- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

- Overview
- Equational Reasoning
- Equational Reasoning and Term Rewriting
- Semantics
- Validity Problem
- Critical Pairs
- Completion
- Efficient Completion

Equational Reasoning and Completion

Definition

An equational system (ES) is pair $(\Sigma, \mathcal{E})$ consisting of

- Σ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\Sigma, \mathcal{X})$

Example

ES $(\Sigma, \mathcal{E})$ with signature Σ

$$0 \text{ (constant)} \qquad \mathsf{s} \text{ (unary)} \qquad + \text{ (binary, infix)}$$

and equations $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

Inference Rules

[r]	reflexivity	$\dfrac{}{t \approx t}$	$\forall\ t$
[s]	symmetry	$\dfrac{s \approx t}{t \approx s}$	
[t]	transitivity	$\dfrac{s \approx t,\ t \approx u}{s \approx u}$	
[a]	application	$\dfrac{}{l\sigma \approx r\sigma}$	$\forall\ l \approx r \in \mathcal{E}\ \forall\ \sigma$
[c]	congruence	$\dfrac{s_1 \approx t_1,\ \ldots,\ s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$	$\forall\ n$-ary f

Definition

$\mathcal{E} \vdash s \approx t \quad (s \approx_{\mathcal{E}} t)$ if equation $s \approx t$ is derivable.

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{\dfrac{[a]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))} \quad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[a]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[c]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\,[t]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[c]$$

Outline

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  - Procedure
- Efficient Completion
- Group Example

Definition

For every ES $(\Sigma, \mathcal{E})$ we define TRS $\mathcal{R}_{\mathcal{E}} = (\Sigma, R)$ with rules:

$$R = \{\ell \to r \mid \ell \approx r \in \mathcal{E} \text{ or } r \approx \ell \in \mathcal{E}\}$$

For $\leftrightarrow_{\mathcal{R}_{\mathcal{E}}}$ we write $\leftrightarrow_{\mathcal{E}}$ for short.

Example

ES $\mathcal{E} = \{0 + y \approx y,\ \mathsf{s}(x) + y \approx \mathsf{s}(x + y)\}$

$$\begin{aligned} R = \{\ & 0 + y \to y, \\ & y \to 0 + y, \\ & \mathsf{s}(x) + y \to \mathsf{s}(x + y), \\ & \mathsf{s}(x + y) \to \mathsf{s}(x) + y\ \} \end{aligned}$$

Theorem

$\forall\ ES\ \mathcal{E} \quad \mathcal{E} \vdash s \approx t \quad \Longleftrightarrow \quad s \leftrightarrow_{\mathcal{E}}^{*} t$

Example

ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{E} \vdash \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\dfrac{\dfrac{\dfrac{[a]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(0 + \mathsf{s}(0))} \quad \dfrac{\dfrac{}{0 + \mathsf{s}(0) \approx \mathsf{s}(0)}\,[a]}{\mathsf{s}(0 + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(0))}\,[c]}{\mathsf{s}(0) + \mathsf{s}(0) \approx \mathsf{s}(\mathsf{s}(0))}\,[t]}{\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))}\,[c]$$

$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow_{\mathcal{E}}^{*} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

$$\mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \leftrightarrow_{\mathcal{E}} \mathsf{s}(\mathsf{s}(0 + \mathsf{s}(0))) \leftrightarrow_{\mathcal{E}} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

Semantics

Definitions

- An equation $s \approx t$ is valid in the Σ-algebra $\mathcal{A}$ $\quad (\mathcal{A} \models s \approx t)$ if

$$[s, \alpha]_{\mathcal{A}} = [t, \alpha]_{\mathcal{A}}$$

  for all assignments α.

- Σ-algebra $\mathcal{A}$ is model of ES $(\Sigma, \mathcal{E})$ if $\mathcal{A} \models s \approx t$ for all equations $s \approx t \in \mathcal{E}$.

