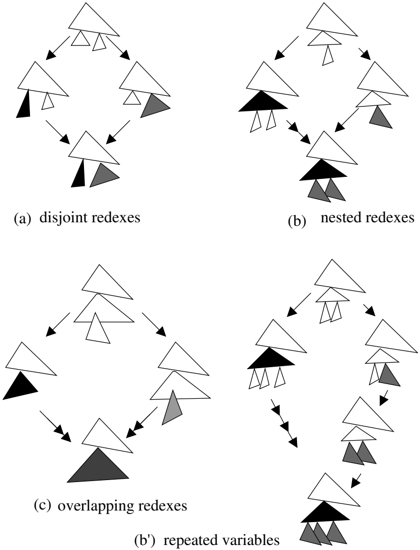
Proof of the Direction $\Rightarrow$.

Consider a peak $t_0 \overset{s_0}{\leftarrow} t \overset{s_1}{\rightarrow} t_1$ (s_0 and s_1 are the contracted redex occurrences).

We have to find a common reduct t_2 of t_0 and t_1.

Distinguish cases according to the relative positions of the redexes s_0 and s_1 in t. ∎

Case (a): Disjoint redexes



(a) disjoint redexes

(b) nested redexes

(c) overlapping redexes

(b') repeated variables

Assume s_0 and s_1 are disjoint.

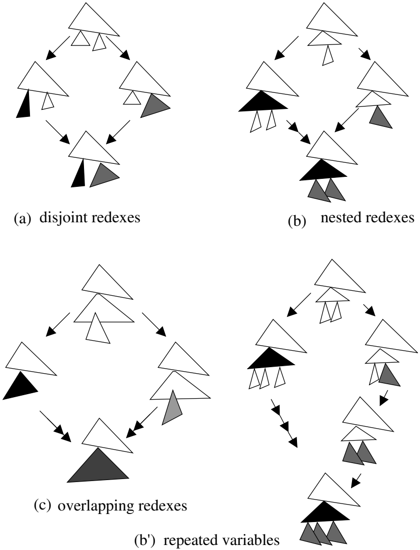
Then $t = C[s_0, s_1]$
for some 2-hole context C.

Let s_0', s_1' be the contracta of s_0, s_1.

Then $t_0 = C[s_0', s_1]$ and $t_1 = C[s_0, s_1']$.

Take $t_2 = C[s_0', s_1']$ as common reduct.

Case (b) and (b'): Nested redexes



(a) disjoint redexes

(b) nested redexes

(c) overlapping redexes

(b') repeated variables

Assume that s_1 is nested below s_0 in t.

Let $C[\,]$ be the prefix of s_0 in t.

Assume s_0 is according to rule $\ell_0 \to r_0$:

- $s_0 = {\ell_0}^{\sigma}$ and $s_0' = {r_0}^{\sigma}$.

By nestedness:

- $\ell_0 = D[x]$,
- $s_0 = D^{\sigma}[x^{\sigma}]$, and $x^{\sigma} = E[s_1]$

Then $t_1 = C[D^{\sigma}[E[s_1']]]$.

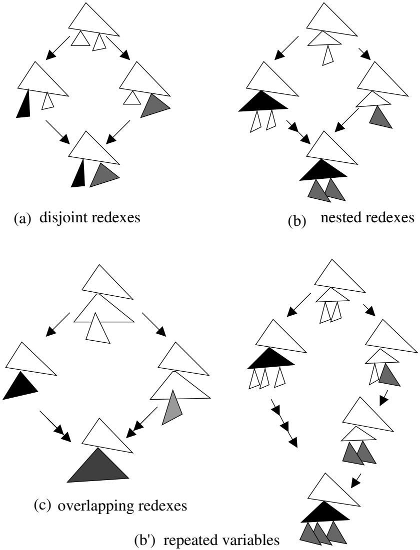
Define σ' by

- $\sigma'(x) = E[s_1']$, and
- $\sigma'(y) = \sigma(y)$ if $y \neq x$.

We can take $t_2 = C[{r_0}^{\sigma'}]$:

$$t_1 = C[{\ell_0}^{\sigma}] \twoheadrightarrow C[{\ell_0}^{\sigma'}]$$
$$\to C[{r_0}^{\sigma'}] \twoheadleftarrow C[{r_0}^{\sigma}] = t_0$$

Case (c): overlapping redexes



(a) disjoint redexes

(b) nested redexes

(c) overlapping redexes

(b') repeated variables

Trivial overlaps are uninteresting.

