Outline

Equational Reasoning and Completion

Definition

An equational system (ES) is pair $(\Sigma, \mathcal{E})$ consisting of

- Σ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\Sigma, \mathcal{X})$

Example

ES $(\Sigma, \mathcal{E})$ with signature Σ

0 (constant) s (unary) $+$ (binary, infix)

and equations $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

Inference Rules

[r]	reflexivity	$\dfrac{}{t \approx t}$	$\forall\ t$
[s]	symmetry	$\dfrac{s \approx t}{t \approx s}$	
[t]	transitivity	$\dfrac{s \approx t,\ t \approx u}{s \approx u}$	
[a]	application	$\dfrac{}{l\sigma \approx r\sigma}$	$\forall\ l \approx r \in \mathcal{E}\ \forall\ \sigma$
[c]	congruence	$\dfrac{s_1 \approx t_1,\ \ldots,\ s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$	$\forall$ n-ary f

Definition

$\mathcal{E} \vdash s \approx t \quad (s \approx_{\mathcal{E}} t)$ if equation $s \approx t$ is derivable.

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{[\mathrm{a}]\ \dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))} \qquad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[\mathrm{a}]}{\mathsf{s}(0 + s(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathrm{c}]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\,[\mathrm{t}]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[\mathrm{c}]$$

Outline

Definition

For every ES $(\Sigma, \mathcal{E})$ we define TRS $\mathcal{R}_\mathcal{E} = (\Sigma, R)$ with rules:

$$R = \{\ell \to r \mid \ell \approx r \in \mathcal{E} \text{ or } r \approx \ell \in \mathcal{E}\}$$

For $\leftrightarrow_{\mathcal{R}_\mathcal{E}}$ we write $\leftrightarrow_\mathcal{E}$ for short.

Example

ES $\mathcal{E} = \{0 + y \approx y,\ \mathsf{s}(x) + y \approx \mathsf{s}(x + y)\}$

$$\begin{aligned} R = \{\ & 0 + y \to y, \\ & y \to 0 + y, \\ & \mathsf{s}(x) + y \to \mathsf{s}(x + y), \\ & \mathsf{s}(x + y) \to \mathsf{s}(x) + y\ \} \end{aligned}$$

Theorem

$$\forall\ ES\ \mathcal{E} \quad \mathcal{E} \vdash s \approx t \quad \Longleftrightarrow \quad s \leftrightarrow^*_\mathcal{E} t$$

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{\dfrac{}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))}[\mathrm{a}] \quad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}[\mathrm{a}]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}[\mathrm{c}]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}[\mathrm{t}]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}[\mathrm{c}]$$

$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow^*_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow_{\mathcal{E}} \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \leftrightarrow_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

Outline

Definitions

- An equation $s \approx t$ is valid in the Σ-algebra $\mathcal{A}$ $\quad (\mathcal{A} \vDash s \approx t)$ if

$$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

  for all assignments α.
- Σ-algebra $\mathcal{A}$ is model of ES $(\Sigma, \mathcal{E})$ if $\mathcal{A} \vDash s \approx t$ for all equations $s \approx t \in \mathcal{E}$.

Example

- $\mathcal{A} = (\mathbb{N}, [\cdot])$ with $\quad [0] = 0, \quad [\mathsf{s}](x) = x + 1, \quad [+](x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, [\cdot])$ with $\quad [0] = 1, \quad [\mathsf{s}](x) = x + 1, \quad [+](x, y) = 2x + y$
- ES $\mathcal{E}$

$$\begin{aligned} 0 + y &\approx y \\ \mathsf{s}(x) + y &\approx \mathsf{s}(x + y) \end{aligned}$$

$\mathcal{A}$ is model of $\mathcal{E}$ $\qquad$ $\mathcal{B}$ is no model of $\mathcal{E}$

Definition

- $\mathcal{E} \models s \approx t$ if equation $s \approx t$ is valid in all models of $\mathcal{E}$.
- The equational theory of $\mathcal{E}$ consists of all equations $s \approx t$ such that $\mathcal{E} \models s \approx t$.

