- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

Confluence

Local Confluence

Critical Pair Lemma

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

We decide WCR as follows:

For every $s \leftarrow \rtimes \rightarrow t \in \mathrm{CP}(\mathcal{R})$ we check $s \downarrow t$ by computing:

- the set of reducts of s: $\rightarrow(s) = \{s' \mid s \rightarrow^* s'\}$, and
- the set of reducts of t: $\rightarrow(t) = \{t' \mid t \rightarrow^* t'\}$.

(In a finite, terminating TRS every term has only only finitely many reducts.)

Then $s \downarrow t$ if and only if $\rightarrow(s) \cap \rightarrow(t) \neq \emptyset$. ∎

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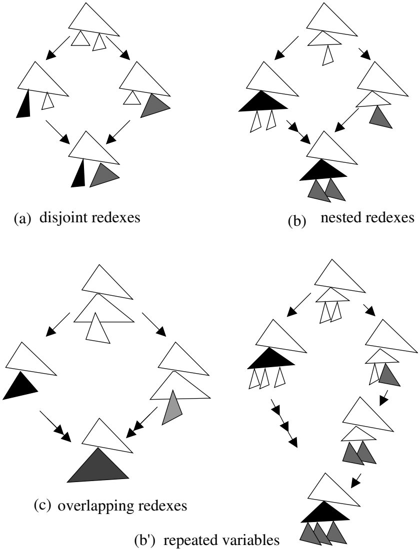

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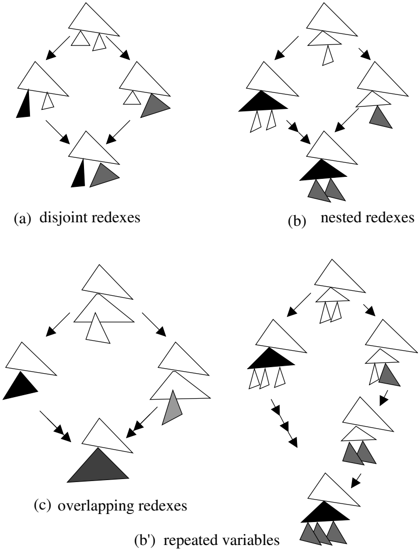

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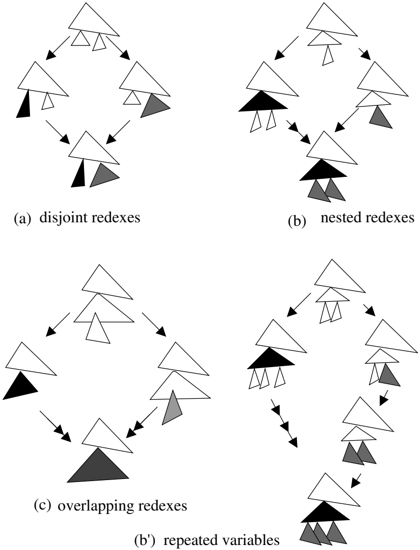

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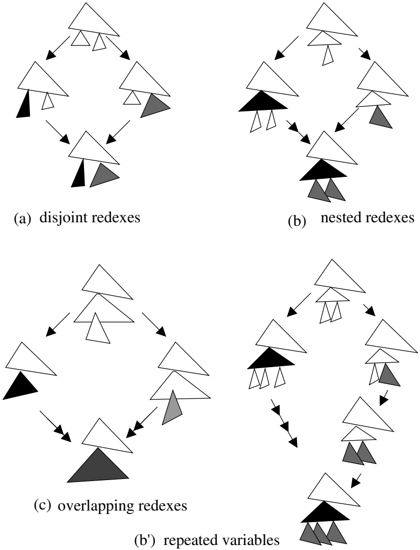

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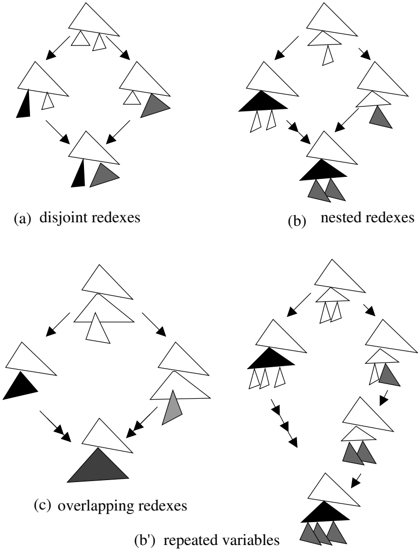

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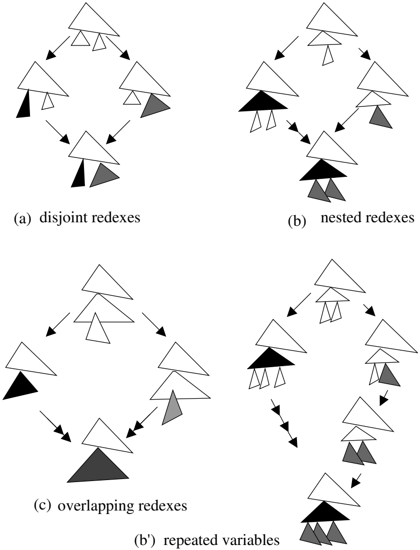

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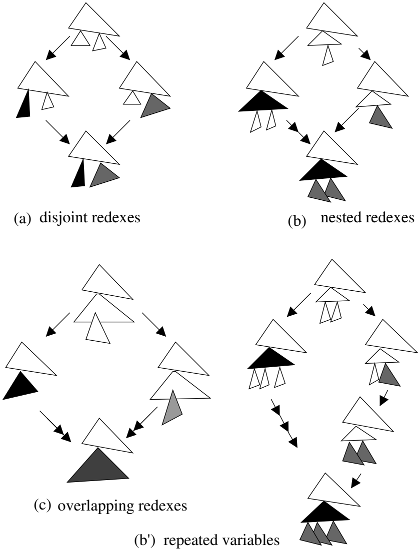

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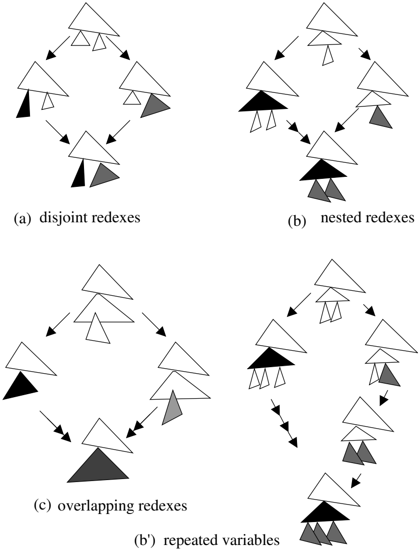