Let s_1 be overlapping s_0
(with s_1 at lower or equal position).

Let $C[\,]$ be the prefix of s_0.

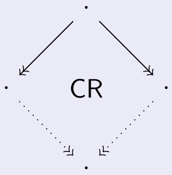
Then t_0, t_1 are instances $C[{c_0}^\rho], C[{c_1}^\rho]$,
for some critical pair $\langle c_1, c_0 \rangle$.

The critical pair is convergent,
say with common reduct s_2.

Take $t_2 = C[{s_2}^\rho]$.

Confluence

every two coinitial rewrite sequences can be joined



- . . . yields uniqueness of normal forms
- . . . is decidable for terminating TRSs
- . . . what about nonterminating TRSs ?

Examples (Non-Confluence)

- no confluence because of critical pairs

$$\mathsf{a} \to \mathsf{b} \qquad\qquad \mathsf{b} \leftarrow \mathsf{a} \to \mathsf{c}$$
$$\mathsf{a} \to \mathsf{c}$$

- no critical pairs but no confluence (Klop 1978)

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad\qquad \mathsf{a} \ {}^*\!\!\leftarrow \mathsf{c} \to \mathsf{g}(\mathsf{c}) \to^* \mathsf{g}(\mathsf{a})$$
$$\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$$
$$\mathsf{c} \to \mathsf{g}(\mathsf{c})$$

- no critical pairs but no confluence (Huet 1980)

$$\mathsf{f}(x, x) \to \mathsf{a} \qquad\qquad \mathsf{a} \leftarrow \mathsf{f}(\mathsf{c},\mathsf{c}) \to \mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{c})) \to \mathsf{b}$$
$$\mathsf{f}(x, \mathsf{g}(x)) \to \mathsf{b}$$
$$\mathsf{c} \to \mathsf{g}(\mathsf{c})$$

Confluence via Critical Pairs

Control interference of rewrite rules:

- Critical Pair Lemma (lecture 5):
  WCR $\iff$ $\leftarrow\rtimes\rightarrow \subseteq \downarrow$ (all critical pairs are convergent)
- Combine with Newman's Lemma:
  SN & $\leftarrow\rtimes\rightarrow \subseteq \downarrow$ $\implies$ CR

Observe from preceding examples:

Confluence via Orthogonality

Forbid interference of rewrite rules

- no critical pairs
- no equality checks (duplicated variables in the left-hand sides)

Definitions

- term t is linear is each variable in $\mathcal{V}\text{ar}(t)$ occurs exactly once in t
- rewrite rule $\ell \to r$ is left-linear if ℓ is linear
- TRS is left-linear if all rewrite rules are left-linear
- rewrite rule $\ell \to r$ is right-linear if r is linear
- rewrite rule $\ell \to r$ is linear if ℓ and r are linear
- TRS is (right-)linear if all rewrite rules are (right-)linear

Examples

- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
  left-linear but not right-linear
- $\mathsf{f}(x, x) \to \mathsf{a}$
  right-linear but not left-linear

Definition

A TRS is orthogonal if it is left-linear and has no criticial pairs.

Examples

$$\mathsf{I} \cdot x \to x \qquad\qquad \mathsf{ack}(0, y) \to \mathsf{s}(y)$$

$$(\mathsf{K} \cdot x) \cdot y \to x \qquad\qquad \mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$

$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \qquad\qquad \mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

Lemma

Every orthogonal TRS is locally confluent.

Proof.

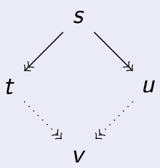
All critical pairs are convergent (are are none).

Hence we have WCR by the Critical Pair Lemma. ■

Theorem

Every orthogonal TRS is confluent

$\forall s, t, u$



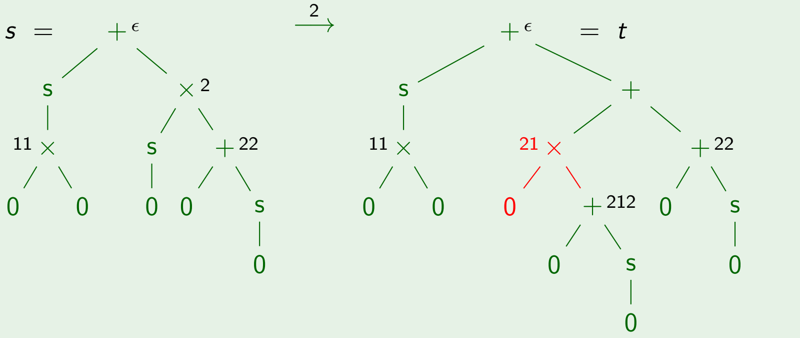
$\exists v$

For orthogonal TRSs there is a canonical way to compute common reduct. . .