Example

- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

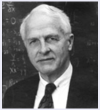
$$\mathcal{E} \models \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \qquad \mathcal{E} \not\models x + y \approx y + x$$

- Consider model $\mathcal{C} = (\mathbb{N}, [\cdot])$ with $[0] = 0$, $[s](x) = x$, $[+](x, y) = y$.

Theorem (Birkhoff)

Equational reasoning is sound and complete

$$\forall \text{ ES } \mathcal{E}: \quad \mathcal{E} \vdash s \approx t \iff \mathcal{E} \models s \approx t$$



Outline

Definition

An ES $\mathcal{E}$ is consistent if $\exists$ terms s, t such that $s \not\approx_{\mathcal{E}} t$.

Validity Problem

instance: ES $(\Sigma, \mathcal{E})$ terms $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$
question: $\mathcal{E} \vDash s \approx t$? (or equivalently $\mathcal{E} \vdash s \approx t$?)

Theorem

The validity problem is undecidable.

Example (Combinatory Logic)

$$\mathsf{I} \cdot x \approx x$$
$$(\mathsf{K} \cdot x) \cdot y \approx x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$$

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $\mathcal{E} \models s \approx t$?

Theorem

The validity problem is *decidable* *for ES $\mathcal{E}$ if $\exists$ finite TRS R such that*

1. *R is complete (confluent and terminating)*
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

Example (Group Theory)

signature	e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)		
ES	$\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$		$\mathcal{E}$
theorems	$\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$		
TRS	$\mathsf{e} \cdot x \to x$	$x \cdot \mathsf{e} \to x$	R
	$x^- \cdot x \to \mathsf{e}$	$x \cdot x^- \to \mathsf{e}$	
	$(x \cdot y) \cdot z \to x \cdot (y \cdot z)$	$x^{--} \to x$	
	$\mathsf{e}^- \to \mathsf{e}$	$(x \cdot y)^- \to y^- \cdot x^-$	
	$x^- \cdot (x \cdot y) \to y$	$x \cdot (x^- \cdot y) \to y$	

- R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}} \quad \Longrightarrow \quad \mathcal{E}$ has decidable validity problem

Example (Group Theory)

signature	e (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)	
ES	$\mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$	$\mathcal{E}$
theorems	$\mathsf{e}^{-} \downarrow_R \mathsf{e} \qquad (x \cdot y)^{-} \downarrow_R y^{-} \cdot z^{-}$	
TRS	see rules below	R

$$
\begin{array}{rclcrcl}
\mathsf{e} \cdot x & \to & x & \qquad & x \cdot \mathsf{e} & \to & x \\
x^{-} \cdot x & \to & \mathsf{e} & & x \cdot x^{-} & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & & x^{--} & \to & x \\
\mathsf{e}^{-} & \to & \mathsf{e} & & (x \cdot y)^{-} & \to & y^{-} \cdot x^{-} \\
x^{-} \cdot (x \cdot y) & \to & y & & x \cdot (x^{-} \cdot y) & \to & y
\end{array}
$$

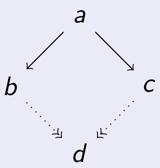
- R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem
- How to compute R ? completion

Newman's Lemma

SN & WCR $\Rightarrow$ CR

Definition (WCR)

$\forall a, b, c$



peak

$\exists d$

Question

How to prove WCR ?

Example (Non-confluence)

$$\begin{array}{lllll} \rho_1: & F(G(x), y) & \rightarrow & x \\ \rho_2: & G(a) & \rightarrow & b \end{array}$$

$F(G(a), x) \rightarrow_{\rho_1} a$
$F(G(a), x) \rightarrow_{\rho_2} F(b, x)$
The term a and $F(b, x)$ are normal forms, without a common reduct.

