- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

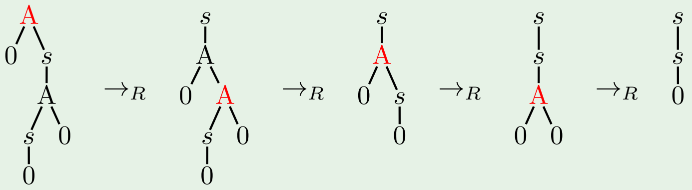
Outline

Termination

Termination, Example 1

Example

$$\mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y))$$
$$\mathrm{A}(x, 0) \to x$$



Looks terminating:

- second rule makes terms smaller
- first rule makes ‘s’ move upwards

Termination, Example 2

Example

$$f(g(x)) \to g(f(x))$$

$$\begin{aligned} & f(f(g(f(g(x))))) \\ \to\ & f(f(g(g(f(x))))) \\ \to\ & f(g(f(g(f(x))))) \\ \to\ & g(f(f(g(f(x))))) \\ \to\ & g(f(g(f(f(x))))) \\ \to\ & g(g(f(f(f(x))))) \end{aligned}$$

Looks terminating:

- f's move to the right
- g's move to the left

Termination, Example 3

Example

$$f(g(x)) \to g(g(f(f(x))))$$

Looks terminating:

- f's move to the right
- g's move to the left

But we have an infinite rewrite sequence:

$$\begin{aligned} & f(g(g(x))) \\ \to\ & g(g(f(f(g(x))))) \\ \to\ & g(g(f(g(g(f(f(x))))))) \\ \to\ & \dots \end{aligned}$$

We need proofs of termination!

Termination, Example 4

Example

$$f(x, g(y)) \to f(x, x)$$

Looks terminating:

- right side is smaller than the left side

But we have an infinite rewrite sequence:

$$\begin{aligned} & f(g(x), g(x)) \\ \to\ & f(g(x), g(x)) \\ \to\ & \ldots \end{aligned}$$

We need proofs of termination!

Definition (Termination SN(R))

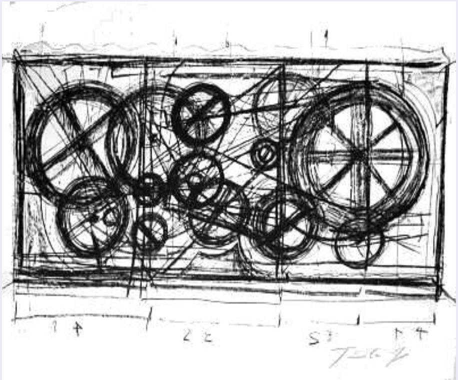
A rewrite system R is terminating if there are no infinite rewrite sequences.

Termination Methods

Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling, monotonic semantic path order, context-dependent interpretations, match-bounds, size-change principle, matrix interpretations, predictive labeling, uncurrying, bounded increase, quasi-periodic interpretations, arctic interpretations, increasing interpretations, root-labeling, . . .

Termination Research

Termination Research



Termination Tools

AProVE, Cariboo, CiME, Jambox, Termptation, Matchbox, MuTerm, NTI, Torpa, TPA, $\mathsf{T_TT_2}$, VMTL, . . .

Outline

Lemma

TRS R is terminating

$\iff$

$\exists$ *well-founded order $>$ on terms such that $s > t$ whenever $s \to_R t$*

Example

- TRS

$$0 + y \to y \qquad\qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- well-founded order $>$

$$s > t \quad\iff\quad \varphi(s) >_{\mathbb{N}} \varphi(t) \text{ with } \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = \mathsf{s}(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

Remark

(very) inconvenient to check all rewrite steps

Definitions

A reduction order is well-founded order $>$ on terms which is

- closed under contexts $\quad s > t \quad \Longrightarrow \quad C[s] > C[t]$
- closed under substitutions $\quad s > t \quad \Longrightarrow \quad s\sigma > t\sigma$

Notation

$R \subseteq >$ if $\ell > r$ for all rules $\ell \to r$ in R

Theorem

TRS R is terminating $\quad \Longleftrightarrow \quad$ $R \subseteq >$ *for reduction order* $>$

Theorem

TRS R is terminating $\iff$ $R \subseteq >$ *for reduction order* $>$

Proof $\Rightarrow$.

Let R be terminating.

We define $> = \rightarrow$.

Then:

- $>$ is well-founded since R is terminating,
- $>$ is closed under substitutions since $\rightarrow$ is, and
- $>$ is closed under contexts since $\rightarrow$ is.

