- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

Outline

Matching and Unification

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Definition (Matching Problem)

instance: terms s, t

question: $\exists$ subsitution σ: $s\sigma = t$?

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Remarks

- term t can be rewritten if left-hand side of rewrite rule matches subterm of t
- matching problem is decidable (in linear time)

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$$\{x \mapsto t\} \cup S \;\Rightarrow\; \bot \text{ if } S \text{ contains } x \mapsto t' \text{ with } t \neq t'$$

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composition of substitutions σ and τ:

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Lemma

$(\rho\sigma)\tau = \rho(\sigma\tau)$ *for all substitutions* ρ, σ, τ

Definitions

- $\leq$ subsumption

  $s \leq t \iff \exists\sigma : s\sigma = t$

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most general unifier (mgu) is at least as general as any other unifier

Definition (Unification Rules)

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v variable elimination $(x \in \mathcal{X})$

$$\frac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\sigma} \quad \text{and} \quad \frac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\sigma}$$

if $\underbrace{x \notin \mathcal{V}\mathrm{ar}(t)}_{\text{occurs check}}$ and $\sigma = \{x \mapsto t\}$

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$$\mathsf{d} \Downarrow$$

$$y \approx z$$

$$\mathsf{v} \Downarrow \ y \mapsto z$$

$$z \approx z$$

$$\mathsf{t} \Downarrow$$

$$\square$$

Example

$$x + (0 + s(y)) \approx s(z) + (0 + x)$$

$$\mathsf{d} \Downarrow$$

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mgu $\{x \mapsto s(z), y \mapsto z\}$

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Optional Failure Rules

$$\frac{E_1,\ f(s_1,\ldots,s_n) \approx g(t_1,\ldots,t_m),\ E_2}{\bot} \qquad \frac{E_1,\ x \approx t,\ E_2}{\bot} \quad \frac{E_1,\ t \approx x,\ E_2}{\bot}$$

if $x \in \mathcal{V}\mathrm{ar}(t)$

Outline

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That is, $s\sigma = t(\gamma\sigma)$, or:

$$s[x \mapsto x,\ y \mapsto s(x)] = t[x \mapsto s(x)]$$