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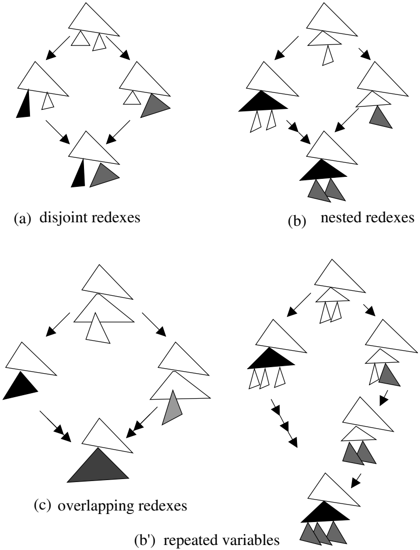

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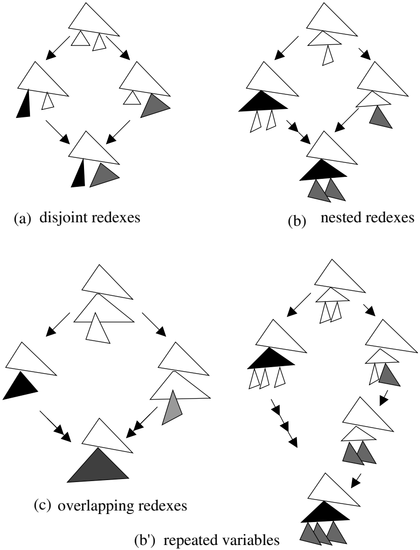

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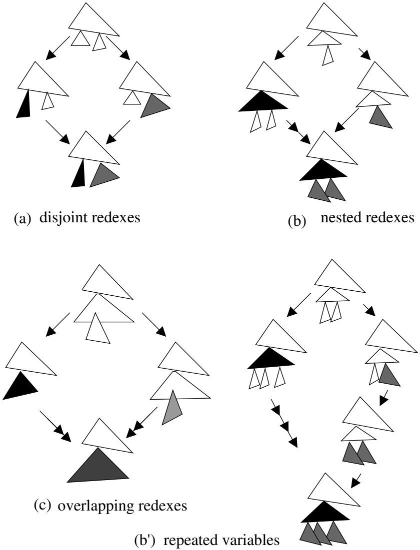

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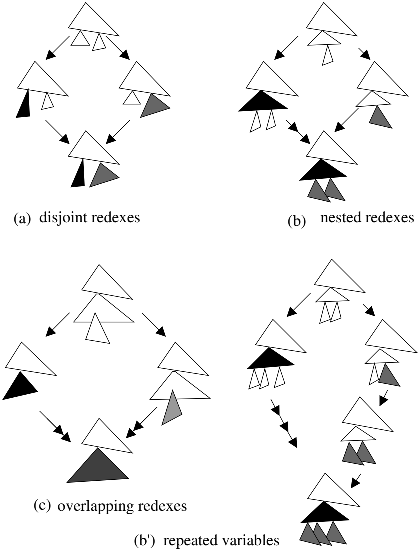

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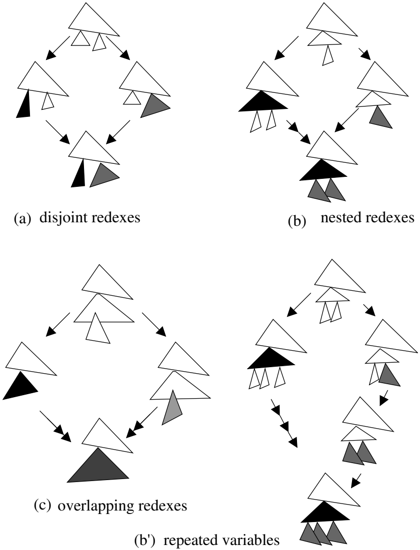

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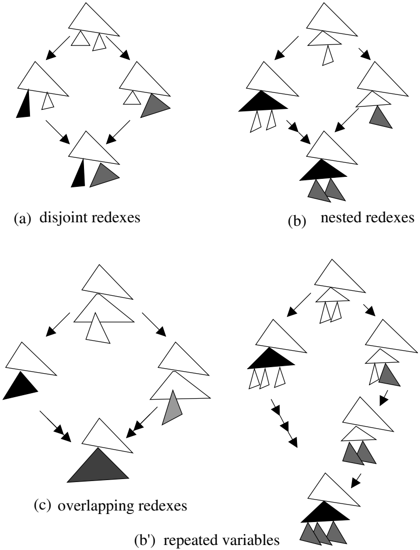

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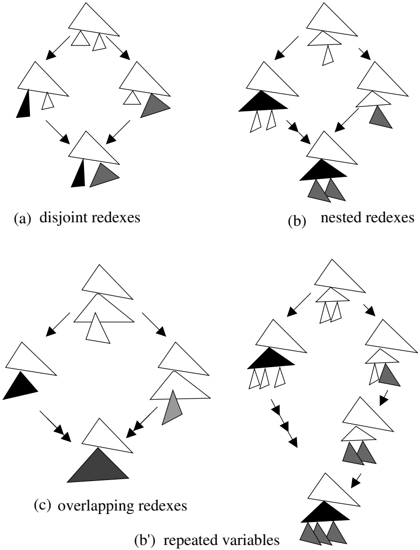

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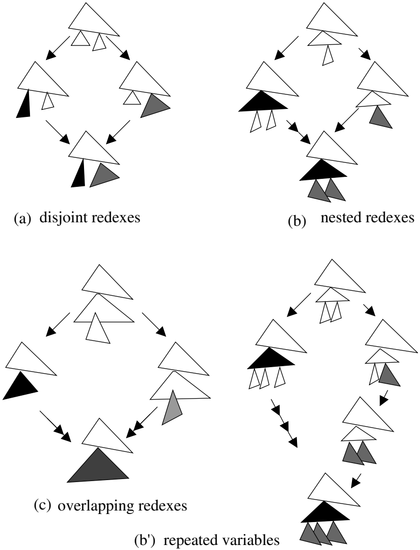

