- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
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Outline

- Overview
- Matching
- Unification
- Common Instances

Matching and Unification

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Definition (Matching Problem)

instance: terms s, t

question: $\exists$ subsitution σ: $s\sigma = t$? (s matches t)

Remarks

- term t can be rewritten if left-hand side of rewrite rule matches subterm of t
- matching problem is decidable (in linear time)

Definition (Matching Problem)

instance: terms s, t

question: $\exists$ subsitution σ: $s\sigma = t$?

Matching Algorithm

1. start with $\{s \mapsto t\}$
2. repeatedly apply following tranformation rules

$$\{f(s_1,\ldots,s_n) \mapsto f(t_1,\ldots,t_n)\} \cup S \;\Rightarrow\; \{s_1 \mapsto t_1,\ldots,s_n \mapsto t_n\} \cup S$$

$$\{f(s_1,\ldots,s_n) \mapsto g(t_1,\ldots,t_n)\} \cup S \;\Rightarrow\; \bot \text{ if } f \neq g$$

$$\{f(s_1,\ldots,s_n) \mapsto x\} \cup S \;\Rightarrow\; \bot$$

$$\{x \mapsto t\} \cup S \;\Rightarrow\; \bot \text{ if } S \text{ contains } x \mapsto t' \text{ with } t \neq t'$$

Examples

- $x + \mathsf{s}(y+z)$ matches $\mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)$:

$$\begin{aligned}
&\{x + \mathsf{s}(y+z) \mapsto \mathsf{s}(y) + \mathsf{s}((x+\mathsf{s}(0))+z)\} \\
&\quad\Longrightarrow\quad \{x \mapsto \mathsf{s}(y),\ \mathsf{s}(y+z) \mapsto \mathsf{s}((x+\mathsf{s}(0))+z)\} \\
&\quad\Longrightarrow\quad \{x \mapsto \mathsf{s}(y),\ y+z \mapsto (x+\mathsf{s}(0))+z\} \\
&\quad\Longrightarrow\quad \{x \mapsto \mathsf{s}(y),\ y \mapsto x+\mathsf{s}(0),\ z \mapsto z\}
\end{aligned}$$

- $x^- \cdot (x \cdot y)$ does not match $(\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)$:

$$\begin{aligned}
&\{x^- \cdot (x \cdot y) \mapsto (\mathsf{e} \cdot x)^- \cdot ((\mathsf{e} \cdot \mathsf{e}) \cdot x)\} \\
&\quad\Longrightarrow\quad \{x^- \mapsto (\mathsf{e} \cdot x)^-,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\quad\Longrightarrow\quad \{x \mapsto \mathsf{e} \cdot x,\ x \cdot y \mapsto (\mathsf{e} \cdot \mathsf{e}) \cdot x\} \\
&\quad\Longrightarrow\quad \{x \mapsto \mathsf{e} \cdot x,\ x \mapsto \mathsf{e} \cdot \mathsf{e},\ y \mapsto x\} \\
&\quad\Longrightarrow\quad \bot
\end{aligned}$$

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Definition

composition of substitutions σ and τ:

$$\sigma\tau = \{x \mapsto \sigma(x)\tau \mid x \in \mathcal{X}\}$$

Example

$\sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\}$ $\quad$ $\tau = \{x \mapsto \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$

- $\sigma\tau = \{x \mapsto \mathsf{s}(y), y \mapsto \mathsf{s}(0) + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(y))\}$
- $\tau\sigma = \{x \mapsto \mathsf{s}(0), y \mapsto x + \mathsf{s}(0), z \mapsto \mathsf{s}(\mathsf{s}(x + \mathsf{s}(0)\}$

Lemma

$(\rho\sigma)\tau = \rho(\sigma\tau)$ *for all substitutions* ρ, σ, τ

Definitions

- $\lessdot$ subsumption

  $s \lessdot t \iff \exists\sigma\colon s\sigma = t$ "s subsumes t" "t is instance of s"

- $\lessdot$ proper subsumption

  $s \lessdot t \iff s \lessdot t \wedge \neg(t \lessdot s)$

Example

$x + y \lessdot \mathsf{s}(y) + \mathsf{s}(0)$ $\mathsf{s}(x) + y \not\lessdot x + \mathsf{s}(0)$ $\mathsf{s}(x) + y \lessdot \mathsf{s}(x) + x$

