- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

Equational Reasoning and Completion

Definition

An equational system (ES) is pair $(\Sigma, \mathcal{E})$ consisting of

- Σ signature
- $\mathcal{E}$ set of equations between terms in $\mathcal{T}(\Sigma, \mathcal{X})$

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Example

ES $(\Sigma, \mathcal{E})$ with signature Σ

$$\mathsf{0} \text{ (constant)} \qquad \mathsf{s} \text{ (unary)} \qquad + \text{ (binary, infix)}$$

and equations $\mathcal{E}$

$$\begin{aligned} \mathsf{0} + y &\approx y \\ \mathsf{s}(x) + y &\approx \mathsf{s}(x + y) \end{aligned}$$

Inference Rules

[r] reflexivity $$\frac{}{t \approx t} \qquad \forall\ t$$

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[c]	congruence	$\dfrac{s_1 \approx t_1,\ \ldots,\ s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$	$\forall$ n-ary f

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Definition

$\mathcal{E} \vdash s \approx t$ $\quad (s \approx_{\mathcal{E}} t)$ if equation $s \approx t$ is derivable.

Example

ES $\mathcal{E}$

$$\begin{aligned} 0 + y &\approx y \\ \mathsf{s}(x) + y &\approx \mathsf{s}(x + y) \end{aligned}$$

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- Equational Reasoning
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Definition

For every ES $(\Sigma, \mathcal{E})$ we define TRS $\mathcal{R}_{\mathcal{E}} = (\Sigma, R)$ with rules:

$$R = \{\ell \to r \mid \ell \approx r \in \mathcal{E} \text{ or } r \approx \ell \in \mathcal{E}\}$$

For $\leftrightarrow_{\mathcal{R}_{\mathcal{E}}}$ we write $\leftrightarrow_{\mathcal{E}}$ for short.

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Theorem

$$\forall\ ES\ \mathcal{E} \quad \mathcal{E} \vdash s \approx t \quad \iff \quad s \leftrightarrow_{\mathcal{E}}^{*} t$$

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Outline

Definitions

- An equation $s \approx t$ is valid in the Σ-algebra $\mathcal{A}$ $\quad (\mathcal{A} \vDash s \approx t)$ $\quad$ if

$$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

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- $\mathcal{A} = (\mathbb{N}, [\cdot])$ with $\quad [0] = 0, \quad [\mathsf{s}](x) = x + 1, \quad [+](x, y) = x + y$
- $\mathcal{B} = (\mathbb{N}, [\cdot])$ with $\quad [0] = 1, \quad [\mathsf{s}](x) = x + 1, \quad [+](x, y) = 2x + y$
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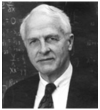

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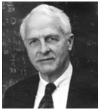

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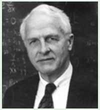

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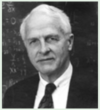

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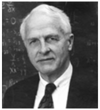

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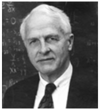

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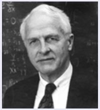
Theorem (Birkhoff)

Equational reasoning is sound and complete

$$\forall \text{ ES } \mathcal{E}: \quad \mathcal{E} \vdash s \approx t \iff \mathcal{E} \vDash s \approx t$$



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An ES $\mathcal{E}$ is consistent if $\exists$ terms s, t such that $s \not\approx_{\mathcal{E}} t$.

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Validity Problem

instance: ES $(\Sigma, \mathcal{E})$ terms $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$
question: $\mathcal{E} \vDash s \approx t$? (or equivalently $\mathcal{E} \vdash s \approx t$?)

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Example (Combinatory Logic)

$$\mathsf{I} \cdot x \approx x$$
$$(\mathsf{K} \cdot x) \cdot y \approx x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \approx (x \cdot z) \cdot (y \cdot z)$$

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The validity problem is decidable for ES $\mathcal{E}$ if $\exists$ finite TRS R such that

1. *R is complete (confluent and terminating)*

Validity Problem

instance: ES $\mathcal{E}$ terms s, t
question: $\mathcal{E} \vDash s \approx t$?

Theorem

The validity problem is decidable *for ES $\mathcal{E}$ if $\exists$ finite TRS R such that*

1. *R is complete (confluent and terminating)*
2. $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*}$

Example (Group Theory)

signature e (constant) $^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e} \qquad (x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot z^-$

TRS

$$\begin{array}{rrr}
\mathsf{e} \cdot x \to x & x \cdot \mathsf{e} \to x & R \\
x^- \cdot x \to \mathsf{e} & x \cdot x^- \to \mathsf{e} & \\
(x \cdot y) \cdot z \to x \cdot (y \cdot z) & x^{--} \to x & \\
\mathsf{e}^- \to \mathsf{e} & (x \cdot y)^- \to y^- \cdot x^- & \\
x^- \cdot (x \cdot y) \to y & x \cdot (x^- \cdot y) \to y &
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TRS

$$\begin{array}{rcl}
\mathsf{e} \cdot x \to x & \qquad x \cdot \mathsf{e} \to x & \qquad R \\
x^{-} \cdot x \to \mathsf{e} & \qquad x \cdot x^{-} \to \mathsf{e} & \\
(x \cdot y) \cdot z \to x \cdot (y \cdot z) & \qquad x^{--} \to x & \\
\mathsf{e}^{-} \to \mathsf{e} & \qquad (x \cdot y)^{-} \to y^{-} \cdot x^{-} & \\
x^{-} \cdot (x \cdot y) \to y & \qquad x \cdot (x^{-} \cdot y) \to y &
\end{array}$$

- R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}} \implies \mathcal{E}$ has decidable validity problem

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TRS $\qquad R$

$$
\begin{aligned}
\mathsf{e} \cdot x &\to x & x \cdot \mathsf{e} &\to x \\
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\end{aligned}
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$$\begin{array}{rcr}
\mathsf{e} \cdot x \to x & \qquad & x \cdot \mathsf{e} \to x \\
x^{-} \cdot x \to \mathsf{e} & & x \cdot x^{-} \to \mathsf{e} \\
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x^{-} \cdot (x \cdot y) \to y & & x \cdot (x^{-} \cdot y) \to y
\end{array} \qquad R$$

- R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*} \quad \Longrightarrow \quad \mathcal{E}$ has decidable validity problem
- How to compute R ?

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signature e (constant) $\,^-$ (unary, postfix) $\cdot$ (binary, infix)

ES $\mathsf{e} \cdot x \approx x \qquad x^- \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

theorems $\mathsf{e}^- \downarrow_R \mathsf{e} \qquad (x \cdot y)^- \downarrow_R y^- \cdot z^-$

TRS

$$\begin{array}{rclcrcl}
\mathsf{e} \cdot x & \to & x & \qquad & x \cdot \mathsf{e} & \to & x \\
x^- \cdot x & \to & \mathsf{e} & & x \cdot x^- & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & & x^{--} & \to & x \\
\mathsf{e}^- & \to & \mathsf{e} & & (x \cdot y)^- & \to & y^- \cdot x^- \\
x^- \cdot (x \cdot y) & \to & y & & x \cdot (x^- \cdot y) & \to & y
\end{array} \qquad R$$

- R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*} \implies \mathcal{E}$ has decidable validity problem
- How to compute R ? completion

Newman's Lemma

SN & WCR $\Rightarrow$ CR

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SN & WCR $\quad\Rightarrow\quad$ CR

Definition (WCR)

$$\begin{tikzcd}
\forall a, b, c & & a \arrow[dl] \arrow[dr] & \\
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Question

How to prove WCR ?