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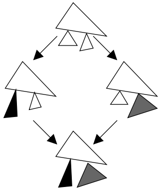

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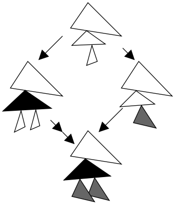

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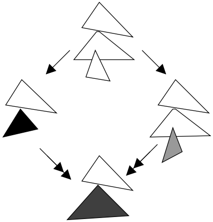

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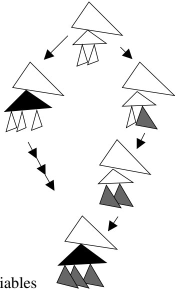

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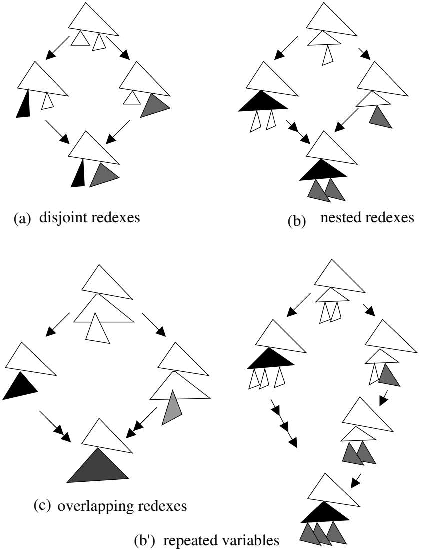

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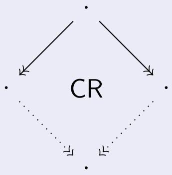

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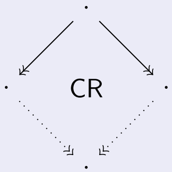

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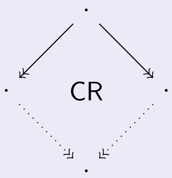

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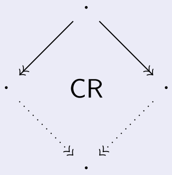

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- ... what about nonterminating TRSs ?

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Forbid interference of rewrite rules

Confluence via Critical Pairs

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Observe from preceding examples:

Confluence via Orthogonality

Forbid interference of rewrite rules

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- no equality checks (duplicated variables in the left-hand sides)

Definitions

- term t is linear is each variable in $\mathcal{V}\text{ar}(t)$ occurs exactly once in t

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- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
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- $\mathsf{g}(x) \to \mathsf{f}(x, \mathsf{g}(x))$
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- $\mathsf{f}(x, x) \to \mathsf{a}$
  right-linear but not left-linear

Definition

A TRS is orthogonal if it is left-linear and has no criticial pairs.

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Examples

$$\mathsf{I} \cdot x \to x$$

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Examples

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

$$\mathsf{ack}(0, y) \to \mathsf{s}(y)$$
$$\mathsf{ack}(\mathsf{s}(x), 0) \to \mathsf{ack}(x, \mathsf{s}(0))$$
$$\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$$

Lemma

Every orthogonal TRS is locally confluent.

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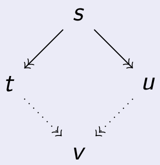
Theorem

Every orthogonal TRS is confluent

$\forall s, t, u$



$\exists v$

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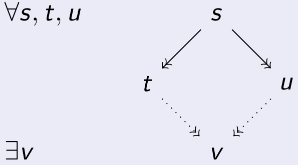


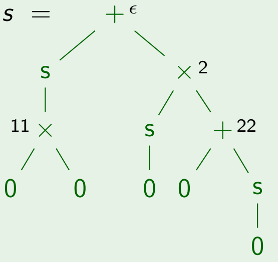
For orthogonal TRSs there is a canonical way to compute common reduct...

Outline

Example

$0 + y \to y \quad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad 0 \times y \to 0 \quad \mathsf{s}(x) \times y \to (x \times y) + y$

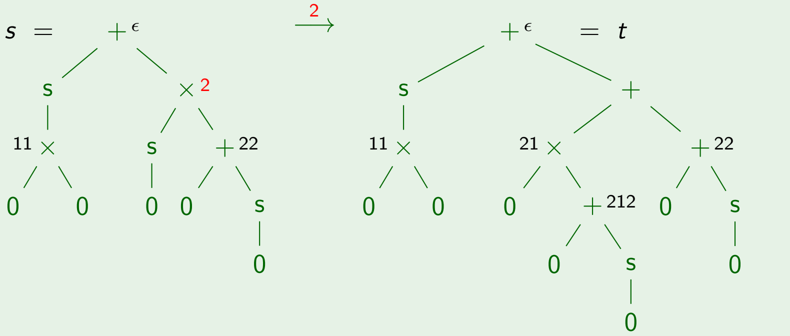
$s =$



redex positions in s: $\epsilon \mid 11 \mid 2 \mid \quad 22 \quad \mid$

Example

$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$



redex positions in s: ϵ | 11 | 2 | 22 |

Example

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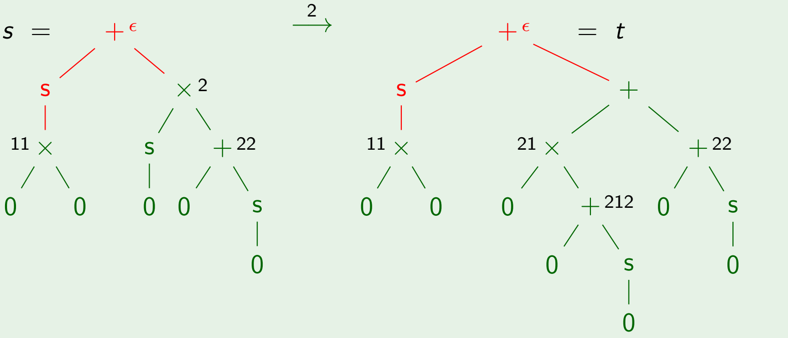
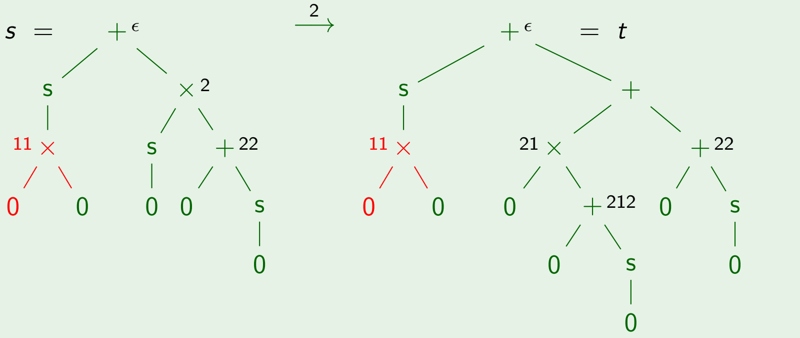