Example

- $\mathcal{A} = (\mathbb{N}, [\cdot])$ with $\quad [0] = 0, \quad [s](x) = x + 1, \quad [+](x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, [\cdot])$ with $\quad [0] = 1, \quad [s](x) = x + 1, \quad [+](x, y) = 2x + y$
- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

$\mathcal{A}$ is model of $\mathcal{E}$ $\qquad$ $\mathcal{B}$ is no model of $\mathcal{E}$

Definition

- $\mathcal{E} \models s \approx t$ if equation $s \approx t$ is valid in all models of $\mathcal{E}$.
- The equational theory of $\mathcal{E}$ consists of all equations $s \approx t$ such that $\mathcal{E} \models s \approx t$.

Example

- ES $\mathcal{E}$

$$0 + y \approx y$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

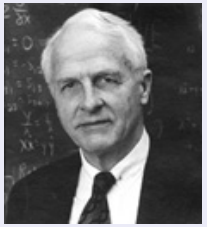
$$\mathcal{E} \models \mathsf{s}(\mathsf{s}(0) + \mathsf{s}(0)) \approx \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \qquad \mathcal{E} \not\models x + y \approx y + x$$

- Consider model $\mathcal{C} = (\mathbb{N}, [\cdot])$ with $[0] = 0, \quad [s](x) = x, \quad [+](x, y) = y.$

Theorem (Birkhoff)

Equational reasoning is sound and complete

$$\forall \text{ ES } \mathcal{E}: \quad \mathcal{E} \vdash s \approx t \iff \mathcal{E} \models s \approx t$$



Validity Problem

Definition

An ES $\mathcal{E}$ is consistent if $\exists$ terms s, t such that $s \not\approx_{\mathcal{E}} t$.

Validity Problem

instance: ES $(\Sigma, \mathcal{E})$ terms $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$
question: $\mathcal{E} \vDash s \approx t$? (or equivalently $\mathcal{E} \vdash s \approx t$?)

Theorem

The validity problem is undecidable.

Example (Combinatory Logic)

$$\mathsf{I} \cdot x \approx x$$
$$(\mathsf{K} \cdot x) \cdot y \approx x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$$

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $\mathcal{E} \models s \approx t$?

Theorem

The validity problem is *decidable* *for ES* $\mathcal{E}$ *if* $\exists$ *finite TRS* R *such that*

1. R *is complete (confluent and terminating)*
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$

TRS

$$\begin{array}{rcl} \mathsf{e} \cdot x \to x & \qquad & x \cdot \mathsf{e} \to x \\ x^- \cdot x \to \mathsf{e} & & x \cdot x^- \to \mathsf{e} \\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) & & x^{--} \to x \\ \mathsf{e}^- \to \mathsf{e} & & (x \cdot y)^- \to y^- \cdot x^- \\ x^- \cdot (x \cdot y) \to y & & x \cdot (x^- \cdot y) \to y \end{array} \qquad R$$

- R is complete and $\overset{*}{\underset{\mathcal{E}}{\leftrightarrow}} = \overset{*}{\underset{R}{\leftrightarrow}} \implies \mathcal{E}$ has decidable validity problem

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \downarrow_R \mathsf{e} \qquad (x \cdot y)^- \downarrow_R y^- \cdot z^-$

TRS

$$\begin{array}{rcl} \mathsf{e} \cdot x \to x & & x \cdot \mathsf{e} \to x \\ x^- \cdot x \to \mathsf{e} & & x \cdot x^- \to \mathsf{e} \\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) & & x^{--} \to x \\ \mathsf{e}^- \to \mathsf{e} & & (x \cdot y)^- \to y^- \cdot x^- \\ x^- \cdot (x \cdot y) \to y & & x \cdot (x^- \cdot y) \to y \end{array} \qquad R$$

- R is complete and $\overset{*}{\underset{\mathcal{E}}{\leftrightarrow}} = \overset{*}{\underset{R}{\leftrightarrow}} \implies \mathcal{E}$ has decidable validity problem
- How to compute R ? completion

Critical Pairs

Newman's Lemma

SN & WCR $\Rightarrow$ CR

Definition (WCR)

$\forall a, b, c$

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr, dotted, "*"] & & c \arrow[dl, dotted, "*"'] \\ & d & \end{tikzcd}$$

peak

$\exists d$

Question

How to prove WCR ?