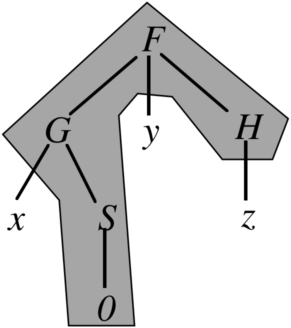
Example (Confluence)

$$\begin{array}{lllll} \rho_1: & F(x, y) & \rightarrow & x \\ \rho_2: & G(a) & \rightarrow & b \end{array}$$

$F(G(a), x) \rightarrow_{\rho_1} G(a)$
$F(G(a), x) \rightarrow_{\rho_2} F(b, x)$
Now $G(a)$ and $F(b, x)$ can both be further reduced to the common reduct b:
$F(G(a), x) \rightarrow_{\rho_1} G(a) \rightarrow_{\rho_2} b$
$F(G(a), x) \rightarrow_{\rho_2} F(b, x) \rightarrow_{\rho_1} b$

Pattern

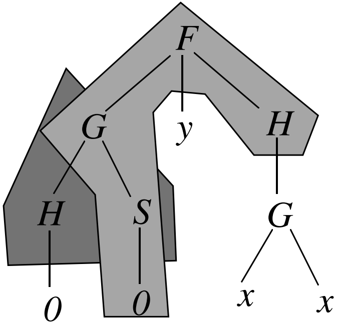
Pattern of a redex or rewrite rule:



$$F(G(x, S(0)), y, H(z)) \rightarrow G(x, z)$$

Overlap

Overlapping redex occurrences:



$$\begin{array}{llll} \rho_1 : & F(G(x, S(0)), y, H(z)) & \rightarrow & x \\ \rho_2 : & G(H(x), S(y)) & \rightarrow & y \end{array}$$

Definition (Redex occurrence)

A redex occurrence in a term t is a pair $\langle p, \ell \to r \rangle$ such that

- $\ell \to r \in R$, and
- ℓ matches $t|_p$.

We write $s \xrightarrow{r} t$ if r is the redex occurrence contracted in the step $s \to t$.

Definition (Pattern)

The pattern of a redex occurrence $\langle p, \ell \to r \rangle$ in t are the symbol occurrences in t at positions pq with $q \in \mathcal{P}os_\Sigma(\ell)$ a non-variable position in ℓ.

Definition (Overlap)

Two redex occurrences:

- overlap if their pattern share at least one symbol occurrence,
- are parallel if their positions are parallel, and
- are nested if they are not parallel and do not overlap.

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

$\mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})})$	$\overline{\mathsf{f}(\mathsf{a}, \mathsf{g}(\underline{\mathsf{g}(\mathsf{b})}))}$	$\overline{\mathsf{f}(\mathsf{a}, \underline{\mathsf{g}(\mathsf{b})})}$
$\swarrow_1 \quad \searrow_2$	$\swarrow_{21} \quad \searrow_{\epsilon}$	$\swarrow_2 \quad \searrow_{\epsilon}$
f(c,g(b)) f(g(b),c)	f(a,g(c)) f(g(b),g(b))	f(a,c) f(b,b)
parallel redexes	nested redexes / variable overlap	overlapping redexes
non-critical	non-critical	critical

Definitions

- An overlap is a triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are rewrite rules
  2. $p \in \mathcal{P}os_\Sigma(\ell_2)$ a non-variable position in ℓ_2
  3. ℓ_1 and $\ell_2|_p$ have a common instance

     Let σ and τ substitutions such that $\ell_1\sigma = \ell_2|_p\tau$ is the mgci.
     (w.l.o.g. $dom(\tau) = \mathcal{V}ar(\ell_2|_p)$ and $\mathcal{V}ar((\ell_2|_p)\tau) \cap \mathcal{V}ar(\ell_2[\square]_p) = \emptyset$)
  4. if $p = \epsilon$ then $(\ell_1 \to r_1) \neq (\ell_2 \to r_2)$

  Then we have steps $\ell_2\tau[r_1\sigma]_p \leftarrow \ell_2\tau[\ell_1\sigma]_p = \ell_2\tau \to r_2\tau$.