redex positions in s: ϵ | 11 | 2 | 22 |
redex positions in t: ϵ |

Example

$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$



redex positions in s:	ϵ	11	2	22	
redex positions in t:	ϵ	11			

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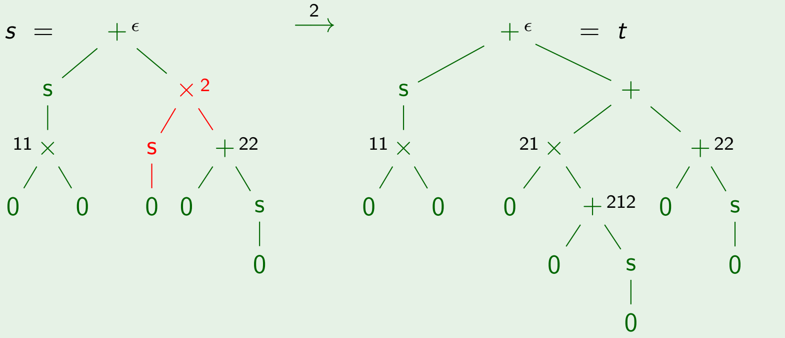

redex positions in s:	ϵ	11	2	22	
redex positions in t:	ϵ	11			

Example

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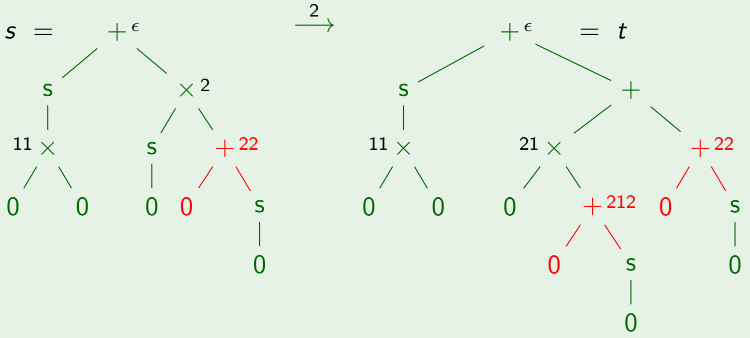

redex positions in s:	ϵ	11	2	22	
redex positions in t:	ϵ	11		212	22

duplicated

Example

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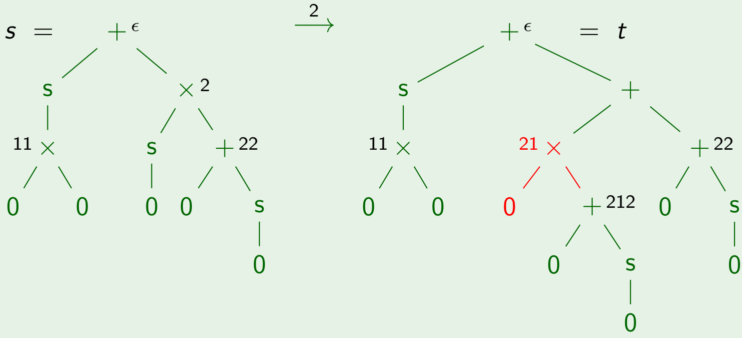


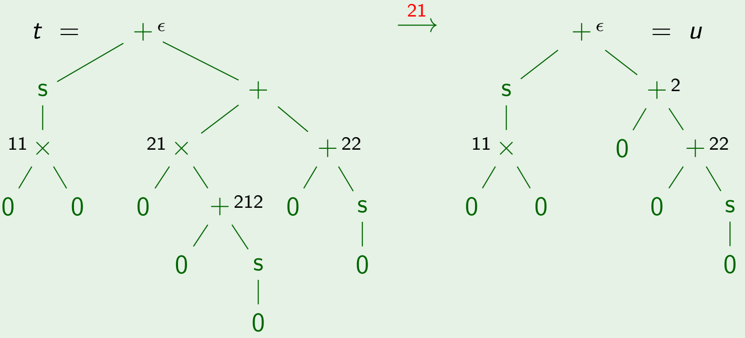
redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

redex at position 22 is duplicated

redex at position 21 is created

Example

$$0 + y \to y \qquad s(x) + y \to s(x + y) \qquad 0 \times y \to 0 \qquad s(x) \times y \to (x \times y) + y$$



redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21

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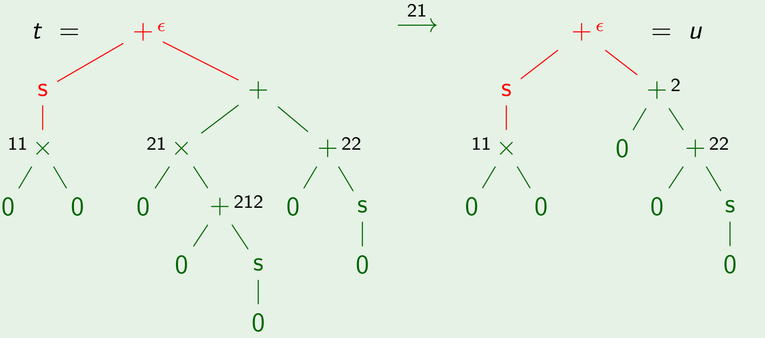

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
redex positions in u:	ϵ					

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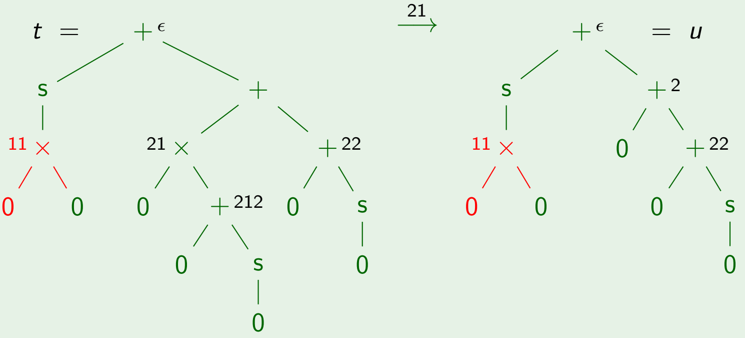