Example (Non-confluence)

$$\begin{array}{lll} \rho_1: & F(G(x), y) & \to & x \\ \rho_2: & G(a) & \to & b \end{array}$$

$F(G(a), x) \to_{\rho_1} a$
$F(G(a), x) \to_{\rho_2} F(b, x)$

The term a and $F(b, x)$ are normal forms, without a common reduct.

Example (Confluence)

$$\begin{array}{lll} \rho_1: & F(x, y) & \to & x \\ \rho_2: & G(a) & \to & b \end{array}$$

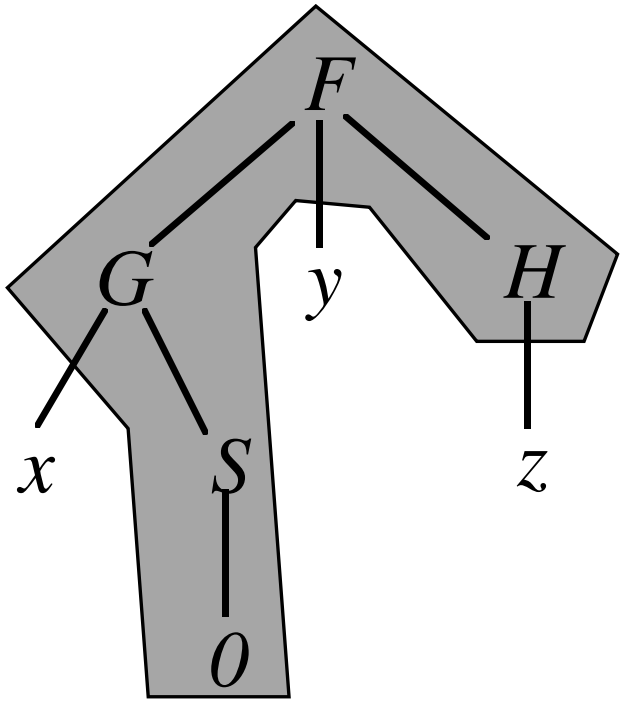
$F(G(a), x) \to_{\rho_1} G(a)$
$F(G(a), x) \to_{\rho_2} F(b, x)$

Now $G(a)$ and $F(b, x)$ can both be further reduced to the common reduct b:

$F(G(a), x) \to_{\rho_1} G(a) \to_{\rho_2} b$
$F(G(a), x) \to_{\rho_2} F(b, x) \to_{\rho_1} b$

Pattern

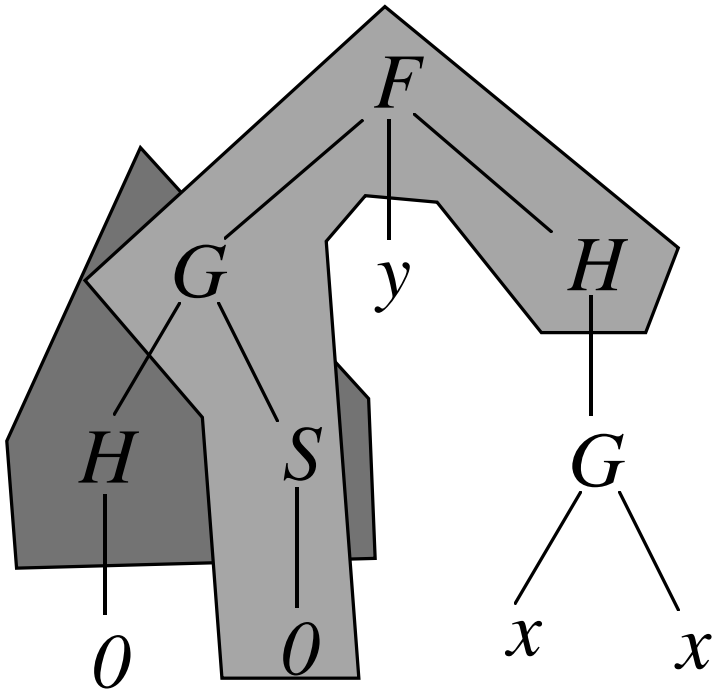
Pattern of a redex or rewrite rule:



$$F(G(x, S(0)), y, H(z)) \to G(x, z)$$

Overlap

Overlapping redex occurrences:



$$\begin{array}{lll} \rho_1: & F(G(x, S(0)), y, H(z)) & \to \quad x \\ \rho_2: & G(H(x), S(y)) & \to \quad y \end{array}$$

Definition (Redex occurrence)

A redex occurrence in a term t is a pair $\langle p,\ \ell \to r\rangle$ such that

- $\ell \to r \in R$, and
- ℓ matches $t|_p$.