  We call $\ell_2\tau[r_1\sigma]_p \leftarrow\rtimes\to r_2\tau$ critical pair.

- A critical pair $s \leftarrow\rtimes\to t$ is convergent if $s \downarrow t$.

Critical Pair Lemma (Huet 1980)

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

Example

$$e \cdot x \to x \qquad x^- \cdot x \to e \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, e \cdot x \to x,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^- \cdot x \to e,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\rtimes\rightarrow e \cdot (y \cdot z)$ convergent
- $e \cdot z \leftarrow\rtimes\rightarrow u^- \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\rtimes\rightarrow (u \cdot v) \cdot (w \cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

A terminating TRS is confluent $\iff$ all critical pairs are convergent.

Outline

Example

TRS R

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

- SN ?

Example

TRS R

①	$x+0$	$\to$	x		②	$x-0$	$\to$	x
③	$x+\mathsf{s}(y)$	$\to$	$\mathsf{s}(x+y)$		④	$x-\mathsf{s}(y)$	$\to$	$\mathsf{p}(x-y)$
⑤	$\mathsf{p}(\mathsf{s}(x))$	$\to$	x		⑥	$\mathsf{s}(\mathsf{p}(x))$	$\to$	x
⑦	$\mathsf{s}(x+\mathsf{p}(y))$	$\to$	$x+y$		⑧	$\mathsf{p}(x-\mathsf{p}(y))$	$\to$	$x-y$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\overline{x+\underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{⑥} x+y \qquad \xrightarrow{③} \mathsf{s}(x+\mathsf{p}(y)) \qquad x+y \xleftarrow{⑦} \mathsf{s}(x+\mathsf{p}(y))$$

$$\overline{x-\underline{\mathsf{s}(\mathsf{p}(y))}} \quad \xrightarrow{⑥} x-y \qquad \xrightarrow{④} \mathsf{p}(x-\mathsf{p}(y)) \qquad x-y \xleftarrow{⑧} \mathsf{p}(x-\mathsf{p}(y))$$

$$\overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \quad \xrightarrow{⑥} \mathsf{p}(x) \qquad \xrightarrow{⑤} \mathsf{p}(x) \qquad \mathsf{p}(x) = \mathsf{p}(x)$$

$$\overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \quad \xrightarrow{⑤} \mathsf{s}(x) \qquad \xrightarrow{⑥} \mathsf{s}(x) \qquad \mathsf{s}(x) = \mathsf{s}(x)$$

Example (cont'd)

- added rewrite rules

$$\textcircled{7}\quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad\qquad \textcircled{8}\quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

  preserve termination and do not change $\overset{*}{\leftrightarrow}$

- new critical pairs

$$\mathsf{p}(x + y) \xleftarrow{\textcircled{7}} \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \xrightarrow{\textcircled{5}} x + \mathsf{p}(y) \qquad \mathsf{p}(x + y) \xleftarrow[\textcircled{9}]{} x + \mathsf{p}(y)$$

$$\mathsf{s}(x + y) \xleftarrow{\textcircled{5}} \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \xrightarrow{\textcircled{7}} x + \mathsf{s}(y) \qquad \mathsf{s}(x + y) \xleftarrow[\textcircled{2}]{} x + \mathsf{s}(y)$$

$$\mathsf{s}(x - y) \xleftarrow{\textcircled{8}} \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \xrightarrow{\textcircled{6}} x - \mathsf{p}(y) \qquad \mathsf{s}(x - y) \xleftarrow[\textcircled{10}]{} x - \mathsf{p}(y)$$

$$\mathsf{p}(x - y) \xleftarrow{\textcircled{5}} \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \xrightarrow{\textcircled{8}} x - \mathsf{s}(y) \qquad \mathsf{p}(x - y) \xleftarrow[\textcircled{4}]{} x - \mathsf{s}(y)$$