<table>
<tr><td>redex positions in s :</td><td>ϵ</td><td>11</td><td>2</td><td colspan="2">22</td><td></td></tr>
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</table>

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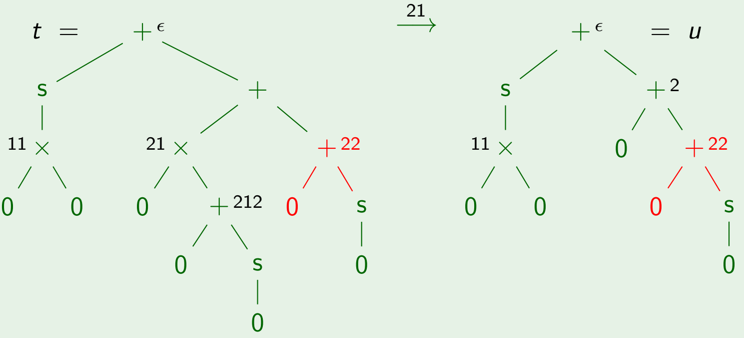

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
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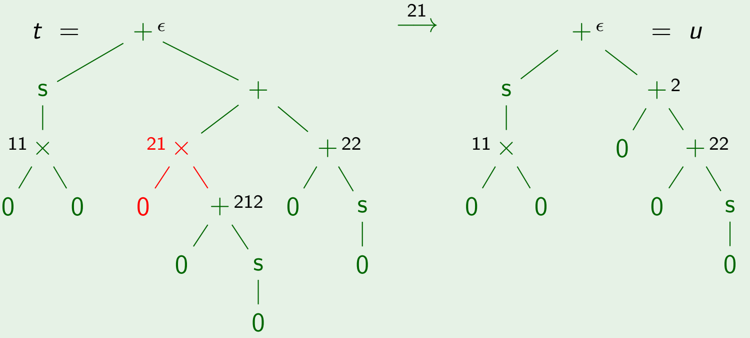

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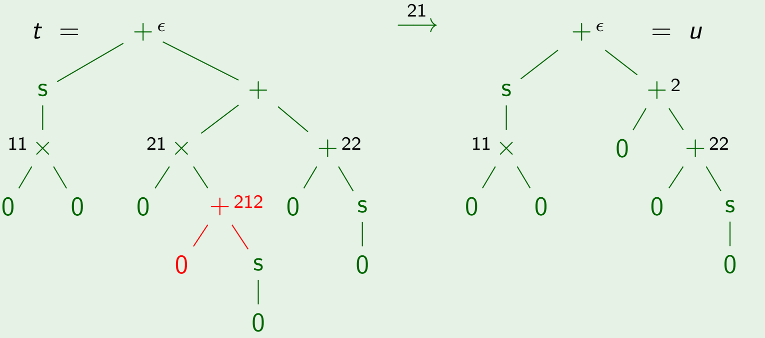

redex positions in s:	ϵ	11	2	22		
redex positions in t:	ϵ	11		212	22	21
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erased

Example

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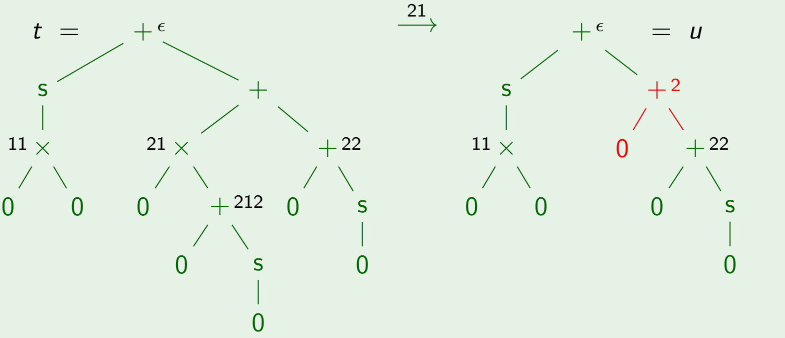

redex positions in s:	ϵ	11	2	22			
redex positions in t:	ϵ	11		212	22	21	
redex positions in u:	ϵ	11			22		2

redex at position 212 is erased redex at position 2 is created rewrite step A: $s \xrightarrow[\ell \to r]{p} t$ at position $p \in \mathcal{P}\mathrm{os}(s)$ set of positions $Q \subseteq \mathcal{P}\mathrm{os}(s)$

Definition (Descendants after Rewrite Step)

- The descendants of q after A in t

$$q/A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p \\ \{pp_3p_2 \mid r|_{p_3} = \ell|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}\mathrm{os}_{\mathcal{X}}(\ell) \\ \varnothing & \text{otherwise} \end{cases}$$

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- The descendants of Q after A in t are

$$Q/A = \bigcup_{q \in Q} q/A$$

Remark

- Information about position is needed to determine descendants:

$$f(x) \to x \qquad\qquad A : f(f(x)) \xrightarrow[f(x)\to x]{?} f(x)$$

Is $1/A = \epsilon$ or $1/A = \varnothing$?

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- Information about rewrite rule is needed to determine descendants:

$$\begin{array}{c}\mathsf{f}(x) \to \mathsf{f}(x)\\ \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a})\end{array} \qquad\qquad A : \mathsf{f}(\mathsf{a}) \xrightarrow[?]{\epsilon} \mathsf{f}(\mathsf{a})$$

  Is $1/A = 1$ or $1/A = \varnothing$?

rewrite sequence $A\colon s \to^* t$ set of positions $Q \subseteq \mathcal{P}os(s)$

Definition (Descendants after Rewrite Sequence)

The descendants of Q after A in t

$$Q/A = \begin{cases} Q & \text{if } A \text{ is empty sequence} \\ (Q/A_1)/A_2 & \text{If } A = A_1; A_2 \text{ with } A_1\colon s \to u \text{ and } A_2\colon u \to^* t \end{cases}$$

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Lemma

For arbitrary TRSs: if Q is parallel ($\forall\, p \neq q \in Q\colon p \parallel q$) then so is Q/A.

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Lemma

For orthogonal TRSs: if Q is set of redex positions then so is Q/A.