We write $s \xrightarrow{r} t$ if r is the redex occurrence contracted in the step $s \to t$.

Definition (Pattern)

The pattern of a redex occurrence $\langle p,\ \ell \to r\rangle$ in t are the symbol occurrences in t at positions pq with $q \in \mathcal{P}\mathrm{os}_\Sigma(\ell)$ a non-variable position in ℓ.

Definition (Overlap)

Two redex occurrences:

- overlap if their pattern share at least one symbol occurrence,
- are parallel if their positions are parallel, and
- are nested if they are not parallel and do not overlap.

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

three peaks

$$\begin{tikzcd}[column sep=small]
 & \mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})}) \arrow[dl, "1"'] \arrow[dr, "2"] & & \overline{\mathsf{f}(\mathsf{a}, \mathsf{g}(\underline{\mathsf{g}(\mathsf{b})}))} \arrow[dl, "21"'] \arrow[dr, "\epsilon"] & & \overline{\mathsf{f}(\mathsf{a}, \underline{\mathsf{g}(\mathsf{b})})} \arrow[dl, "2"'] \arrow[dr, "\epsilon"] & \\
\mathsf{f}(\mathsf{c},\mathsf{g}(\mathsf{b})) & & \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{c}) & \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{c})) & \mathsf{f}(\mathsf{g}(\mathsf{b}),\mathsf{g}(\mathsf{b})) & \mathsf{f}(\mathsf{a},\mathsf{c}) & \mathsf{f}(\mathsf{b},\mathsf{b})
\end{tikzcd}$$

parallel redexes	nested redexes / variable overlap	overlapping redexes
non-critical	non-critical	critical

Definitions

- An overlap is a triple $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ such that
  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are rewrite rules
  2. $p \in \mathcal{P}os_\Sigma(\ell_2)$ a non-variable position in ℓ_2
  3. ℓ_1 and $\ell_2|_p$ have a common instance

     Let σ and τ substitutions such that $\ell_1\sigma = \ell_2|_p\tau$ is the mgci.
     (w.l.o.g. $dom(\tau) = \mathcal{V}ar(\ell_2|_p)$ and $\mathcal{V}ar((\ell_2|_p)\tau) \cap \mathcal{V}ar(\ell_2[\Box]_p) = \emptyset$)
  4. if $p = \epsilon$ then $(\ell_1 \to r_1) \neq (\ell_2 \to r_2)$

  Then we have steps $\ell_2\tau[r_1\sigma]_p \leftarrow \ell_2\tau[\ell_1\sigma]_p = \ell_2\tau \to r_2\tau$.

  We call $\ell_2\tau[r_1\sigma]_p \leftarrow \rtimes \to r_2\tau$ critical pair.
- A critical pair $s \leftarrow \rtimes \to t$ is convergent if $s \downarrow t$.

Critical Pair Lemma (Huet 1980)

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

Example

$$\mathrm{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathrm{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathrm{e} \cdot x \to x,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^{-} \cdot x \to \mathrm{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z),\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow (\mathrm{e} \cdot y) \cdot z \to \mathrm{e} \cdot (y \cdot z)$ convergent
- $\mathrm{e} \cdot z \leftarrow (x^{-} \cdot x) \cdot z \to x^{-} \cdot (x \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow ((u \cdot v) \cdot w) \cdot z \to (u \cdot v) \cdot (w \cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

A terminating TRS is confluent $\iff$ all critical pairs are convergent.

Completion

Example

TRS R

①	$x + 0$	$\to$	x	②	$x - 0 \to x$
③	$x + \mathsf{s}(y)$	$\to$	$\mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x))$	$\to$	x	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

- SN ?