Example (Non-confluence)

$$\begin{array}{llcl} \rho_1 : & F(G(x), y) & \to & x \\ \rho_2 : & G(a) & \to & b \end{array}$$

$F(G(a), x) \to_{\rho_1} G(a)$
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Now $G(a)$ and $F(b, x)$ can both be further reduced to the common reduct b:

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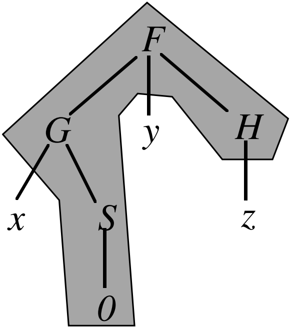
Pattern

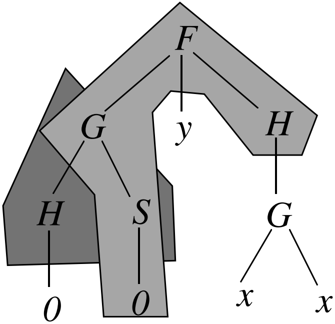
Pattern of a redex or rewrite rule:



$$F(G(x, S(0)), y, H(z)) \to G(x, z)$$

Overlap

Overlapping redex occurrences:



$$\begin{array}{lll} \rho_1: & F(G(x,S(0)),y,H(z)) & \to \quad x \\ \rho_2: & G(H(x),S(y)) & \to \quad y \end{array}$$

Definition (Redex occurrence)

A redex occurrence in a term t is a pair $\langle p, \ell \to r\rangle$ such that

- $\ell \to r \in R$, and
- ℓ matches $t|_p$.

We write $s \xrightarrow{r} t$ if r is the redex occurrence contracted in the step $s \to t$.

Definition (Pattern)

The pattern of a redex occurrence $\langle p, \ell \to r\rangle$ in t are the symbol occurrences in t at positions pq with $q \in \mathcal{P}\mathrm{os}_\Sigma(\ell)$ a non-variable position in ℓ.

Definition (Overlap)

Two redex occurrences:

- overlap if their pattern share at least one symbol occurrence,
- are parallel if their positions are parallel, and
- are nested if they are not parallel and do not overlap.

Example

$$\mathsf{f}(\mathsf{a}, \mathsf{g}(x)) \to \mathsf{f}(x, x) \qquad \mathsf{g}(\mathsf{b}) \to \mathsf{c}$$

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three peaks

$$\begin{tikzcd}
& \mathsf{f}(\underline{\mathsf{g}(\mathsf{b})}, \overline{\mathsf{g}(\mathsf{b})}) \arrow[dl, "1"'] \arrow[dr, "2"] & \\
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parallel redexes

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parallel redexes

nested redexes / variable overlap

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parallel redexes nested redexes / variable overlap overlapping redexes

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parallel redexes

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nested redexes / variable overlap

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overlapping redexes

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Critical Pair Lemma (Huet 1980)

A TRS is locally confluent (WCR) $\iff$ all critical pairs are convergent.

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- $\langle\, \mathsf{e} \cdot x \to x,\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^- \cdot x \to \mathsf{e},\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^{-} \cdot x \to \mathsf{e},\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

Example

$$\mathsf{e} \cdot x \to x \qquad x^- \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^- \cdot x \to \mathsf{e},\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\!\rtimes\!\to \mathsf{e} \cdot (y \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^{-} \cdot x \to \mathsf{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\!\rtimes\!\to \mathsf{e} \cdot (y \cdot z)$
- $\mathsf{e} \cdot z \leftarrow\!\rtimes\!\to u^{-} \cdot (u \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^- \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^- \cdot x \to \mathsf{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\!\rtimes\!\rightarrow \mathsf{e} \cdot (y \cdot z)$
- $\mathsf{e} \cdot z \leftarrow\!\rtimes\!\rightarrow u^- \cdot (u \cdot z)$
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\!\rtimes\!\rightarrow (u \cdot v) \cdot (w \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^- \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^- \cdot x \to \mathsf{e},\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\!\rtimes\!\rightarrow \mathsf{e} \cdot (y \cdot z)$ convergent
- $\mathsf{e} \cdot z \leftarrow\!\rtimes\!\rightarrow u^- \cdot (u \cdot z)$
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\!\rtimes\!\rightarrow (u \cdot v) \cdot (w \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^{-} \cdot x \to \mathsf{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\!\rtimes\!\rightarrow \mathsf{e} \cdot (y \cdot z)$ convergent
- $\mathsf{e} \cdot z \leftarrow\!\rtimes\!\rightarrow u^{-} \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\!\rtimes\!\rightarrow (u \cdot v) \cdot (w \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^- \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x,\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^- \cdot x \to \mathsf{e},\ 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1,\ (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\!\rtimes\!\rightarrow \mathsf{e} \cdot (y \cdot z)$ convergent
- $\mathsf{e} \cdot z \leftarrow\!\rtimes\!\rightarrow u^- \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\!\rtimes\!\rightarrow (u \cdot v) \cdot (w \cdot z)$ convergent

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

overlaps

- $\langle\, \mathsf{e} \cdot x \to x, 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, x^{-} \cdot x \to \mathsf{e}, 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$
- $\langle\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\, 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \,\rangle$

critical pairs

- $y \cdot z \leftarrow\rtimes\rightarrow \mathsf{e} \cdot (y \cdot z)$ convergent
- $\mathsf{e} \cdot z \leftarrow\rtimes\rightarrow u^{-} \cdot (u \cdot z)$ not convergent
- $(u \cdot (v \cdot w)) \cdot z \leftarrow\rtimes\rightarrow (u \cdot v) \cdot (w \cdot z)$ convergent

Theorem (Knuth & Bendix 1970)

A terminating TRS is confluent $\iff$ all critical pairs are convergent.

Outline

- Overview
- Equational Reasoning
- Equational Reasoning and Term Rewriting
- Semantics
- Validity Problem
- Critical Pairs
- Completion
  - Example
  - Procedure
- Efficient Completion
- Group Example

Example

TRS R

$$\begin{array}{llcl@{\qquad}llcl}
① & x + 0 & \to & x & ② & x - 0 & \to & x \\
③ & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x
\end{array}$$

Example

TRS R

①	$x + 0 \rightarrow x$	②	$x - 0 \rightarrow x$
③	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

- SN ?

Example

TRS R

①	$x + 0 \rightarrow x$	②	$x - 0 \rightarrow x$
③	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$

Example

TRS R

①	$x + 0 \rightarrow x$	②	$x - 0 \rightarrow x$
③	$x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$	④	$x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \rightarrow x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ?

Example

TRS R

$$
\begin{array}{clcl}
① & x + 0 & \rightarrow & x \\
③ & x + \mathsf{s}(y) & \rightarrow & \mathsf{s}(x + y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \rightarrow & x
\end{array}
\qquad
\begin{array}{clcl}
② & x - 0 & \rightarrow & x \\
④ & x - \mathsf{s}(y) & \rightarrow & \mathsf{p}(x - y) \\
⑥ & \mathsf{s}(\mathsf{p}(x)) & \rightarrow & x
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{③}"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y))
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{④}"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y))
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{⑤}"] & \\
\mathsf{p}(x) & & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑥}"] & \\
\mathsf{s}(x) & & \mathsf{s}(x)
\end{tikzcd}$$

Example

TRS R

$$
\begin{array}{rrcl@{\qquad}rrcl}
\text{①} & x + 0 & \to & x & \text{②} & x - 0 & \to & x \\
\text{③} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & \text{④} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\text{⑤} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \text{⑥} & \mathsf{s}(\mathsf{p}(x)) & \to & x
\end{array}
$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{③}"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y))
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{④}"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y))
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{⑤}"] & \\
\mathsf{p}(x) & = & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑥}"] & \\
\mathsf{s}(x) & = & \mathsf{s}(x)
\end{tikzcd}$$