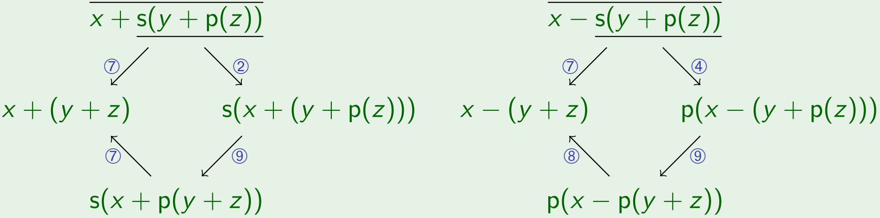
Example (cont'd)

- added rewrite rules

$$⑨ \quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩ \quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

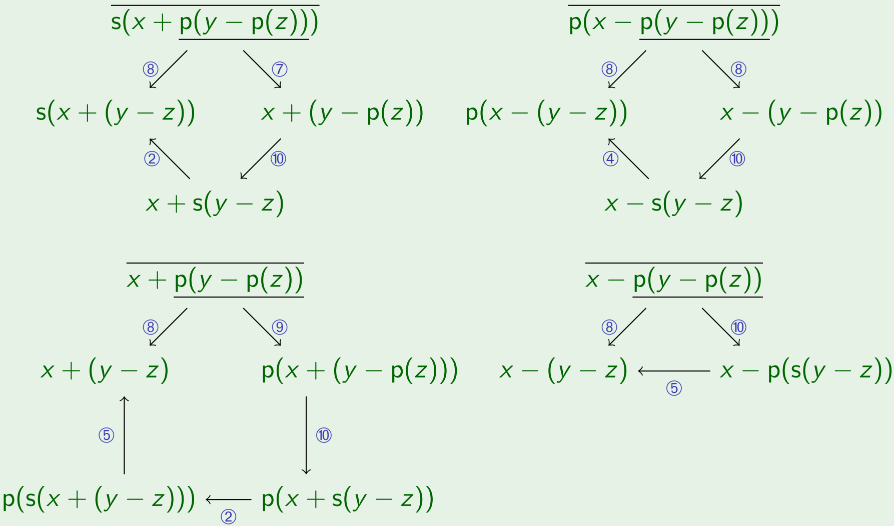
  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs



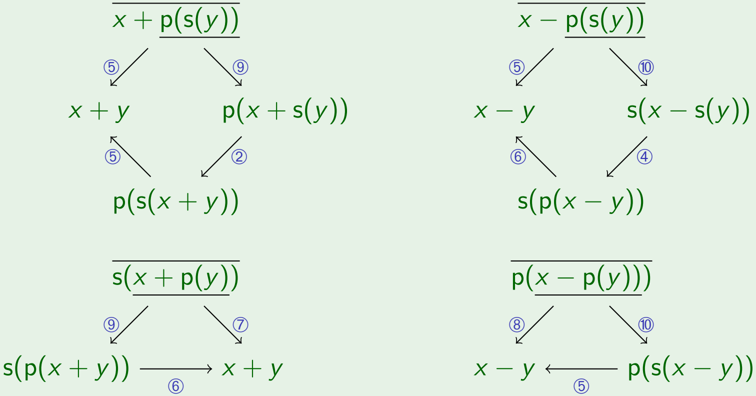
Example (cont'd)

- new critical pairs



Example (cont'd)

- new critical pairs



Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\overset{*}{\underset{\mathcal{S}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

Outline

Knuth–Bendix Completion Procedure (Simple Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete TRS R such that $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*}$