Example

TRS R

①	$x + 0$	$\to$	x	②	$x - 0 \to x$
③	$x + \mathsf{s}(y)$	$\to$	$\mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x))$	$\to$	x	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y))$	$\to$	$x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd}
 & \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "⑥"'] \arrow[dr, "③"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y)) \arrow[ll, "⑦"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "⑥"'] \arrow[dr, "④"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y)) \arrow[ll, "⑧"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "⑥"'] \arrow[dr, "⑤"] & \\
\mathsf{p}(x) \arrow[rr, equal] & & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "⑤"'] \arrow[dr, "⑥"] & \\
\mathsf{s}(x) \arrow[rr, equal] & & \mathsf{s}(x)
\end{tikzcd}$$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- added rewrite rules ⑦ and ⑧ preserve termination and do not change $\overset{*}{\longleftrightarrow}$
- new critical pairs

$$\begin{tikzcd}
 & \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "⑦"'] \arrow[dr, "⑤"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y) \arrow[ll, "⑨"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑦"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "②"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "⑧"'] \arrow[dr, "⑥"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y) \arrow[ll, "⑩"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑧"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "④"]
\end{tikzcd}$$

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑦	$\mathsf{s}(x + \mathsf{p}(y)) \to x + y$	⑧	$\mathsf{p}(x - \mathsf{p}(y)) \to x - y$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- added rewrite rules

  ⑨ $x + \mathsf{p}(y) \to \mathsf{p}(x + y)$ ⑩ $x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- rules ⑦ and ⑧ are now superfluous:
  - $\mathsf{s}(x + \mathsf{p}(y)) \xrightarrow{⑨} \mathsf{s}(\mathsf{p}(x + y)) \xrightarrow{⑥} x + y$
  - $\mathsf{p}(x - \mathsf{p}(y)) \xrightarrow{⑩} \mathsf{p}(\mathsf{s}(x - y)) \xrightarrow{⑤} x - y$

  Thus we can drop the rules ⑦ and ⑧.

Example (cont'd)

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- new critical pairs

$$\begin{tikzcd}
 & x + \underline{\mathsf{p}(\mathsf{s}(y))} \arrow[dl, "⑤"'] \arrow[dr, "⑨"] & \\
x + y & & \mathsf{p}(x + \mathsf{s}(y)) \arrow[dl, "②"] \\
 & \mathsf{p}(\mathsf{s}(x + y)) \arrow[ul, "⑤"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x - \underline{\mathsf{p}(\mathsf{s}(y))} \arrow[dl, "⑤"'] \arrow[dr, "⑩"] & \\
x - y & & \mathsf{s}(x - \mathsf{s}(y)) \arrow[dl, "④"] \\
 & \mathsf{s}(\mathsf{p}(x - y)) \arrow[ul, "⑥"] &
\end{tikzcd}$$

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$

TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑨, ⑩\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$
⑨	$x + \mathsf{p}(y) \to \mathsf{p}(x + y)$	⑩	$x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent
- $\overset{*}{\underset{\mathcal{S}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

Completion Procedure

Knuth–Bendix Completion Procedure (Simple Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete TRS R such that $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

$R := \varnothing \quad C := \mathcal{E}$

while $C \neq \varnothing$ do

  choose $s \approx t \in C \quad C := C \setminus \{s \approx t\}$

  compute R-normal forms s' and t' of s and t

  if $s' \neq t'$ then

    if $s' > t'$ then

      $\alpha := s' \quad \beta := t'$

    else if $t' > s'$ then

      $\alpha := t' \quad \beta := s'$

    else

      failure (try another reduction order $>$)

    $R := R \cup \{\alpha \to \beta\}$

    $C := C \cup \{e \in \mathrm{CP}(R) \mid \alpha \to \beta$ was used to generate $e\}$

Invariants

1. $R \subseteq >$
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R \cup C}{\longleftrightarrow}}$
3. equations in $\mathrm{CP}(R) \setminus C$ are convergent with respect to R

Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x, y) \to \mathsf{g}(x) \qquad \mathsf{f}(x, y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow \rtimes \to \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow \rtimes \to \mathsf{g}(x)$$

- no orientation possible $\implies$ failure

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$
$$\mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a})))) \to \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b})))$$
$$\cdots$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))$$

Efficient Completion

Removal of Redundant Rules

In every step of the we are allowed to remove redundant rules from R.