Example

TRS R

$$\begin{array}{lrcl@{\qquad}lrcl}
① & x + 0 & \to & x & ② & x - 0 & \to & x \\
③ & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
⑦ & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & & & &
\end{array}$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "⑥"'] \arrow[dr, "③"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y)) \arrow[ll, "⑦"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "⑥"'] \arrow[dr, "④"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y))
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "⑥"'] \arrow[dr, "⑤"] & \\
\mathsf{p}(x) & = & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "⑤"'] \arrow[dr, "⑥"] & \\
\mathsf{s}(x) & = & \mathsf{s}(x)
\end{tikzcd}$$

Example

TRS R

$$\begin{array}{clcl@{\qquad}clcl}
\text{①} & x + 0 & \to & x & \text{②} & x - 0 & \to & x \\
\text{③} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & \text{④} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\text{⑤} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \text{⑥} & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
\text{⑦} & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & \text{⑧} & \mathsf{p}(x - \mathsf{p}(y)) & \to & x - y
\end{array}$$

- SN (e.g.) LPO with precedence $+ > \mathsf{s}$ and $- > \mathsf{p}$
- WCR ? 4 critical pairs

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{③}"] & \\
x + y & & \mathsf{s}(x + \mathsf{p}(y)) \arrow[ll, "\text{⑦}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{x - \underline{\mathsf{s}(\mathsf{p}(y))}} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{④}"] & \\
x - y & & \mathsf{p}(x - \mathsf{p}(y)) \arrow[ll, "\text{⑧}"]
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(\mathsf{p}(x))})} \arrow[dl, "\text{⑥}"'] \arrow[dr, "\text{⑤}"] & \\
\mathsf{p}(x) \arrow[rr, phantom, "="] & & \mathsf{p}(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(\mathsf{s}(x))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑥}"] & \\
\mathsf{s}(x) \arrow[rr, phantom, "="] & & \mathsf{s}(x)
\end{tikzcd}$$

Example (cont'd)

- added rewrite rules

  ⑦ $\mathsf{s}(x + \mathsf{p}(y)) \to x + y$ ⑧ $\mathsf{p}(x - \mathsf{p}(y)) \to x - y$

  preserve termination

Example (cont'd)

- added rewrite rules

$$⑦\quad \mathsf{s}(x + \mathsf{p}(y)) \rightarrow x + y \qquad ⑧\quad \mathsf{p}(x - \mathsf{p}(y)) \rightarrow x - y$$

preserve termination and do not change $\overset{*}{\leftrightarrow}$

Example (cont'd)

- added rewrite rules

$$⑦\quad \mathsf{s}(x+\mathsf{p}(y)) \to x+y \qquad ⑧\quad \mathsf{p}(x-\mathsf{p}(y)) \to x-y$$

  preserve termination and do not change $\xleftrightarrow{*}$

- new critical pairs

$$\begin{tikzcd}[column sep=small]
 & \overline{\mathsf{p}(\underline{\mathsf{s}(x+\mathsf{p}(y))})} \arrow[dl, "\text{⑦}"'] \arrow[dr, "\text{⑤}"] & \\
\mathsf{p}(x+y) & & x+\mathsf{p}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \overline{\mathsf{s}(x+\underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑦}"] & \\
\mathsf{s}(x+y) & & x+\mathsf{s}(y)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \overline{\mathsf{s}(\underline{\mathsf{p}(x-\mathsf{p}(y))})} \arrow[dl, "\text{⑧}"'] \arrow[dr, "\text{⑥}"] & \\
\mathsf{s}(x-y) & & x-\mathsf{p}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \overline{\mathsf{p}(x-\underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑧}"] & \\
\mathsf{p}(x-y) & & x-\mathsf{s}(y)
\end{tikzcd}$$

Example (cont'd)

- added rewrite rules

$$⑦ \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad ⑧ \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

  preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "⑦"'] \arrow[dr, "⑤"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y)
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑦"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "②"]
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "⑧"'] \arrow[dr, "⑥"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y)
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑧"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "④"]
\end{tikzcd}$$

Example (cont'd)

- added rewrite rules

$$\text{⑦}\quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad \text{⑧}\quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

  preserve termination and do not change $\overset{*}{\leftrightarrow}$

- new critical pairs

$$\begin{tikzcd}
& \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "\text{⑦}"'] \arrow[dr, "\text{⑤}"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y) \arrow[ll, "\text{⑨}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑦}"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "\text{②}"]
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "\text{⑧}"'] \arrow[dr, "\text{⑥}"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑧}"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "\text{④}"]
\end{tikzcd}$$

Example (cont'd)

- added rewrite rules

$$⑦ \quad \mathsf{s}(x + \mathsf{p}(y)) \to x + y \qquad\qquad ⑧ \quad \mathsf{p}(x - \mathsf{p}(y)) \to x - y$$

preserve termination and do not change $\overset{*}{\longleftrightarrow}$

- new critical pairs

$$\begin{tikzcd}
 & \overline{\mathsf{p}(\underline{\mathsf{s}(x + \mathsf{p}(y))})} \arrow[dl, "⑦"'] \arrow[dr, "⑤"] & \\
\mathsf{p}(x + y) & & x + \mathsf{p}(y) \arrow[ll, red, "⑨"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathsf{s}(x + \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑦"] & \\
\mathsf{s}(x + y) & & x + \mathsf{s}(y) \arrow[ll, "②"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{\mathsf{s}(\underline{\mathsf{p}(x - \mathsf{p}(y))})} \arrow[dl, "⑧"'] \arrow[dr, "⑥"] & \\
\mathsf{s}(x - y) & & x - \mathsf{p}(y) \arrow[ll, red, "⑩"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathsf{p}(x - \underline{\mathsf{p}(\mathsf{s}(y))})} \arrow[dl, "⑤"'] \arrow[dr, "⑧"] & \\
\mathsf{p}(x - y) & & x - \mathsf{s}(y) \arrow[ll, "④"]
\end{tikzcd}$$

Example (cont'd)

- added rewrite rules

⑨ $x + \mathsf{p}(y) \to \mathsf{p}(x + y)$ ⑩ $x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

Example (cont'd)

- added rewrite rules

$$⑨\quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩\quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs

$$\begin{tikzcd} & \overline{x + \underline{\mathsf{s}(y + \mathsf{p}(z))}} \arrow[dl, "\text{⑦}"'] \arrow[dr, "\text{②}"] & & & \overline{x - \underline{\mathsf{s}(y + \mathsf{p}(z))}} \arrow[dl, "\text{⑦}"'] \arrow[dr, "\text{④}"] & \\ x + (y + z) & & \mathsf{s}(x + (y + \mathsf{p}(z))) & x - (y + z) & & \mathsf{p}(x - (y + \mathsf{p}(z))) \end{tikzcd}$$

Example (cont'd)

- added rewrite rules

$$⑨ \quad x + \mathsf{p}(y) \to \mathsf{p}(x + y) \qquad ⑩ \quad x - \mathsf{p}(y) \to \mathsf{s}(x - y)$$

  preserve termination (extend LPO precedence with $+ > \mathsf{p}$ and $- > \mathsf{s}$)