Outline

Example

$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$



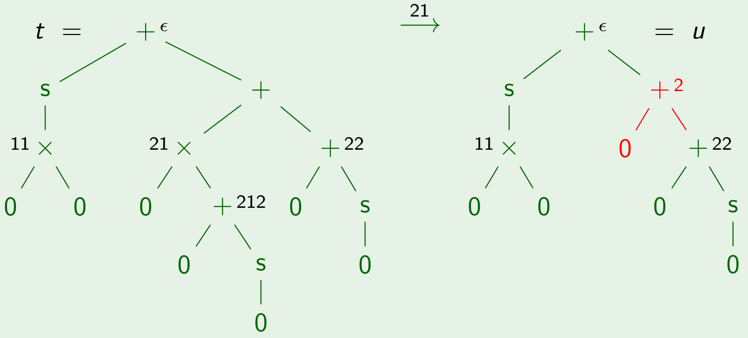
redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

redex at position 22 is duplicated

redex at position 21 is created

Example

$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$



redex positions in s :	ϵ	11	2	22			
redex positions in t :	ϵ	11		212	22	21	
redex positions in u :	ϵ	11			22		2

redex at position 212 is erased redex at position 2 is created rewrite step A: $s \xrightarrow[\ell \to r]{p} t$ at position $p \in \mathcal{P}\mathrm{os}(s)$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Step)

- The descendants of q after A in t

$$q/A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}\mathrm{os}_{\mathcal{X}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

- The descendants of Q after A in t are

$$Q/A = \bigcup_{q \in Q} q/A$$

Remark

- Information about position is needed to determine descendants:

$$\mathsf{f}(x) \to x \qquad\qquad A : \mathsf{f}(\mathsf{f}(x)) \xrightarrow[\mathsf{f}(x)\to x]{?} \mathsf{f}(x)$$

  Is $1/A = \epsilon$ or $1/A = \varnothing$?

- Information about rewrite rule is needed to determine descendants:

$$\begin{array}{c} \mathsf{f}(x) \to \mathsf{f}(x) \\ \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \end{array} \qquad\qquad A : \mathsf{f}(\mathsf{a}) \xrightarrow[?]{\epsilon} \mathsf{f}(\mathsf{a})$$

  Is $1/A = 1$ or $1/A = \varnothing$?

rewrite sequence $A\colon s \to^* t$ set of positions $Q \subseteq \mathcal{P}os(s)$

Definition (Descendants after Rewrite Sequence)

The descendants of Q after A in t

$$Q/A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q/A_1)/A_2 & \text{If } A = A_1; A_2 \text{ with } A_1\colon s \to u \text{ and } A_2\colon u \to^* t \end{cases}$$

Lemma

For arbitrary TRSs: if Q is parallel ($\forall\, p \neq q \in Q\colon p \parallel q$) then so is Q/A.

Lemma

For orthogonal TRSs: if Q is set of redex positions then so is Q/A.

Terminology

A descendant of redex is called residual.

Remark

- In non-left-linear TRS descendant of redex is not necessarily redex

$$\mathsf{a} \to \mathsf{b} \qquad\qquad \mathsf{f(a,a)} \to \mathsf{f(b,a)}$$
$$\mathsf{f}(x,x) \to \mathsf{b}$$

- In TRS with critical pairs descendant of redex is not necessarily redex

$$\mathsf{a} \to \mathsf{b} \qquad\qquad \mathsf{f(a)} \to \mathsf{f(b)}$$
$$\mathsf{f(a)} \to \mathsf{b}$$