That is, rules $\ell \to r \in R$ such that:

$$\ell \to^*_{R'} r$$

where $R' = R \setminus \{\ell \to r\}$.

Observation

- less rewrite rules $\implies$ less critical pairs

Group Example

Group Example

Group example:

$$\begin{array}{lcl} e \cdot x & = & x \\ I(x) \cdot x & = & e \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \end{array}$$

First we give these equations a 'sensible' orientation:

$$\begin{array}{llcl} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

$$
\begin{array}{llll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z)
\end{array}
$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$ not convergent

$$
\begin{tikzcd}
 & (I(x) \cdot x) \cdot z \arrow[dl, "2"'] \arrow[dr, "3"] & \\
e \cdot z \arrow[d, "1"'] & & I(x) \cdot (x \cdot z) \arrow[dll, dashed, "4"'] \\
z & &
\end{tikzcd}
$$

Add a rule:

(4) $\quad I(x) \cdot (x \cdot z) \to z$

$$
\begin{array}{llcl}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z
\end{array}
$$

Overlap of the rules (1) and (4), (2) and (4), (3) and (4), (4) and (4).

We start with (4) and (1) with critical pair: $\langle I(e) \cdot z,\ z \rangle$

$$
\begin{tikzcd}
 & I(e) \cdot (e \cdot z) \arrow[dl, "1"'] \arrow[dr, "4"] & \\
I(e) \cdot z \arrow[rr, dashed, "5"] & & z
\end{tikzcd}
$$

Add a rule:

(5) $\quad I(e) \cdot z \to z$

$$
\begin{array}{llll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z
\end{array}
$$

Overlap between rules (4) and (2) with critical pair:

$$
\begin{tikzcd}
 & I(I(y)) \cdot (I(y) \cdot y) \arrow[dl, "2"'] \arrow[dr, "4"] & \\
I(I(y)) \cdot e \arrow[rr, dashed, "6"] & & y
\end{tikzcd}
$$

Add a rule:

$$(6) \qquad I(I(y)) \cdot e \to y$$

$$
\begin{array}{llll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y
\end{array}
$$

Overlap between rules (3) and (6) with critical pair:

$$
\begin{tikzcd}
 & (I(I(y)) \cdot e) \cdot x \arrow[dl, "6"'] \arrow[dr, "3"] & \\
y \cdot x & & I(I(y)) \cdot (e \cdot x) \arrow[dl, dashed, "7"'] \arrow[dr, "1"] & \\
 & & & I(I(y)) \cdot x
\end{tikzcd}
$$

Add a rule:

$$(7) \qquad I(I(y)) \cdot x \to y \cdot x$$

$$
\begin{array}{llcl}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y \\
(7) & I(I(y)) \cdot x & \to & y \cdot x
\end{array}
$$

Overlap between rules (7) and (6) with critical pair:

$$
\begin{tikzcd}
 & I(I(y)) \cdot e \arrow[dl, "7"'] \arrow[dr, "6"] & \\
y \cdot e \arrow[rr, dashed, "8"] & & y
\end{tikzcd}
$$

Add a rule:

(8) $\quad y \cdot e \to y$

$$
\begin{array}{llcl}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y \\
(7) & I(I(y)) \cdot x & \to & y \cdot x \\
(8) & y \cdot e & \to & y
\end{array}
$$

Overlap between rules (8) and (6) with critical pair:

$$
\begin{tikzcd}
 & I(I(y)) \cdot e \arrow[dl, "8"'] \arrow[dr, "6"] & \\
I(I(y)) \arrow[rr, dashed, "9"] & & y
\end{tikzcd}
$$

Add a rule:

(9) $\quad I(I(y)) \to y$

Removing Redundant Reduction Rules

$$
\begin{array}{llcl}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y \\
(7) & I(I(y)) \cdot x & \to & y \cdot x \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y
\end{array}
$$