- new critical pairs

$$\begin{tikzcd}
 & \overline{x + \underline{\mathsf{s}(y + \mathsf{p}(z))}} \arrow[dl, "\textcircled{7}"'] \arrow[dr, "\textcircled{2}"] & \\
x + (y + z) & & \mathsf{s}(x + (y + \mathsf{p}(z))) \arrow[dl, "\textcircled{9}"] \\
 & \mathsf{s}(x + \mathsf{p}(y + z)) \arrow[ul, "\textcircled{7}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{x - \underline{\mathsf{s}(y + \mathsf{p}(z))}} \arrow[dl, "\textcircled{7}"'] \arrow[dr, "\textcircled{4}"] & \\
x - (y + z) & & \mathsf{p}(x - (y + \mathsf{p}(z))) \arrow[dl, "\textcircled{9}"] \\
 & \mathsf{p}(x - \mathsf{p}(y + z)) \arrow[ul, "\textcircled{8}"] &
\end{tikzcd}$$

Example (cont'd)

- new critical pairs

$$\begin{tikzcd} & \overline{\mathsf{s}(x + \underline{\mathsf{p}(y - \mathsf{p}(z))})} \arrow[dl, "⑧"'] \arrow[dr, "⑦"] & \\ \mathsf{s}(x + (y - z)) & & x + (y - \mathsf{p}(z)) \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\mathsf{p}(x - \underline{\mathsf{p}(y - \mathsf{p}(z))})} \arrow[dl, "⑧"'] \arrow[dr, "⑧"] & \\ \mathsf{p}(x - (y - z)) & & x - (y - \mathsf{p}(z)) \end{tikzcd}$$

$$\begin{tikzcd} & \overline{x + \underline{\mathsf{p}(y - \mathsf{p}(z))}} \arrow[dl, "⑧"'] \arrow[dr, "⑨"] & \\ x + (y - z) & & \mathsf{p}(x + (y - \mathsf{p}(z))) \end{tikzcd}$$

$$\begin{tikzcd} & \overline{x - \underline{\mathsf{p}(y - \mathsf{p}(z))}} \arrow[dl, "⑧"'] \arrow[dr, "⑩"] & \\ x - (y - z) & & x - \mathsf{p}(\mathsf{s}(y - z)) \end{tikzcd}$$

Example (cont'd)

- new critical pairs

$$\begin{tikzcd}
& \overline{\mathsf{s}(x + \underline{\mathsf{p}(y - \mathsf{p}(z))})} \arrow[dl, "⑧"'] \arrow[dr, "⑦"] & \\
\mathsf{s}(x + (y - z)) & & x + (y - \mathsf{p}(z)) \arrow[dl, "⑩"] \\
& x + \mathsf{s}(y - z) \arrow[ul, "②"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{\mathsf{p}(x - \underline{\mathsf{p}(y - \mathsf{p}(z))})} \arrow[dl, "⑧"'] \arrow[dr, "⑧"] & \\
\mathsf{p}(x - (y - z)) & & x - (y - \mathsf{p}(z)) \arrow[dl, "⑩"] \\
& x - \mathsf{s}(y - z) \arrow[ul, "④"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{x + \underline{\mathsf{p}(y - \mathsf{p}(z))}} \arrow[dl, "⑧"'] \arrow[dr, "⑨"] & \\
x + (y - z) & & \mathsf{p}(x + (y - \mathsf{p}(z))) \arrow[d, "⑩"] \\
\mathsf{p}(\mathsf{s}(x + (y - z))) \arrow[u, "⑤"] & & \mathsf{p}(x + \mathsf{s}(y - z)) \arrow[ll, "②"]
\end{tikzcd}$$

$$\begin{tikzcd}
& \overline{x - \underline{\mathsf{p}(y - \mathsf{p}(z))}} \arrow[dl, "⑧"'] \arrow[dr, "⑩"] & \\
x - (y - z) & & x - \mathsf{p}(\mathsf{s}(y - z)) \arrow[ll, "⑤"]
\end{tikzcd}$$

Example (cont'd)

- new critical pairs

$$\begin{tikzcd} & \overline{x + \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{9}"] & \\ x + y & & \mathsf{p}(x + \mathsf{s}(y)) \end{tikzcd}$$

$$\begin{tikzcd} & \overline{x - \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "\textcircled{5}"'] \arrow[dr, "\textcircled{10}"] & \\ x - y & & \mathsf{s}(x - \mathsf{s}(y)) \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\mathsf{s}(\underline{x + \mathsf{p}(y)})} \arrow[dl, "\textcircled{9}"'] \arrow[dr, "\textcircled{7}"] & \\ \mathsf{s}(\mathsf{p}(x + y)) & & x + y \end{tikzcd}$$

$$\begin{tikzcd} & \overline{\mathsf{p}(\underline{x - \mathsf{p}(y)})} \arrow[dl, "\textcircled{8}"'] \arrow[dr, "\textcircled{10}"] & \\ x - y & & \mathsf{p}(\mathsf{s}(x - y)) \end{tikzcd}$$

Example (cont'd)

- new critical pairs

$$\begin{tikzcd}
 & \overline{x + \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑨}"] & \\
x + y & & \mathsf{p}(x + \mathsf{s}(y)) \arrow[dl, "\text{②}"] \\
 & \mathsf{p}(\mathsf{s}(x + y)) \arrow[ul, "\text{⑤}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{x - \underline{\mathsf{p}(\mathsf{s}(y))}} \arrow[dl, "\text{⑤}"'] \arrow[dr, "\text{⑩}"] & \\
x - y & & \mathsf{s}(x - \mathsf{s}(y)) \arrow[dl, "\text{④}"] \\
 & \mathsf{s}(\mathsf{p}(x - y)) \arrow[ul, "\text{⑥}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{\mathsf{s}(\underline{x + \mathsf{p}(y)})} \arrow[dl, "\text{⑨}"'] \arrow[dr, "\text{⑦}"] & \\
\mathsf{s}(\mathsf{p}(x + y)) \arrow[rr, "\text{⑥}"'] & & x + y
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overline{\mathsf{p}(\underline{x - \mathsf{p}(y))})} \arrow[dl, "\text{⑧}"'] \arrow[dr, "\text{⑩}"] & \\
x - y & & \mathsf{p}(\mathsf{s}(x - y)) \arrow[ll, "\text{⑤}"]
\end{tikzcd}$$

Example (cont'd)

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$

①	$x + 0 \to x$	②	$x + \mathsf{s}(y) \to \mathsf{s}(x + y)$
③	$x - 0 \to x$	④	$x - \mathsf{s}(y) \to \mathsf{p}(x - y)$
⑤	$\mathsf{p}(\mathsf{s}(x)) \to x$	⑥	$\mathsf{s}(\mathsf{p}(x)) \to x$

Example (cont'd)

TRS $R = \{$①, ②, ③, ④, ⑤, ⑥$\}$

TRS $\mathcal{S} = \{$①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩$\}$

① $x + 0 \to x$

② $x + \mathsf{s}(y) \to \mathsf{s}(x + y)$

③ $x - 0 \to x$

④ $x - \mathsf{s}(y) \to \mathsf{p}(x - y)$

⑤ $\mathsf{p}(\mathsf{s}(x)) \to x$

⑥ $\mathsf{s}(\mathsf{p}(x)) \to x$

⑦ $\mathsf{s}(x + \mathsf{p}(y)) \to x + y$

⑧ $\mathsf{p}(x - \mathsf{p}(y)) \to x - y$

⑨ $x + \mathsf{p}(y) \to \mathsf{p}(x + y)$

⑩ $x - \mathsf{p}(y) \to \mathsf{s}(x - y)$

Example (cont'd)

TRS $R = \{$①, ②, ③, ④, ⑤, ⑥$\}$

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④ $x - \mathsf{s}(y) \to \mathsf{p}(x - y)$

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Example (cont'd)

TRS $R = \{$①, ②, ③, ④, ⑤, ⑥$\}$ TRS $\mathcal{S} = \{$①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩$\}$