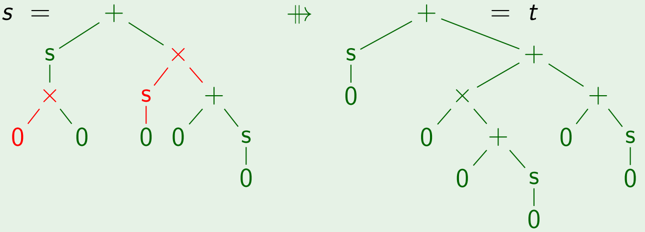
Outline

Definition (Parallel Rewriting)

1. $= \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow}$
2. $\xrightarrow{\epsilon} \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow}$
3. $f(s_1, \ldots, s_n) \mathrel{\parallel\!\!\!\rightarrow} f(t_1, \ldots, t_n)$ if $s_i \mathrel{\parallel\!\!\!\rightarrow} t_i$ for each $1 \leqslant i \leqslant n$

Example

$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$



Lemma

$s \mathrel{\parallel\!\!\!\rightarrow} t \quad \Longleftrightarrow \quad s \to^* t$ *by contracting redexes at pairwise parallel positions in* s

Definition (Projection)

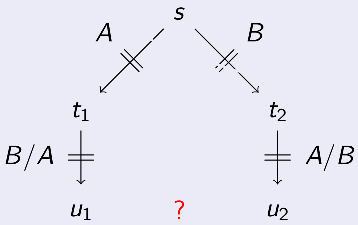
$A\colon\ s \mathrel{-\!\!\!\Vert\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P

$B\colon\ s \mathrel{-\!\!\!\Vert\!\!\!\rightarrow} t_2$ by contracting redexes at positions in Q

- $B/A\colon\ t_1 \mathrel{-\!\!\!\Vert\!\!\!\rightarrow} u_1$ by contracting redexes at positions in Q/A
- $A/B\colon\ t_2 \mathrel{-\!\!\!\Vert\!\!\!\rightarrow} u_2$ by contracting redexes at positions in P/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Parallel Moves Lemma

For every orthogonal TRS:



Definition (Projection)

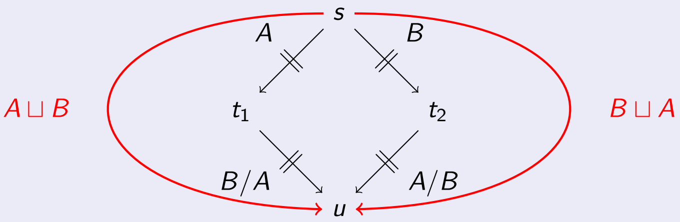
A: $s \mathrel{+\!\!\!\!\!\twoheadrightarrow} t_1$ by contracting redexes at positions in P

B: $s \mathrel{+\!\!\!\!\!\twoheadrightarrow} t_2$ by contracting redexes at positions in Q

- B/A: $t_1 \mathrel{+\!\!\!\!\!\twoheadrightarrow} u_1$ by contracting redexes at positions in Q/A
- A/B: $t_2 \mathrel{+\!\!\!\!\!\twoheadrightarrow} u_2$ by contracting redexes at positions in P/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Parallel Moves Lemma

For every orthogonal TRS:

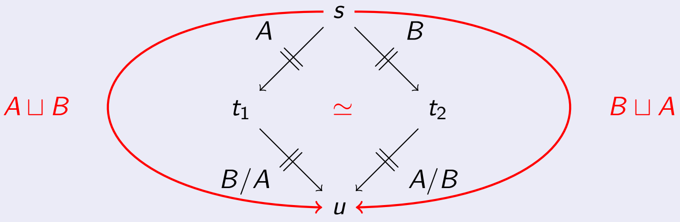


Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

1. $s_1 = s_2$
2. $t_1 = t_2$
3. $p/A = p/B$ for all redex positions p in s_1

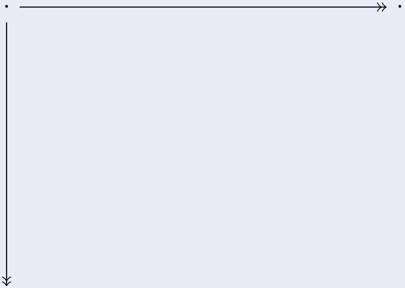
Parallel Moves Lemma (with Permutation Equivalence)



Corollary

Every orthogonal TRSs is confluent.

Proof.