Rule (7) is now no longer necessary:

$$I(I(y)) \cdot x \ \to_9 \ y \cdot x$$

Likewise for rule (6):

$$I(I(y)) \cdot e \ \to_9 \ y \cdot e \ \to_8 \ y$$

We remove the rules (6) and (7).

$$
\begin{array}{llcl}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y
\end{array}
$$

Overlap between rules (8) and (5) with critical pair:

$$
\begin{tikzcd}
 & I(e) \cdot e \arrow[dl, "8"'] \arrow[dr, "5"] & \\
I(e) \arrow[rr, dashed, "10"] & & e
\end{tikzcd}
$$

Add a rule:

(10) $\quad I(e) \to e$

Removing Another Redundant Reduction Rule

(1)	$e \cdot x$	$\rightarrow$	x
(2)	$I(x) \cdot x$	$\rightarrow$	e
(3)	$(x \cdot y) \cdot z$	$\rightarrow$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\rightarrow$	z
(5)	$I(e) \cdot z$	$\rightarrow$	z
(8)	$y \cdot e$	$\rightarrow$	y
(9)	$I(I(y))$	$\rightarrow$	y
(10)	$I(e)$	$\rightarrow$	e

Rule (5) can now be dropped:

$$I(e) \cdot z \ \rightarrow_{10} \ e \cdot z \ \rightarrow_{1} \ z$$

(1)	$e \cdot x$	$\to$	x
(2)	$I(x) \cdot x$	$\to$	e
(3)	$(x \cdot y) \cdot z$	$\to$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\to$	z
(8)	$y \cdot e$	$\to$	y
(9)	$I(I(y))$	$\to$	y
(10)	$I(e)$	$\to$	e

Overlap between rules (9) and (2) with critical pair:

$$\begin{tikzcd}
 & I(I(y)) \cdot I(y) \arrow[dl, "9"'] \arrow[dr, "2"] & \\
y \cdot I(y) \arrow[rr, dashed, "11"] & & e
\end{tikzcd}$$

Add a rule:

(11) $\quad y \cdot I(y) \to e$

(1)	$e \cdot x$	$\to$	x
(2)	$I(x) \cdot x$	$\to$	e
(3)	$(x \cdot y) \cdot z$	$\to$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\to$	z
(8)	$y \cdot e$	$\to$	y
(9)	$I(I(y))$	$\to$	y
(10)	$I(e)$	$\to$	e
(11)	$y \cdot I(y)$	$\to$	e

Overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
 & (y \cdot I(y)) \cdot x \arrow[dl, "11"'] \arrow[dr, "3"] & \\
e \cdot x \arrow[d, "1"'] & & y \cdot (I(y) \cdot x) \arrow[dll, dashed, "12"'] \\
x & &
\end{tikzcd}$$

Add a rule:

(12) $\quad y \cdot (I(y) \cdot x) \to x$

(1)	$e \cdot x$	$\to$	x
(2)	$I(x) \cdot x$	$\to$	e
(3)	$(x \cdot y) \cdot z$	$\to$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\to$	z
(8)	$y \cdot e$	$\to$	y
(9)	$I(I(y))$	$\to$	y
(10)	$I(e)$	$\to$	e
(11)	$y \cdot I(y)$	$\to$	e
(12)	$y \cdot (I(y) \cdot x)$	$\to$	x

Another overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
 & (x \cdot y) \cdot I(x \cdot y) \arrow[dl, "3"'] \arrow[dr, "11"] & \\
x \cdot (y \cdot I(x \cdot y)) \arrow[rr, dashed, "13"] & & e
\end{tikzcd}$$

Add a rule:

(13) $\quad x \cdot (y \cdot I(x \cdot y)) \to e$

(1)	$e \cdot x$	$\rightarrow$	x
(2)	$I(x) \cdot x$	$\rightarrow$	e
(3)	$(x \cdot y) \cdot z$	$\rightarrow$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\rightarrow$	z
(8)	$y \cdot e$	$\rightarrow$	y
(9)	$I(I(y))$	$\rightarrow$	y
(10)	$I(e)$	$\rightarrow$	e
(11)	$y \cdot I(y)$	$\rightarrow$	e
(12)	$y \cdot (I(y) \cdot x)$	$\rightarrow$	x
(13)	$x \cdot (y \cdot I(x \cdot y))$	$\rightarrow$	e