① $x + 0 \to x$

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③ $x - 0 \to x$

④ $x - \mathsf{s}(y) \to \mathsf{p}(x - y)$

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- $\mathcal{S}$ is SN LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$

Example (cont'd)

TRS $R = \{$①, ②, ③, ④, ⑤, ⑥$\}$

TRS $\mathcal{S} = \{$①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩$\}$

① $x + 0 \to x$

② $x + \mathsf{s}(y) \to \mathsf{s}(x + y)$

③ $x - 0 \to x$

④ $x - \mathsf{s}(y) \to \mathsf{p}(x - y)$

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- $\mathcal{S}$ is SN — LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR — all critical pairs of $\mathcal{S}$ are convergent

Example (cont'd)

TRS $R = \{$①, ②, ③, ④, ⑤, ⑥$\}$

TRS $\mathcal{S} = \{$①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩$\}$

① $x + 0 \rightarrow x$

③ $x - 0 \rightarrow x$

⑤ $\mathsf{p}(\mathsf{s}(x)) \rightarrow x$

⑦ $\mathsf{s}(x + \mathsf{p}(y)) \rightarrow x + y$

⑨ $x + \mathsf{p}(y) \rightarrow \mathsf{p}(x + y)$

② $x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$

④ $x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$

⑥ $\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

⑧ $\mathsf{p}(x - \mathsf{p}(y)) \rightarrow x - y$

⑩ $x - \mathsf{p}(y) \rightarrow \mathsf{s}(x - y)$

- $\mathcal{S}$ is SN   LPO with precedence $+ > \mathsf{s}, \mathsf{p}$ and $- > \mathsf{s}, \mathsf{p}$
- $\mathcal{S}$ is WCR   all critical pairs of $\mathcal{S}$ are convergent
- $\xleftrightarrow[\mathcal{S}]{*} = \xleftrightarrow[R]{*}$

Outline

- Overview
- Equational Reasoning
- Equational Reasoning and Term Rewriting
- Semantics
- Validity Problem
- Critical Pairs
- Completion
  - Example
  - Procedure
- Efficient Completion
- Group Example

Knuth–Bendix Completion Procedure (Simple Version)

input ES $\mathcal{E}$ and reduction order $>$

output complete TRS R such that $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$

```
$R := \varnothing$    $C := \mathcal{E}$
while $C \neq \varnothing$ do
    choose $s \approx t \in C$    $C := C \setminus \{s \approx t\}$
    compute $R$-normal forms $s'$ and $t'$ of $s$ and $t$
    if $s' \neq t'$ then
        if $s' > t'$ then
            $\alpha := s'$    $\beta := t'$
        else if $t' > s'$ then
            $\alpha := t'$    $\beta := s'$
        else
            failure      (try another reduction order $>$)
        $R := R \cup \{\alpha \to \beta\}$
        $C := C \cup \{e \in \mathrm{CP}(R) \mid \alpha \to \beta$ was used to generate $e\}$
```

Invariants

1. $R \subseteq >$

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Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure

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Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*}$

Invariants

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Three Possibilities

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1. terminate without failure $\implies$ R is complete and $\xleftrightarrow[\mathcal{E}]{*} = \xleftrightarrow[R]{*}$
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Invariants

1. $R \subseteq >$
2. $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}} \cup \overset{*}{\underset{C}{\longleftrightarrow}}$
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Three Possibilities

Knuth-Bendix completion procedure may

1. terminate without failure $\implies$ R is complete and $\overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} = \overset{*}{\underset{R}{\longleftrightarrow}}$
2. terminate with failure
3. not terminate (divergence)

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$f(x, y) \to g(x) \qquad f(x, y) \to h(y)$$

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow\rtimes\rightarrow \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow\rtimes\rightarrow \mathsf{g}(x)$$

Three Possibilities

Knuth-Bendix completion procedure may

2. terminate with failure

Example

- rewrite rules

$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{h}(y)$$

- two critical pairs

$$\mathsf{g}(x) \leftarrow\rtimes\rightarrow \mathsf{h}(y) \qquad \mathsf{h}(y) \leftarrow\rtimes\rightarrow \mathsf{g}(x)$$

- no orientation possible $\implies$ failure

Three Possibilities

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Example

- rewrite rules

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$

Three Possibilities

Knuth-Bendix completion procedure may

3 not terminate (divergence)

Example

- rewrite rules

$$\begin{aligned} \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pair

$$\mathsf{f}(\mathsf{b}) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$f(g(x)) \to g(h(x)) \qquad g(h(a)) \to f(b)$$
$$g(a) \to b$$

- LPO with precedence $a > f > g > h > b$
- critical pair

$$f(b) \leftarrow \rtimes \rightarrow g(h(a))$$

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow \rtimes \to \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$

Three Possibilities

Knuth-Bendix completion procedure may

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Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$

$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$

$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{f}(\mathsf{b})$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \to \mathsf{f}(\mathsf{f}(\mathsf{b}))$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$\mathsf{f}(\mathsf{b}) \leftarrow \rtimes \rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{b})) \leftarrow \rtimes \rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a})))$$
$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \leftarrow \rtimes \rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))$$

Three Possibilities

Knuth-Bendix completion procedure may

3. not terminate (divergence)

Example

- rewrite rules

$$
\begin{aligned}
\mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{h}(x)) \\
\mathsf{g}(\mathsf{a}) &\to \mathsf{b}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{g}(\mathsf{h}(\mathsf{a})) &\to \mathsf{f}(\mathsf{b}) \\
\mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) &\to \mathsf{f}(\mathsf{f}(\mathsf{b})) \\
\mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a})))) &\to \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) \\
&\cdots
\end{aligned}
$$

- LPO with precedence $\mathsf{a} > \mathsf{f} > \mathsf{g} > \mathsf{h} > \mathsf{b}$
- critical pairs

$$
\begin{aligned}
\mathsf{f}(\mathsf{b}) &\leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{a})) \\
\mathsf{f}(\mathsf{f}(\mathsf{b})) &\leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{a}))) \\
\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{b}))) &\leftarrow\rtimes\rightarrow \mathsf{g}(\mathsf{h}(\mathsf{h}(\mathsf{h}(\mathsf{a}))))
\end{aligned}
$$

Efficient Completion

Removal of Redundant Rules

In every step of the we are allowed to remove redundant rules from R.

That is, rules $\ell \to r \in R$ such that:

$$\ell \to_{R'}^{*} r$$

where $R' = R \setminus \{\ell \to r\}$.

Efficient Completion

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That is, rules $\ell \to r \in R$ such that:

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where $R' = R \setminus \{\ell \to r\}$.

Observation

- less rewrite rules $\implies$ less critical pairs

Example

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$

$$\begin{array}{llcll} ① & x + 0 \to x & & ② & x - 0 \to x \\ ③ & x + \mathsf{s}(y) \to \mathsf{s}(x + y) & & ④ & x - \mathsf{s}(y) \to \mathsf{p}(x - y) \\ ⑤ & \mathsf{p}(\mathsf{s}(x)) \to x & & ⑥ & \mathsf{s}(\mathsf{p}(x)) \to x \end{array}$$