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Lemma

For orthogonal TRSs: if Q is set of redex positions then so is Q/A.

Terminology

A descendant of redex is called residual.

Remark

- In non-left-linear TRS descendant of redex is not necessarily redex

$$a \to b \qquad f(a,a) \to f(b,a)$$
$$f(x, x) \to b$$

Remark

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$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(x,x) &\to \mathsf{b} \end{aligned} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{f}(\mathsf{b},\mathsf{a})$$

- In TRS with critical pairs descendant of redex is not necessarily redex

$$\begin{aligned} \mathsf{a} &\to \mathsf{b} \\ \mathsf{f}(\mathsf{a}) &\to \mathsf{b} \end{aligned} \qquad \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{b})$$

Outline

- Overview
- Local Confluence
- Orthogonality
  - Definitions
  - Descendants
  - Parallel Moves Lemma
- Beyond Orthogonality

Definition (Parallel Rewriting)

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1. $= \ \subseteq \ \twoheadrightarrow\!\!\!\!\!\!\!+$
2. $\xrightarrow{\epsilon} \ \subseteq \ \twoheadrightarrow\!\!\!\!\!\!\!+$
3. $f(s_1, \ldots, s_n) \twoheadrightarrow\!\!\!\!\!\!\!+ \ f(t_1, \ldots, t_n)$ if $s_i \twoheadrightarrow\!\!\!\!\!\!\!+ \ t_i$ for each $1 \leqslant i \leqslant n$

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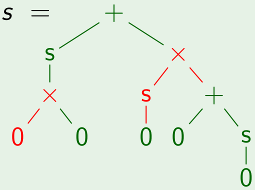
Example

$0 + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad 0 \times y \to 0 \qquad \mathsf{s}(x) \times y \to (x \times y) + y$

$s =$



Definition (Parallel Rewriting)

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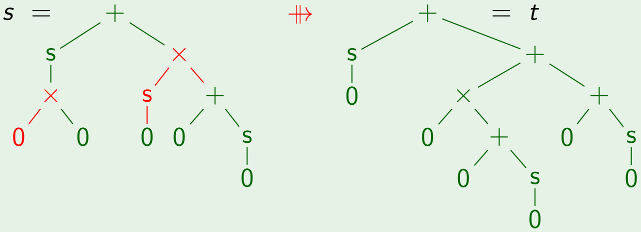

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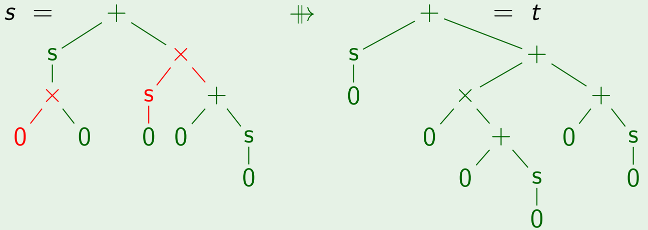

Lemma

$s \mathrel{+\!\!\!\!\!\!\rightarrow} t \iff s \to^* t$ by contracting redexes at pairwise parallel positions in s

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\!+\!\!\!\!\rightarrow} t_1$ by contracting redexes at positions in P

B: $s \mathrel{+\!\!\!\!\!\!+\!\!\!\!\rightarrow} t_2$ by contracting redexes at positions in Q

Definition (Projection)

A: $s \twoheadrightarrow t_1$ by contracting redexes at positions in P

B: $s \twoheadrightarrow t_2$ by contracting redexes at positions in Q

- B/A: $t_1 \twoheadrightarrow u_1$ by contracting redexes at positions in Q/A

Definition (Projection)

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B: $s \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} t_2$ by contracting redexes at positions in Q

- B/A: $t_1 \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} u_1$ by contracting redexes at positions in Q/A
- A/B: $t_2 \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} u_2$ by contracting redexes at positions in P/B

Definition (Projection)

$A\colon\ s \mathrel{+\!\!\!\!\!\Rightarrow} t_1$ by contracting redexes at positions in P

$B\colon\ s \mathrel{+\!\!\!\!\!\Rightarrow} t_2$ by contracting redexes at positions in Q

- $B/A\colon\ t_1 \mathrel{+\!\!\!\!\!\Rightarrow} u_1$ by contracting redexes at positions in Q/A
- $A/B\colon\ t_2 \mathrel{+\!\!\!\!\!\Rightarrow} u_2$ by contracting redexes at positions in P/B
- $A \sqcup B = A; B/A$ and $B \sqcup A = B; A/B$

Definition (Projection)

A: $s \mathrel{+\!\!\!\!\!\to} t_1$ by contracting redexes at positions in P

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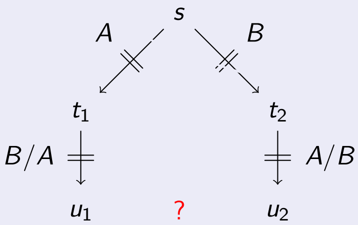
Parallel Moves Lemma

For every orthogonal TRS:



Definition (Projection)

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B: $s \mathrel{-\!\!\!\!\parallel\!\!\!\!\to} t_2$ by contracting redexes at positions in Q

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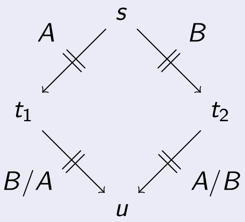

Definition (Projection)

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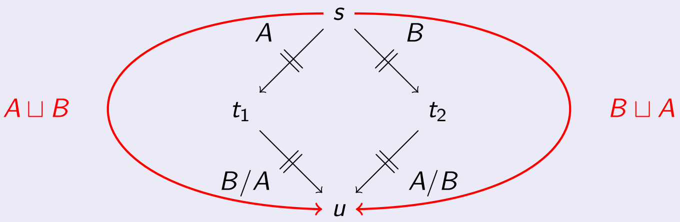

Definition

$A\colon s_1 \to^* t_1$ and $B\colon s_2 \to^* t_2$ are permutation equivalent ($A \simeq B$) if

1. $s_1 = s_2$
2. $t_1 = t_2$
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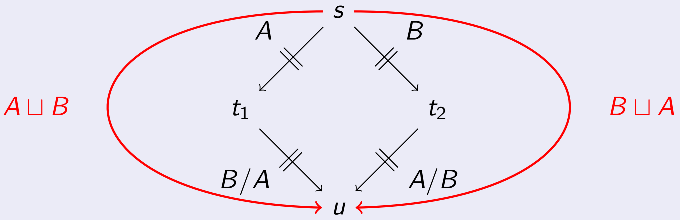
Parallel Moves Lemma (with Permutation Equivalence)