Overlap between rules (4) and (13) with critical pair:

$$\begin{tikzcd}
 & I(x) \cdot (x \cdot (y \cdot I(x \cdot y))) \arrow[dl, "13"'] \arrow[dr, "4"] & \\
I(x) \cdot e \arrow[r, "8"] & I(x) & y \cdot I(x \cdot y) \arrow[l, dashed, "14"']
\end{tikzcd}$$

(14) $\quad y \cdot I(x \cdot y) \rightarrow I(x)$

(1)	$e \cdot x$	$\rightarrow$	x
(2)	$I(x) \cdot x$	$\rightarrow$	e
(3)	$(x \cdot y) \cdot z$	$\rightarrow$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\rightarrow$	z
(8)	$y \cdot e$	$\rightarrow$	y
(9)	$I(I(y))$	$\rightarrow$	y
(10)	$I(e)$	$\rightarrow$	e
(11)	$y \cdot I(y)$	$\rightarrow$	e
(12)	$y \cdot (I(y) \cdot x)$	$\rightarrow$	x
(13)	$x \cdot (y \cdot I(x \cdot y))$	$\rightarrow$	e
(14)	$y \cdot I(x \cdot y)$	$\rightarrow$	$I(x)$

Removing the redundant reduction rule (13):

$$x \cdot (y \cdot I(x \cdot y)) \rightarrow_{14} x \cdot I(x) \rightarrow_{11} e$$

(1)	$e \cdot x$	$\rightarrow$	x
(2)	$I(x) \cdot x$	$\rightarrow$	e
(3)	$(x \cdot y) \cdot z$	$\rightarrow$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\rightarrow$	z
(8)	$y \cdot e$	$\rightarrow$	y
(9)	$I(I(y))$	$\rightarrow$	y
(10)	$I(e)$	$\rightarrow$	e
(11)	$y \cdot I(y)$	$\rightarrow$	e
(12)	$y \cdot (I(y) \cdot x)$	$\rightarrow$	x
(14)	$y \cdot I(x \cdot y)$	$\rightarrow$	$I(x)$

Overlap between rules (14) and (4) with critical pair:

$$\begin{tikzcd}
 & I(y) \cdot (y \cdot I(x \cdot y)) \arrow[dl, "14"'] \arrow[dr, "4"] & \\
I(y) \cdot I(x) & & I(x \cdot y) \arrow[ll, dashed, "15"']
\end{tikzcd}$$

(15) $\quad I(x \cdot y) \rightarrow I(y) \cdot I(x)$

(1)	$e \cdot x$	$\rightarrow$	x
(2)	$I(x) \cdot x$	$\rightarrow$	e
(3)	$(x \cdot y) \cdot z$	$\rightarrow$	$x \cdot (y \cdot z)$
(4)	$I(x) \cdot (x \cdot z)$	$\rightarrow$	z
(8)	$y \cdot e$	$\rightarrow$	y
(9)	$I(I(y))$	$\rightarrow$	y
(10)	$I(e)$	$\rightarrow$	e
(11)	$y \cdot I(y)$	$\rightarrow$	e
(12)	$y \cdot (I(y) \cdot x)$	$\rightarrow$	x
(14)	$y \cdot I(x \cdot y)$	$\rightarrow$	$I(x)$
(15)	$I(x \cdot y)$	$\rightarrow$	$I(y) \cdot I(x)$

Removing the redundant reduction rule (14):

$$y \cdot I(x \cdot y) \ \rightarrow_{15} \ y \cdot (I(y) \cdot I(x)) \ \rightarrow_{12} \ I(x)$$

Finally!

We have constructed a complete rewrite system:

- all critical pairs are convergent
- the system is terminating

$$
\begin{array}{llcl}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y \\
(10) & I(e) & \to & e \\
(11) & y \cdot I(y) & \to & e \\
(12) & y \cdot (I(y) \cdot x) & \to & x \\
(15) & I(x \cdot y) & \to & I(y) \cdot I(x)
\end{array}
$$

Completion Tools

- Waldmeister
- Slothrop
- mkbTT
- KBCV