Example

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

① $x + 0 \rightarrow x$

③ $x + \mathsf{s}(y) \rightarrow \mathsf{s}(x + y)$

⑤ $\mathsf{p}(\mathsf{s}(x)) \rightarrow x$

⑦ $\mathsf{s}(x + \mathsf{p}(y)) \rightarrow x + y$

⑨ $x + \mathsf{p}(y) \rightarrow \mathsf{p}(x + y)$

② $x - 0 \rightarrow x$

④ $x - \mathsf{s}(y) \rightarrow \mathsf{p}(x - y)$

⑥ $\mathsf{s}(\mathsf{p}(x)) \rightarrow x$

⑧ $\mathsf{p}(x - \mathsf{p}(y)) \rightarrow x - y$

⑩ $x - \mathsf{p}(y) \rightarrow \mathsf{s}(x - y)$

Example

TRS $R = \{\text{①}, \text{②}, \text{③}, \text{④}, \text{⑤}, \text{⑥}\}$ TRS $\mathcal{S} = \{\text{①}, \text{②}, \text{③}, \text{④}, \text{⑤}, \text{⑥}, \text{⑦}, \text{⑧}, \text{⑨}, \text{⑩}\}$

$$\begin{array}{lrcl@{\qquad}lrcl}
\text{①} & x + 0 & \to & x & \text{②} & x - 0 & \to & x \\
\text{③} & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & \text{④} & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
\text{⑤} & \mathsf{p}(\mathsf{s}(x)) & \to & x & \text{⑥} & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
\text{⑦} & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & \text{⑧} & \mathsf{p}(x - \mathsf{p}(y)) & \to & x - y \\
\text{⑨} & x + \mathsf{p}(y) & \to & \mathsf{p}(x + y) & \text{⑩} & x - \mathsf{p}(y) & \to & \mathsf{s}(x - y)
\end{array}$$

rewrite rules ⑦ and ⑧ are redundant:

$$\begin{tikzcd}
\mathsf{s}(x + \mathsf{p}(y)) \arrow[rr, "\text{⑦}"] \arrow[dr, "\text{⑨}"'] & & x + y \\
& \mathsf{s}(\mathsf{p}(x + y)) \arrow[ur, "\text{⑥}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{p}(x - \mathsf{p}(y)) \arrow[rr, "\text{⑧}"] \arrow[dr, "\text{⑩}"'] & & x - y \\
& \mathsf{p}(\mathsf{s}(x - y)) \arrow[ur, "\text{⑤}"'] &
\end{tikzcd}$$

Example

TRS $R = \{①, ②, ③, ④, ⑤, ⑥\}$ TRS $\mathcal{S} = \{①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩\}$

$$\begin{array}{lrcllrcl}
① & x + 0 & \to & x & ② & x - 0 & \to & x \\
③ & x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) & ④ & x - \mathsf{s}(y) & \to & \mathsf{p}(x - y) \\
⑤ & \mathsf{p}(\mathsf{s}(x)) & \to & x & ⑥ & \mathsf{s}(\mathsf{p}(x)) & \to & x \\
⑦ & \mathsf{s}(x + \mathsf{p}(y)) & \to & x + y & ⑧ & \mathsf{p}(x - \mathsf{p}(y)) & \to & x - y \\
⑨ & x + \mathsf{p}(y) & \to & \mathsf{p}(x + y) & ⑩ & x - \mathsf{p}(y) & \to & \mathsf{s}(x - y)
\end{array}$$

rewrite rules ⑦ and ⑧ are redundant:

$$\begin{tikzcd}
\mathsf{s}(x + \mathsf{p}(y)) \arrow[rr, "⑦"] \arrow[dr, "⑨"'] & & x + y & \mathsf{p}(x - \mathsf{p}(y)) \arrow[rr, "⑧"] \arrow[dr, "⑩"'] & & x - y \\
& \mathsf{s}(\mathsf{p}(x + y)) \arrow[ur, "⑥"'] & & & \mathsf{p}(\mathsf{s}(x - y)) \arrow[ur, "⑤"'] &
\end{tikzcd}$$

Group Example

Group example:

$$
\begin{array}{lcl}
e \cdot x & = & x \\
I(x) \cdot x & = & e \\
(x \cdot y) \cdot z & = & x \cdot (y \cdot z)
\end{array}
$$

Group Example

Group example:

$$\begin{array}{lcl} e \cdot x & = & x \\ I(x) \cdot x & = & e \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \end{array}$$

First we give these equations a 'sensible' orientation:

$$\begin{array}{llcl} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z)
\end{array}$$

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z)
\end{array}
$$

Critical pairs:

$$\begin{array}{lllll} (1) & e \cdot x & \to & x \\ (2) & I(x) \cdot x & \to & e \\ (3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$

$$\begin{array}{lllll} (1) & e \cdot x & \to & x \\ (2) & I(x) \cdot x & \to & e \\ (3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent

$$\begin{array}{lllll} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$

$$\begin{array}{lrcl} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$ not convergent

$$\begin{array}{llcl} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$ not convergent

$$\begin{tikzcd}
 & (I(x) \cdot x) \cdot z \arrow[dl, "2"'] \arrow[dr, "3"] & \\
e \cdot z & & I(x) \cdot (x \cdot z)
\end{tikzcd}$$

$$\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z)
\end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$ not convergent

$$\begin{tikzcd}
& (I(x) \cdot x) \cdot z \arrow[dl, "2"'] \arrow[dr, "3"] & \\
e \cdot z \arrow[d, "1"'] & & I(x) \cdot (x \cdot z) \\
z & &
\end{tikzcd}$$

$$\begin{array}{llcl} (1) & e \cdot x & \to & x \\ (2) & I(x) \cdot x & \to & e \\ (3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \end{array}$$

Critical pairs:

- between (1) and (3): $\langle y \cdot z,\ e \cdot (y \cdot z) \rangle$ convergent
- between (2) and (3): $\langle e \cdot z,\ I(x) \cdot (x \cdot z) \rangle$ not convergent

$$\begin{tikzcd}
 & (I(x) \cdot x) \cdot z \arrow[dl, "2"'] \arrow[dr, "3"] & \\
e \cdot z \arrow[d, "1"'] & & I(x) \cdot (x \cdot z) \arrow[dll, dashed, "4"'] \\
z & &
\end{tikzcd}$$

Add a rule:

$$(4) \qquad I(x) \cdot (x \cdot z) \to z$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z
\end{array}$$

$$\begin{array}{lllll}(1) & e \cdot x & \to & x \\ (2) & I(x) \cdot x & \to & e \\ (3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\ (4) & I(x) \cdot (x \cdot z) & \to & z\end{array}$$

Overlap of the rules (1) and (4), (2) and (4), (3) and (4), (4) and (4).

$$\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z
\end{array}$$

Overlap of the rules (1) and (4), (2) and (4), (3) and (4), (4) and (4).

We start with (4) and (1) with critical pair: $\langle I(e) \cdot z,\ z \rangle$

$$\begin{tikzcd}
 & I(e) \cdot (e \cdot z) \arrow[dl, "1"'] \arrow[dr, "4"] & \\
I(e) \cdot z & & z
\end{tikzcd}$$

$$\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z
\end{array}$$

Overlap of the rules (1) and (4), (2) and (4), (3) and (4), (4) and (4).