Lemma

$\gtrdot$ is well-founded order on terms

Definitions

- $\doteq$ literal similarity

  $s \doteq t \iff s \leq t \ \wedge\ t \leq s$
- variable substitution is substitution from $\mathcal{X}$ to $\mathcal{X}$
- renaming is bijective variable substitution
- terms s and t are variants if $s = t\sigma$ for some renaming σ

Lemma

terms s and t are variants $\iff$ $s \doteq t$

Example

$\mathsf{s}(x) + \mathsf{s}(y + 0) \doteq \mathsf{s}(y) + \mathsf{s}(z + 0)$ $\qquad$ $\mathsf{s}(x) + \mathsf{s}(y + 0) \not\doteq \mathsf{s}(x) + \mathsf{s}(x + 0)$

Definition (Unification Problem)

instance: terms s, t

question: $\exists \underbrace{\text{substitution } \sigma}_{\text{unifier}}\colon\ s\sigma = t\sigma$?

Definition

substitution σ is at least as general as τ ($\sigma \lesssim \tau$) if $\exists$ substitution ρ: $\sigma\rho = \tau$

Lemma

$\gtrdot$ is well-founded order on substitutions (with a finite domain)

Definition

most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

d decomposition

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n),\ E_2}{E_1,\ s_1 \approx t_1,\ \ldots,\ s_n \approx t_n,\ E_2}$$

t removal of trivial equations $(x \in \mathcal{X})$

$$\frac{E_1,\ x \approx x,\ E_2}{E_1,\ E_2}$$

v variable elimination $(x \in \mathcal{X})$

$$\frac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\sigma} \quad \text{and} \quad \frac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

Example

$$x + (0 + \mathsf{s}(y)) \approx \mathsf{s}(z) + (0 + x)$$
$$\mathsf{d} \Downarrow$$
$$x \approx \mathsf{s}(z),\ 0 + \mathsf{s}(y) \approx 0 + x$$
$$\mathsf{v} \Downarrow \quad x \mapsto \mathsf{s}(z)$$
$$0 + \mathsf{s}(y) \approx 0 + \mathsf{s}(z)$$
$$\mathsf{d} \Downarrow$$
$$0 \approx 0,\ \mathsf{s}(y) \approx \mathsf{s}(z)$$
$$\mathsf{d} \Downarrow$$
$$\mathsf{s}(y) \approx \mathsf{s}(z)$$
$$\mathsf{d} \Downarrow$$
$$y \approx z$$
$$\mathsf{v} \Downarrow \quad y \mapsto z$$
$$z \approx z$$
$$\mathsf{t} \Downarrow$$
$$\square$$

mgu $\{x \mapsto \mathsf{s}(z), y \mapsto z\}$

Theorem

- *there are no infinite derivations*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots$$

- *if s and t are unifiable then for every maximal derivation*

$$s \approx t \Rightarrow_{\sigma_1} E_1 \Rightarrow_{\sigma_2} E_2 \Rightarrow_{\sigma_3} \cdots \Rightarrow_{\sigma_n} E_n$$

  - $E_n = \square$
  - $\sigma_1\sigma_2\sigma_3\cdots\sigma_n$ *is mgu of s and t*

Optional Failure Rules

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx g(t_1,\ldots,t_m),\ E_2}{\bot} \qquad \frac{E_1,\ x \approx t,\ E_2}{\bot} \quad \frac{E_1,\ t \approx x,\ E_2}{\bot}$$

if $x \in \mathcal{V}\mathrm{ar}(t)$

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Definition (Common Instance)

instance: terms s, t

question: $\exists$ substitutions σ, τ: $s\sigma = t\tau$?

Definition

most general common instance (mgci) is at least as general as any other common instance

We can compute the most general common instance as follows:

- let γ be a renaming of the variables in t
  such that s and $t\gamma$ have no variables in common
- the mgu for s and $t\gamma$ is the mgci for s and t

Example

We determine the most general common instance of

$$s = f(s(x), y) \qquad\qquad t = f(x, s(0))$$

- we rename the variables in t with $\gamma = \{x \mapsto z\}$

$$t' = t\gamma = f(z, s(0))$$

- the mgu of s and t' is $f(s(x), s(0))$
  (using the substitution $\sigma = \{x \mapsto x,\ y \mapsto s(0),\ z \mapsto s(x)\}$)

Hence the mgci of s and t is $f(s(x), s(0))$.

That is, $s\sigma = t(\gamma\sigma)$, or:

$$s[x \mapsto x,\ y \mapsto s(x)] = t[x \mapsto s(x)]$$