$R := \varnothing \quad C := \mathcal{E}$

while $C \neq \varnothing$ do

choose $s \approx t \in C \quad C := C \setminus \{s \approx t\}$

compute R-normal forms s' and t' of s and t

if $s' \neq t'$ then

if $s' > t'$ then

$\alpha := s' \quad \beta := t'$

else if $t' > s'$ then

$\alpha := t' \quad \beta := s'$

else

failure (try another reduction order $>$)

$R := R \cup \{\alpha \to \beta\}$

$C := C \cup \{e \in \mathrm{CP}(R) \mid \alpha \to \beta$ was used to generate $e\}$

Invariants

1. $R \subseteq >$
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}} \cup \overset{*}{\underset{C}{\longleftrightarrow}}$
3. equations in $\mathrm{CP}(R) \setminus C$ are convergent with respect to R

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow\rtimes\rightarrow \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow\rtimes\rightarrow \mathsf{g}(x)$$

- no orientation possible $\implies$ failure

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(h(a)) \to f(b)$$
$$g(a) \to b \qquad g(h(h(a))) \to f(f(b))$$
$$g(h(h(h(a)))) \to f(f(f(b)))$$

- LPO with precedence $a > f > g > h > b$ $\cdots$

- critical pairs

$$f(b) \leftarrow \rtimes \rightarrow g(h(a))$$
$$f(f(b)) \leftarrow \rtimes \rightarrow g(h(h(a)))$$
$$f(f(f(b))) \leftarrow \rtimes \rightarrow g(h(h(h(a))))$$

Efficient Completion

Removal of Redundant Rules

In every step of the we are allowed to remove redundant rules from R.

That is, rules $\ell \to r \in R$ such that:

$$\ell \to^*_{R'} r$$

where $R' = R \setminus \{\ell \to r\}$.

Observation

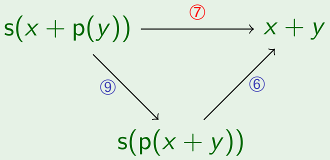
- less rewrite rules $\implies$ less critical pairs

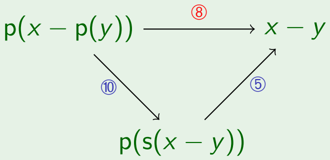
Example

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0$	$\to$	x		②	$x - 0$	$\to$	x
③	$x + \mathsf{s}(y)$	$\to$	$\mathsf{s}(x + y)$		④	$x - \mathsf{s}(y)$	$\to$	$\mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x))$	$\to$	x		⑥	$\mathsf{s}(\mathsf{p}(x))$	$\to$	x
⑦	$\mathsf{s}(x + \mathsf{p}(y))$	$\to$	$x + y$		⑧	$\mathsf{p}(x - \mathsf{p}(y))$	$\to$	$x - y$
⑨	$x + \mathsf{p}(y)$	$\to$	$\mathsf{p}(x + y)$		⑩	$x - \mathsf{p}(y)$	$\to$	$\mathsf{s}(x - y)$

rewrite rules ⑦ and ⑧ are redundant:



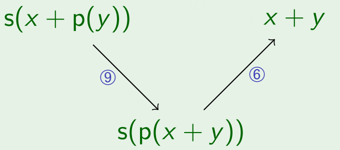


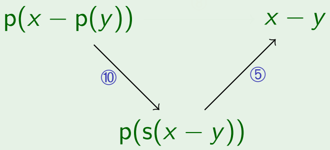
Example

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x - 0 \to x$
③	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

rewrite rules ⑦ and ⑧ are redundant:





Group Example

Group example:

$$\begin{array}{lcl} e \cdot x & = & x \\ I(x) \cdot x & = & e \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \end{array}$$