We start with (4) and (1) with critical pair: $\langle I(e) \cdot z,\ z \rangle$

$$\begin{tikzcd}
& I(e) \cdot (e \cdot z) \arrow[dl, "1"] \arrow[dr, "4"] & \\
I(e) \cdot z \arrow[rr, dashed, "5"] & & z
\end{tikzcd}$$

Add a rule:

$$(5) \qquad I(e) \cdot z \rightarrow z$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z
\end{array}$$

Overlap between rules (4) and (2) with critical pair:

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z
\end{array}$$

Overlap between rules (4) and (2) with critical pair:

$$\begin{tikzcd}
 & I(I(y)) \cdot (I(y) \cdot y) \arrow[dl, "2"'] \arrow[dr, "4"] & \\
I(I(y)) \cdot e & & y
\end{tikzcd}$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z
\end{array}$$

Overlap between rules (4) and (2) with critical pair:

$$\begin{tikzcd}
& I(I(y)) \cdot (I(y) \cdot y) \arrow[dl, "2"] \arrow[dr, "4"] & \\
I(I(y)) \cdot e \arrow[rr, dashed, "6"] & & y
\end{tikzcd}$$

Add a rule:

(6) $I(I(y)) \cdot e \rightarrow y$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y
\end{array}
$$

Overlap between rules (3) and (6) with critical pair:

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y
\end{array}$$

Overlap between rules (3) and (6) with critical pair:

$$\begin{tikzcd}
& (I(I(y)) \cdot e) \cdot x \arrow[dl, "6"'] \arrow[dr, "3"] & \\
y \cdot x & & I(I(y)) \cdot (e \cdot x)
\end{tikzcd}$$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y
\end{array}
$$

Overlap between rules (3) and (6) with critical pair:

$$\begin{tikzcd}
 & (I(I(y)) \cdot e) \cdot x \arrow[dl, "6"'] \arrow[dr, "3"] & & \\
y \cdot x & & I(I(y)) \cdot (e \cdot x) \arrow[dr, "1"] & \\
 & & & I(I(y)) \cdot x
\end{tikzcd}$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y
\end{array}$$

Overlap between rules (3) and (6) with critical pair:

$$\begin{tikzcd}
 & (I(I(y)) \cdot e) \cdot x \arrow[dl, "6"'] \arrow[dr, "3"] & & \\
y \cdot x & & I(I(y)) \cdot (e \cdot x) \arrow[dr, "1"] & \\
 & & & I(I(y)) \cdot x \arrow[ulll, dashed, "7"]
\end{tikzcd}$$

Add a rule:

(7) $I(I(y)) \cdot x \rightarrow y \cdot x$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x
\end{array}$$

Overlap between rules (7) and (6) with critical pair:

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x
\end{array}
$$

Overlap between rules (7) and (6) with critical pair:

$$\begin{tikzcd}
 & I(I(y)) \cdot e \arrow[dl, "7"'] \arrow[dr, "6"] & \\
y \cdot e & & y
\end{tikzcd}$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x
\end{array}$$

Overlap between rules (7) and (6) with critical pair:

$$\begin{tikzcd}
& I(I(y)) \cdot e \arrow[dl, "7"'] \arrow[dr, "6"] & \\
y \cdot e \arrow[rr, dashed, "8"] & & y
\end{tikzcd}$$

Add a rule:

$$(8) \qquad y \cdot e \rightarrow y$$

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\
(8) & y \cdot e & \rightarrow & y
\end{array}
$$

Overlap between rules (8) and (6) with critical pair:

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\
(8) & y \cdot e & \rightarrow & y
\end{array}$$

Overlap between rules (8) and (6) with critical pair:

$$\begin{tikzcd}
& I(I(y)) \cdot e \arrow[dl, "8"] \arrow[dr, "9"] & \\
I(I(y)) & & y
\end{tikzcd}$$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\
(8) & y \cdot e & \rightarrow & y
\end{array}
$$

Overlap between rules (8) and (6) with critical pair:

$$\begin{tikzcd}
 & I(I(y)) \cdot e \arrow[dl, "8"'] \arrow[dr, "9"] & \\
I(I(y)) \arrow[rr, dashed, "9"] & & y
\end{tikzcd}$$

Add a rule:

(9) $I(I(y)) \rightarrow y$

Removing Redundant Reduction Rules

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y
\end{array}$$

Removing Redundant Reduction Rules

$$\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(6) & I(I(y)) \cdot e & \rightarrow & y \\
(7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y
\end{array}$$

Rule (7) is now no longer necessary:

Removing Redundant Reduction Rules

$$\begin{array}{lllll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y \\
(7) & I(I(y)) \cdot x & \to & y \cdot x \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y
\end{array}$$

Rule (7) is now no longer necessary:

$$I(I(y)) \cdot x \;\to_9\; y \cdot x$$

Removing Redundant Reduction Rules

$$\begin{array}{lllll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(6) & I(I(y)) \cdot e & \to & y \\
(7) & I(I(y)) \cdot x & \to & y \cdot x \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y
\end{array}$$

Rule (7) is now no longer necessary:

$$I(I(y)) \cdot x \ \to_9 \ y \cdot x$$

Likewise for rule (6):

$$I(I(y)) \cdot e \ \to_9 \ y \cdot e \ \to_8 \ y$$

Removing Redundant Reduction Rules

$$\begin{array}{lll} (1) & e \cdot x & \rightarrow & x \\ (2) & I(x) \cdot x & \rightarrow & e \\ (3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\ (4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\ (5) & I(e) \cdot z & \rightarrow & z \\ (6) & I(I(y)) \cdot e & \rightarrow & y \\ (7) & I(I(y)) \cdot x & \rightarrow & y \cdot x \\ (8) & y \cdot e & \rightarrow & y \\ (9) & I(I(y)) & \rightarrow & y \end{array}$$

Rule (7) is now no longer necessary:

$$I(I(y)) \cdot x \ \rightarrow_9 \ y \cdot x$$

Likewise for rule (6):

$$I(I(y)) \cdot e \ \rightarrow_9 \ y \cdot e \ \rightarrow_8 \ y$$

We remove the rules (6) and (7).

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y
\end{array}$$

Overlap between rules (8) and (5) with critical pair:

$$\begin{array}{lllll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(5) & I(e) \cdot z & \to & z \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y
\end{array}$$

Overlap between rules (8) and (5) with critical pair:

$$\begin{tikzcd}
 & I(e) \cdot e \arrow[dl, "8"'] \arrow[dr, "5"] & \\
I(e) & & e
\end{tikzcd}$$

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y
\end{array}
$$

Overlap between rules (8) and (5) with critical pair:

$$\begin{tikzcd}
 & I(e) \cdot e \arrow[dl, "8"'] \arrow[dr, "5"] & \\
I(e) \arrow[rr, dashed, "10"] & & e
\end{tikzcd}$$

Add a rule:

(10) $I(e) \rightarrow e$

Removing Another Redundant Reduction Rule

$$\begin{array}{llll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}$$

Removing Another Redundant Reduction Rule

$$\begin{array}{llll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}$$

Rule (5) can now be dropped:

Removing Another Redundant Reduction Rule

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(5) & I(e) \cdot z & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}$$

Rule (5) can now be dropped:

$$I(e) \cdot z \ \rightarrow_{10} \ e \cdot z \ \rightarrow_{1} \ z$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}$$

Overlap between rules (9) and (2) with critical pair:

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}$$

Overlap between rules (9) and (2) with critical pair:

$$\begin{tikzcd}
& I(I(y)) \cdot I(y) \arrow[dl, "9"'] \arrow[dr, "2"] & \\
y \cdot I(y) & & e
\end{tikzcd}$$

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e
\end{array}
$$

Overlap between rules (9) and (2) with critical pair:

$$\begin{tikzcd}
 & I(I(y)) \cdot I(y) \arrow[dl, "9"'] \arrow[dr, "2"] & \\
y \cdot I(y) \arrow[rr, dashed, "11"] & & e
\end{tikzcd}$$

Add a rule:

(11) $y \cdot I(y) \rightarrow e$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e
\end{array}$$

Overlap between rules (11) and (3) with critical pair:

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e
\end{array}
$$

Overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
 & (y \cdot I(y)) \cdot x \arrow[dl, "11"'] \arrow[dr, "3"] & \\
e \cdot x & & y \cdot (I(y) \cdot x)
\end{tikzcd}$$

$$
\begin{array}{lllll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y \\
(10) & I(e) & \to & e \\
(11) & y \cdot I(y) & \to & e
\end{array}
$$

Overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
 & & (y \cdot I(y)) \cdot x \arrow[dl, "11"'] \arrow[dr, "3"] & \\
 & e \cdot x \arrow[dl, "1"'] & & y \cdot (I(y) \cdot x) \\
x & & &
\end{tikzcd}$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e
\end{array}$$

Overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
 & (y \cdot I(y)) \cdot x \arrow[dl, "11"'] \arrow[dr, "3"] & \\
e \cdot x \arrow[d, "1"'] & & y \cdot (I(y) \cdot x) \arrow[dll, dashed, "12"] \\
x & &
\end{tikzcd}$$

Add a rule:

$$(12) \qquad y \cdot (I(y) \cdot x) \rightarrow x$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x
\end{array}$$

Another overlap between rules (11) and (3) with critical pair:

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x
\end{array}$$

Another overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
& (x \cdot y) \cdot I(x \cdot y) \arrow[dl, "3"'] \arrow[dr, "11"] & \\
x \cdot (y \cdot I(x \cdot y)) & & e
\end{tikzcd}$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x
\end{array}$$

Another overlap between rules (11) and (3) with critical pair:

$$\begin{tikzcd}
& (x \cdot y) \cdot I(x \cdot y) \arrow[dl, "3"'] \arrow[dr, "11"] & \\
x \cdot (y \cdot I(x \cdot y)) \arrow[rr, dashed, "13"] & & e
\end{tikzcd}$$

Add a rule:

$$(13) \qquad x \cdot (y \cdot I(x \cdot y)) \rightarrow e$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow & e
\end{array}$$

Overlap between rules (4) and (13) with critical pair:

$$\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow & e
\end{array}$$

Overlap between rules (4) and (13) with critical pair:

$$\begin{tikzcd}
& I(x) \cdot (x \cdot (y \cdot I(x \cdot y))) \arrow[dl, "13"'] \arrow[dr, "4"] & \\
I(x) \cdot e & & y \cdot I(x \cdot y)
\end{tikzcd}$$

$$
\begin{array}{lll}
(1) & e \cdot x & \rightarrow \quad x \\
(2) & I(x) \cdot x & \rightarrow \quad e \\
(3) & (x \cdot y) \cdot z & \rightarrow \quad x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow \quad z \\
(8) & y \cdot e & \rightarrow \quad y \\
(9) & I(I(y)) & \rightarrow \quad y \\
(10) & I(e) & \rightarrow \quad e \\
(11) & y \cdot I(y) & \rightarrow \quad e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow \quad x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow \quad e
\end{array}
$$

Overlap between rules (4) and (13) with critical pair:

$$\begin{tikzcd}
& & I(x) \cdot (x \cdot (y \cdot I(x \cdot y))) \arrow[dll, "13"'] \arrow[drr, "4"] & & \\
I(x) \cdot e \arrow[r, "8"] & I(x) & & & y \cdot I(x \cdot y)
\end{tikzcd}$$

$$
\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow & e
\end{array}
$$

Overlap between rules (4) and (13) with critical pair:

$$\begin{tikzcd}
 & I(x) \cdot (x \cdot (y \cdot I(x \cdot y))) \arrow[dl, "13"'] \arrow[dr, "4"] & \\
I(x) \cdot e \arrow[r, "8"] & I(x) & y \cdot I(x \cdot y) \arrow[l, dashed, "14"']
\end{tikzcd}$$

$$(14) \qquad y \cdot I(x \cdot y) \rightarrow I(x)$$

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \rightarrow & e \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x)
\end{array}$$

Removing the redundant reduction rule (13):

$$
\begin{array}{lll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y \\
(10) & I(e) & \to & e \\
(11) & y \cdot I(y) & \to & e \\
(12) & y \cdot (I(y) \cdot x) & \to & x \\
(13) & x \cdot (y \cdot I(x \cdot y)) & \to & e \\
(14) & y \cdot I(x \cdot y) & \to & I(x)
\end{array}
$$

Removing the redundant reduction rule (13):

$$x \cdot (y \cdot I(x \cdot y)) \;\to_{14}\; x \cdot I(x) \;\to_{11}\; e$$

$$\begin{array}{lllll}
(1) & e \cdot x & \to & x \\
(2) & I(x) \cdot x & \to & e \\
(3) & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \to & z \\
(8) & y \cdot e & \to & y \\
(9) & I(I(y)) & \to & y \\
(10) & I(e) & \to & e \\
(11) & y \cdot I(y) & \to & e \\
(12) & y \cdot (I(y) \cdot x) & \to & x \\
(14) & y \cdot I(x \cdot y) & \to & I(x)
\end{array}$$

Overlap between rules (14) and (4) with critical pair:

$$
\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x)
\end{array}
$$

Overlap between rules (14) and (4) with critical pair:

$$\begin{tikzcd}
& I(y) \cdot (y \cdot I(x \cdot y)) \arrow[dl, "14"'] \arrow[dr, "4"] & \\
I(y) \cdot I(x) & & I(x \cdot y)
\end{tikzcd}$$

$$
\begin{array}{lll}
(1) & e \cdot x & \rightarrow \quad x \\
(2) & I(x) \cdot x & \rightarrow \quad e \\
(3) & (x \cdot y) \cdot z & \rightarrow \quad x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow \quad z \\
(8) & y \cdot e & \rightarrow \quad y \\
(9) & I(I(y)) & \rightarrow \quad y \\
(10) & I(e) & \rightarrow \quad e \\
(11) & y \cdot I(y) & \rightarrow \quad e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow \quad x \\
(14) & y \cdot I(x \cdot y) & \rightarrow \quad I(x)
\end{array}
$$

Overlap between rules (14) and (4) with critical pair:

$$\begin{tikzcd}
 & I(y) \cdot (y \cdot I(x \cdot y)) \arrow[dl, "14"'] \arrow[dr, "4"] & \\
I(y) \cdot I(x) & & I(x \cdot y) \arrow[ll, dashed, "15"']
\end{tikzcd}$$

(15) $I(x \cdot y) \rightarrow I(y) \cdot I(x)$

$$
\begin{array}{lll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x) \\
(15) & I(x \cdot y) & \rightarrow & I(y) \cdot I(x)
\end{array}
$$

Removing the redundant reduction rule (14):

$$\begin{array}{lllll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x) \\
(15) & I(x \cdot y) & \rightarrow & I(y) \cdot I(x)
\end{array}$$

Removing the redundant reduction rule (14):

$$y \cdot I(x \cdot y)$$

$$\begin{array}{llcl}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(14) & y \cdot I(x \cdot y) & \rightarrow & I(x) \\
(15) & I(x \cdot y) & \rightarrow & I(y) \cdot I(x)
\end{array}$$

Removing the redundant reduction rule (14):

$$y \cdot I(x \cdot y) \ \rightarrow_{15} \ y \cdot (I(y) \cdot I(x)) \ \rightarrow_{12} \ I(x)$$

Finally!

We have constructed a complete rewrite system:

- all critical pairs are convergent
- the system is terminating

$$\begin{array}{lll}
(1) & e \cdot x & \rightarrow & x \\
(2) & I(x) \cdot x & \rightarrow & e \\
(3) & (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \\
(4) & I(x) \cdot (x \cdot z) & \rightarrow & z \\
(8) & y \cdot e & \rightarrow & y \\
(9) & I(I(y)) & \rightarrow & y \\
(10) & I(e) & \rightarrow & e \\
(11) & y \cdot I(y) & \rightarrow & e \\
(12) & y \cdot (I(y) \cdot x) & \rightarrow & x \\
(15) & I(x \cdot y) & \rightarrow & I(y) \cdot I(x)
\end{array}$$

Completion Tools

- Waldmeister
- Slothrop
- mkbTT
- KBCV