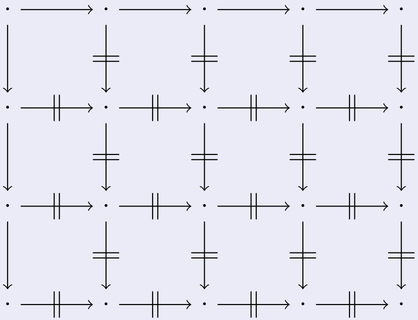


∎

Corollary

Every orthogonal TRSs is confluent.

Proof.



Outline

Definition

A critical pair $s \leftarrow \rtimes \rightarrow t$ is trivial if $s = t$.

Definition

A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

Examples

$$x \vee \mathsf{T} \to \mathsf{T}$$
$$\mathsf{T} \vee x \to \mathsf{T}$$
$$\mathsf{F} \vee \mathsf{F} \to \mathsf{F}$$

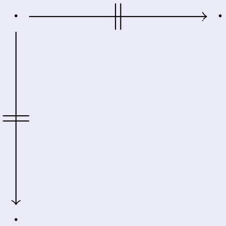
$$\mathsf{p}(\mathsf{s}(x)) \to x$$
$$\mathsf{s}(\mathsf{p}(x)) \to x$$

Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\leftarrow\!\!\!\!\!\!\parallel \cdot \parallel\!\!\!\!\!\!\rightarrow \subseteq \parallel\!\!\!\!\!\!\rightarrow \cdot \leftarrow\!\!\!\!\!\!\parallel$

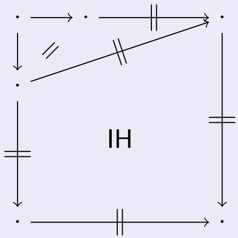


Theorem

Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \subseteq \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\parallel}$



interesting case

- $\rightarrow \subseteq \mathrel{\parallel\!\!\!\!\!\!\rightarrow} \subseteq \rightarrow^*$

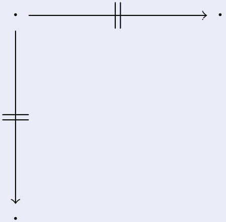
∎

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\rightarrow} \quad \Longrightarrow \quad \mathit{CR}$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\parallel}$

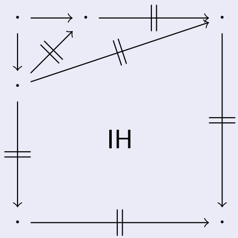


Theorem (Huet 1980)

left-linearity & $\leftarrow\rtimes\rightarrow \subseteq \mathrel{+\!\!\!\Vdash\!\!\!\rightarrow} \implies$ *CR*

Proof Sketch.

- $\leftarrow\!\!\Vdash \cdot \Vdash\!\!\rightarrow \subseteq \Vdash\!\!\rightarrow \cdot \leftarrow\!\!\Vdash$



interesting case

- $\rightarrow \subseteq \Vdash\!\!\rightarrow \subseteq \rightarrow^*$

∎

Open Problem

left-linearity & $\leftarrow\rtimes\rightarrow \subseteq \leftarrow\!\!\Vdash \implies$ *CR ?*

Example

$$\begin{array}{rcl} \mathsf{f}(\mathsf{g}(x,\mathsf{a},\mathsf{b})) & \to & x \\ \mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d})),x,y) & \to & \mathsf{h}(\mathsf{k}(x),\mathsf{k}(y)) \\ \mathsf{k}(\mathsf{a}) & \to & \mathsf{c} \\ \mathsf{k}(\mathsf{b}) & \to & \mathsf{d} \end{array}$$

The only critical pair is:

$$\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}),\mathsf{k}(\mathsf{b}))) \quad \leftarrow \quad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d})),\mathsf{a},\mathsf{b})) \quad \to \quad \mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d}))$$

Since $\mathcal{R}$ is left-linear and $\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}),\mathsf{k}(\mathsf{b}))) \not\twoheadrightarrow \mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d}))$, the system is confluent.

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

Notation

- $s \leftarrow\!\bowtie\!\rightarrow t$ for critical pair originating from overlap $\langle l_1 \rightarrow r_1, \epsilon, l_2 \rightarrow r_2 \rangle$
- $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;=\; \leftarrow\!\rtimes\!\rightarrow \setminus \leftarrow\!\bowtie\!\rightarrow$

Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (van Oostrom 1996)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \color{red}{\mathrel{\circ\!\!\!\rightarrow}}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\circ\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Confluence Tool

- ACP