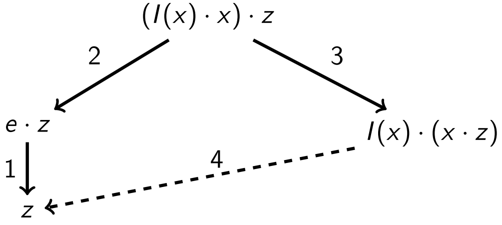
First we give these equations a 'sensible' orientation:

$$\begin{array}{llcl} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

$$
\begin{array}{llll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z)
\end{array}
$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$ not convergent



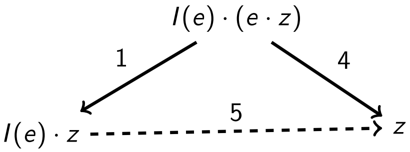
Add a rule:

$$(4) \qquad I(x) \cdot (x \cdot z) \rightarrow z$$

$$\begin{array}{llll} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\ (4) & I(x) \cdot (x \cdot z) & \rightarrow & z \end{array}$$

Overlap of the rules (1) and (4), (2) and (4), (3) and (4), (4) and (4).

We start with (4) and (1) with critical pair: $\langle I(e) \cdot z,\ z \rangle$

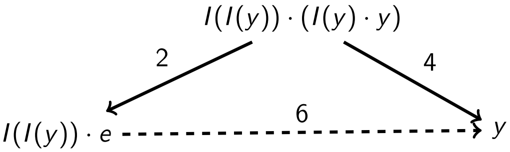


Add a rule:

(5) $I(e) \cdot z \rightarrow z$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z
\end{array}
$$

Overlap between rules (4) and (2) with critical pair:

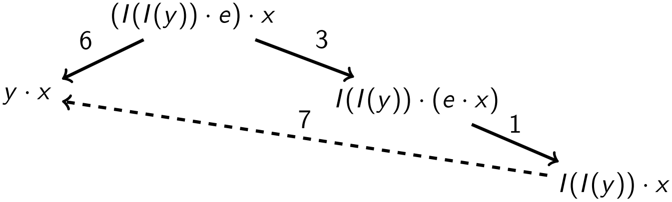


Add a rule:

(6) $I(I(y)) \cdot e \rightarrow y$

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y
\end{array}
$$

Overlap between rules (3) and (6) with critical pair:

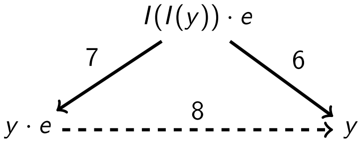


Add a rule:

$$(7) \qquad I(I(y)) \cdot x \rightarrow y \cdot x$$

$$
\begin{array}{lll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x
\end{array}
$$

Overlap between rules (7) and (6) with critical pair:

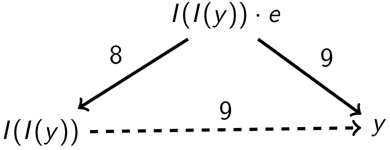


Add a rule:

(8) $y \cdot e \rightarrow y$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\
(8) & y \cdot e & \rightarrow & y
\end{array}
$$

Overlap between rules (8) and (6) with critical pair:



Add a rule:

(9) $I(I(y)) \rightarrow y$

Removing Redundant Reduction Rules

$$
\begin{array}{lll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y \\
(7) & I(I(y)) \cdot x & \to & y \cdot x \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y
\end{array}
$$

Rule (7) is now no longer necessary:

$$I(I(y)) \cdot x \ \to_9 \ y \cdot x$$

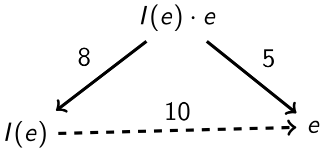
Likewise for rule (6):

$$I(I(y)) \cdot e \ \to_9 \ y \cdot e \ \to_8 \ y$$

We remove the rules (6) and (7).

$$
\begin{array}{llll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y
\end{array}
$$

Overlap between rules (8) and (5) with critical pair:



Add a rule:

(10) $I(e) \rightarrow e$

Removing Another Redundant Reduction Rule

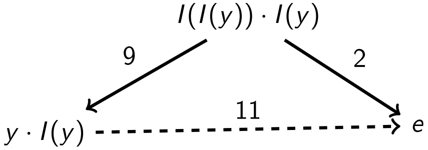
$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}
$$

Rule (5) can now be dropped:

$$I(e) \cdot z \;\rightarrow_{10}\; e \cdot z \;\rightarrow_{1}\; z$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}$$

Overlap between rules (9) and (2) with critical pair:

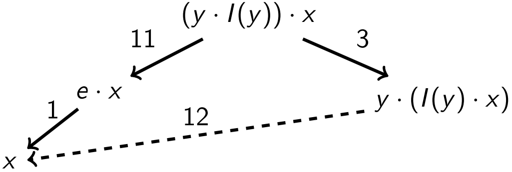


Add a rule:

$$(11) \qquad y \cdot I(y) \rightarrow e$$

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e
\end{array}
$$

Overlap between rules (11) and (3) with critical pair:

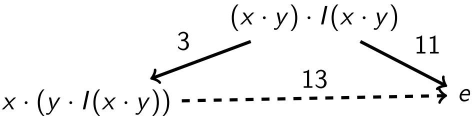


Add a rule:

$$(12) \qquad y \cdot (I(y) \cdot x) \rightarrow x$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x
\end{array}$$

Another overlap between rules (11) and (3) with critical pair:

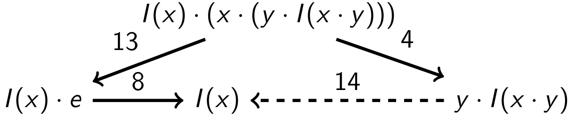


Add a rule:

(13) $\quad x \cdot (y \cdot I(x \cdot y)) \rightarrow e$

$$
\begin{array}{lll}
(1) & e \cdot x & \rightarrow \; x \\
(2) & I(x) \cdot x & \rightarrow \; e \\
(3) & (x \cdot y) \cdot z & \rightarrow \; x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow \; z \\
(8) & y \cdot e & \rightarrow \; y \\
(9) & I(I(y)) & \rightarrow \; y \\
(10) & I(e) & \rightarrow \; e \\
(11) & y \cdot I(y) & \rightarrow \; e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow \; x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow \; e
\end{array}
$$

Overlap between rules (4) and (13) with critical pair:



$$(14) \qquad y \cdot I(x \cdot y) \rightarrow I(x)$$

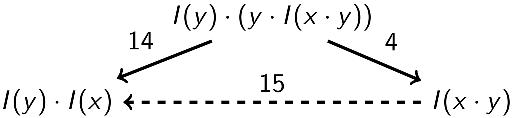
$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow & e \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x)
\end{array}
$$

Removing the redundant reduction rule (13):

$$x \cdot (y \cdot I(x \cdot y)) \rightarrow_{14} x \cdot I(x) \rightarrow_{11} e$$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x)
\end{array}
$$

Overlap between rules (14) and (4) with critical pair:



$$(15) \qquad I(x \cdot y) \rightarrow I(y) \cdot I(x)$$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x) \\
(15) & I(x \cdot y) & \rightarrow & I(y) \cdot I(x)
\end{array}
$$

Removing the redundant reduction rule (14):

$$
y \cdot I(x \cdot y) \ \rightarrow_{15} \ y \cdot (I(y) \cdot I(x)) \ \rightarrow_{12} \ I(x)
$$

Finally!

We have constructed a complete rewrite system:

- all critical pairs are convergent
- the system is terminating

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(15) & I(x \cdot y) & \rightarrow & I(y) \cdot I(x)
\end{array}
$$

Completion Tools

- Waldmeister
- Slothrop
- mkbTT
- KBCV