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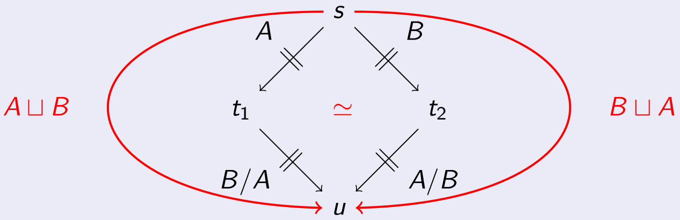
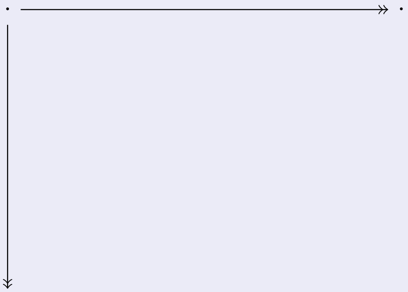


Corollary

Every orthogonal TRSs is confluent.

Proof.



∎

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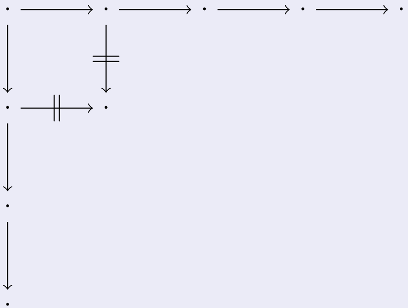

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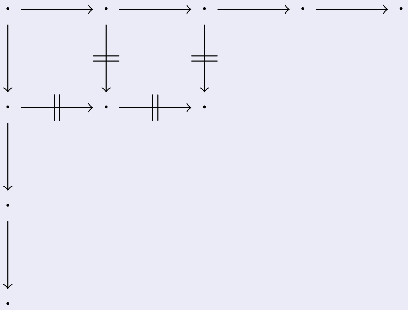

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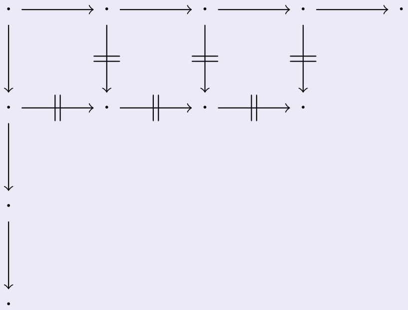

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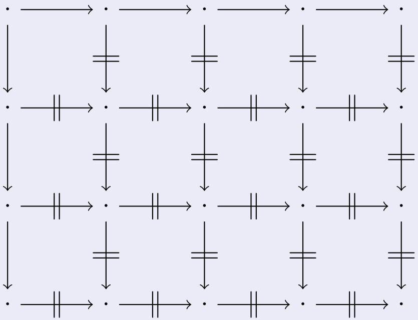

■

Outline

Definition

A critical pair $s \leftarrow \rtimes \rightarrow t$ is trivial if $s = t$.

Definition

A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

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Examples

$$x \vee \mathsf{T} \to \mathsf{T}$$
$$\mathsf{T} \vee x \to \mathsf{T}$$
$$\mathsf{F} \vee \mathsf{F} \to \mathsf{F}$$

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A critical pair $s \leftarrow \rtimes \rightarrow t$ is trivial if $s = t$.

Definition

A weakly orthogonal TRS is left-linear and has only trivial critical pairs.

Examples

$$\begin{aligned} x \vee \mathsf{T} &\to \mathsf{T} \\ \mathsf{T} \vee x &\to \mathsf{T} \\ \mathsf{F} \vee \mathsf{F} &\to \mathsf{F} \end{aligned}$$

$$\begin{aligned} \mathsf{p}(\mathsf{s}(x)) &\to x \\ \mathsf{s}(\mathsf{p}(x)) &\to x \end{aligned}$$

Theorem

Every weakly orthogonal TRSs is confluent.

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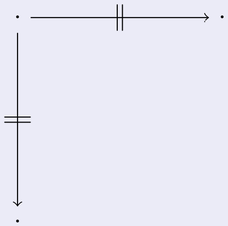
Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\!\!\;\|} \cdot \mathrel{\|\!\!\!\!\!\to} \subseteq \mathrel{\|\!\!\!\!\!\to} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\;\|}$



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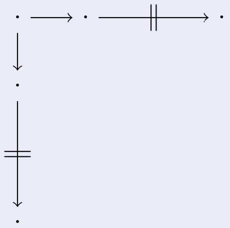

interesting case

∎

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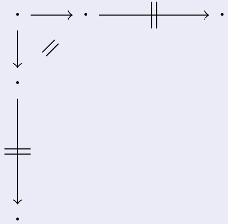

interesting case

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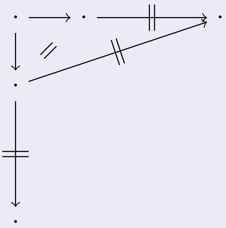

interesting case

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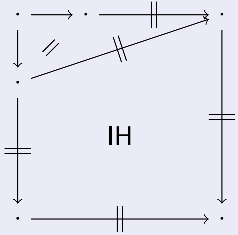

interesting case

■

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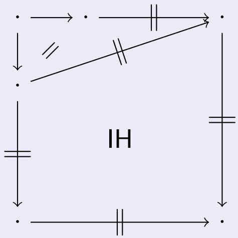
Every weakly orthogonal TRSs is confluent.

Proof Sketch.

- $\leftarrow\!\!\!\!\!\!+\!\!\!\!+ \cdot +\!\!\!\!+\!\!\!\!\!\!\rightarrow \subseteq +\!\!\!\!+\!\!\!\!\!\!\rightarrow \cdot \leftarrow\!\!\!\!\!\!+\!\!\!\!+$



interesting case

- $\rightarrow \subseteq +\!\!\!\!+\!\!\!\!\!\!\rightarrow \subseteq \rightarrow^*$

■

Theorem (Huet 1980)

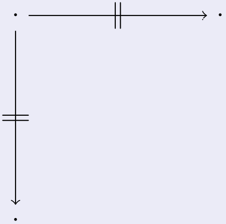
left-linearity & $\leftarrow \rtimes \rightarrow \; \subseteq \; \twoheadrightarrow\!\!\!\!\!\!\!\!+\!\!\!+ \quad \Longrightarrow \quad \mathit{CR}$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\parallel}$



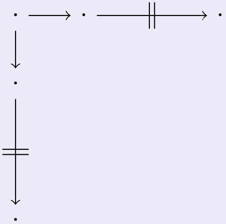
■

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; ⇻$ $\implies$ *CR*

Proof Sketch.

- $⇺ \cdot ⇻ \subseteq ⇻ \cdot ⇺$



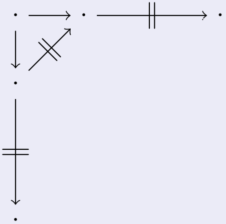
interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \parallel\!\!\!\to \quad\Longrightarrow\quad CR$

Proof Sketch.

- $\leftarrow\!\!\!\parallel \cdot \parallel\!\!\!\to \;\subseteq\; \parallel\!\!\!\to \cdot \leftarrow\!\!\!\parallel$



interesting case

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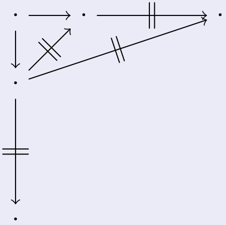

interesting case

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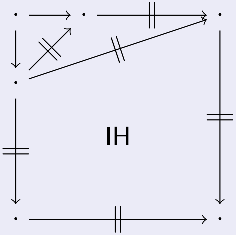
Proof Sketch.

- $\mathrel{\leftarrow\!\!\!\!\!\!\|} \cdot \mathrel{\|\!\!\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\|\!\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\!\|}$



interesting case

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow} \quad\Longrightarrow\quad CR$

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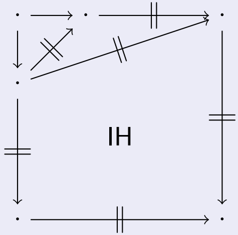
- $\mathrel{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow} \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot \mathrel{\leftarrow\!\!\!\!\!\parallel}$



interesting case

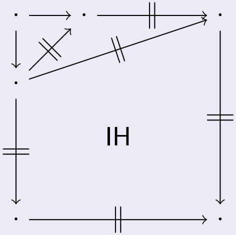
- $\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\!\!\rightarrow} \;\subseteq\; \rightarrow^*$

Theorem (Huet 1980)

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{-\!\!\|\!\!\to} \quad\Longrightarrow\quad \mathit{CR}$

Proof Sketch.

- $\mathrel{\leftarrow\!\!\|\!\!-} \cdot \mathrel{-\!\!\|\!\!\to} \;\subseteq\; \mathrel{-\!\!\|\!\!\to} \cdot \mathrel{\leftarrow\!\!\|\!\!-}$



interesting case

- $\to \;\subseteq\; \mathrel{-\!\!\|\!\!\to} \;\subseteq\; \to^*$

■

Open Problem

left-linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\leftarrow\!\!\|\!\!-} \quad\Longrightarrow\quad \mathit{CR}$?

Example

$$
\begin{array}{rcl}
\mathsf{f}(\mathsf{g}(x,\mathsf{a},\mathsf{b})) & \to & x \\
\mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d})),x,y) & \to & \mathsf{h}(\mathsf{k}(x),\mathsf{k}(y)) \\
\mathsf{k}(\mathsf{a}) & \to & \mathsf{c} \\
\mathsf{k}(\mathsf{b}) & \to & \mathsf{d}
\end{array}
$$

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The only critical pair is:

$$\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}), \mathsf{k}(\mathsf{b}))) \quad \leftarrow \quad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c}, \mathsf{d})), \mathsf{a}, \mathsf{b})) \quad \to \quad \mathsf{f}(\mathsf{h}(\mathsf{c}, \mathsf{d}))$$

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$$\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}),\mathsf{k}(\mathsf{b}))) \quad \leftarrow \quad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d})),\mathsf{a},\mathsf{b})) \quad \to \quad \mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d}))$$

Since $\mathcal{R}$ is left-linear and $\mathsf{f}(\mathsf{h}(\mathsf{k}(\mathsf{a}),\mathsf{k}(\mathsf{b}))) \not\twoheadrightarrow \mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d}))$, the system is confluent.

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \implies CR$

Theorem (Huet 1980)

$$\textit{linearity} \quad \& \quad \leftarrow\rtimes\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\!\leftarrow) \quad \Longrightarrow \quad CR$$

Notation

- $s \leftarrow\bowtie\rightarrow t$ for critical pair originating from overlap $\langle l_1 \to r_1, \epsilon, l_2 \to r_2 \rangle$
- $\leftarrow\cdot\!\!\rtimes\rightarrow \;=\; \leftarrow\rtimes\rightarrow \setminus \leftarrow\bowtie\rightarrow$

Theorem (Huet 1980)

linearity & $\leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\leftarrow) \quad\Longrightarrow\quad CR$

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Theorem (Toyama 1988)

left-linearity & $\leftarrow\!\cdot\!\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\rightarrow}$ & $\leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{+\!\!\!\!\rightarrow} \cdot {}^{*}\!\leftarrow \quad\Longrightarrow\quad CR$

Theorem (Huet 1980)

$\textit{linearity} \quad \& \quad \leftarrow\!\rtimes\!\rightarrow \;\subseteq\; (\rightarrow^{=} \cdot {}^{*}\!\!\leftarrow) \cap (\rightarrow^{*} \cdot {}^{=}\!\!\leftarrow) \quad \Longrightarrow \quad CR$

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Theorem (Toyama 1988)

$\textit{left-linearity} \quad \& \quad \leftarrow\!\cdot\!\rtimes\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow} \quad \& \quad \leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\parallel\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow \quad \Longrightarrow \quad CR$

Theorem (van Oostrom 1996)

$\textit{left-linearity} \quad \& \quad \leftarrow\!\cdot\!\rtimes\!\rightarrow \;\subseteq\; {\color{red}\mathrel{\circ\!\!\!\rightarrow}} \quad \& \quad \leftarrow\!\bowtie\!\rightarrow \;\subseteq\; \mathrel{\circ\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow \quad \Longrightarrow \quad CR$

Confluence Tool

- ACP