Hence $>$ is a reduction order. Moreover $R \subseteq \rightarrow$. ∎

Theorem

TRS R is terminating $\iff$ $R \subseteq >$ *for reduction order* $>$

Proof $\Leftarrow$.

Let $>$ be reduction order such that $R \subseteq >$.

Recall that $\rightarrow$ is the smallest relation S such that:

- $R \subseteq S$,
- S is closed under contexts, and
- S is closed under substitutions.

Then $\rightarrow \subseteq >$ since $>$ has these properties.

Assume there exists an infinite rewrite sequence: $t_0 \rightarrow t_1 \rightarrow t_2 \ldots$

Then also $t_0 > t_1 > t_2 \ldots$ since $\rightarrow \subseteq >$

However, this contradicts well-foundedness of $>$. ∎

Constructing Reduction Orders

Idea: give semantics to terms by interpreting them into an algebra.

Definition

A Σ-algebra $(A, [\cdot])$ consists of:

- a non-empty set A,
- and for every $f \in \Sigma$ an interpretation function $[f]\colon A^{\mathrm{ar}(f)} \to A$.

Constructing Reduction Orders

Idea: give semantics to terms by interpreting them into an algebra.

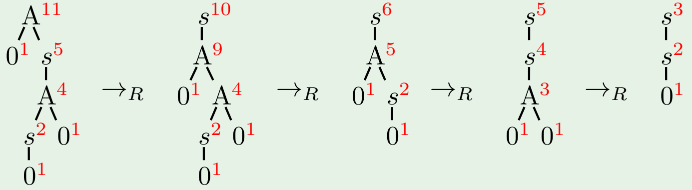
Definition

A Σ-algebra $(A, [\cdot])$ consists of:

- a non-empty set A,
- and for every $f \in \Sigma$ an interpretation function $[f]\colon A^{\mathrm{ar}(f)} \to A$.

Example (We use the Σ-algebra $(\mathbb{N}, [\cdot])$ with)

$$[0] = 1 \qquad [\mathrm{s}](x) = x + 1 \qquad [\mathrm{A}](x, y) = x + 2 \cdot y$$



Interpretation of terms

Definition (Interpretation of Terms)

Let $\alpha : \mathcal{X} \to A$ be an interpretation of the variables.
We define the evaluation of terms

$$[\cdot, \alpha] : \mathcal{T}(\Sigma, \mathcal{X}) \to A$$

inductively:

$$[x, \alpha] = \alpha(x) \qquad \text{if } x \in \mathcal{X}$$
$$[f(t_1, \ldots, t_n), \alpha] = [f]([t_1, \alpha], \ldots, [t_n, \alpha])$$

Example

$$[0] = 1 \qquad [\mathrm{s}](x) = x + 1 \qquad [\mathrm{A}](x, y) = x + 2 \cdot y$$

Let $\alpha(x) = 1$, $\alpha(y) = 3$, we calculate:

- $[\mathrm{A}(0, \mathrm{s}(0)), \alpha] = 5$,
- $[\mathrm{A}(\mathrm{s}(x), y), \alpha] = 8$.

Monotone Σ-algebras

A function $[f]$ is monotone (w.r.t. $>$) if

$$a > b \text{ implies } [f](\ldots, a, \ldots) \; > \; [f](\ldots, b, \ldots)$$

Definition

A well-founded monotone Σ-algebra $(A, [\cdot], >)$:

- a Σ-algebra $(A, [\cdot])$,
- a well-founded order $>$ on A
  (no infinite chains $a_1 > a_2 > \ldots$),
- such that for all $f \in \Sigma$ the function $[f]_{\mathcal{A}}$ is monotone.

Relation $>_{\mathcal{A}}$ on terms: $s > t$ if $[s, \alpha] > [t, \alpha]$ for all assignments α.

Lemma

$>_{\mathcal{A}}$ is a reduction order for every well-founded monotone Σ-algebra $(A, [\cdot], >)$

Lemma

Let $\mathcal{A}$ well-founded monotone Σ-algebra. Then $>_{\mathcal{A}}$ is closed under substitutions.

Proof.

Let $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$ such that $t >_{\mathcal{A}} s$.

Let σ be a substitution. We show $t\sigma >_{\mathcal{A}} s\sigma$.

That is: $[t\sigma, \alpha] > [s\sigma, \alpha]$ for every $\alpha : \mathcal{X} \to A$.

Define $\beta : \mathcal{X} \to A$ by $\beta(x) = [\sigma(x), \alpha]$. Claim: $[u\sigma, \alpha] = [u, \beta]$ for all u.

- Proof of the claim by induction over the term structure of u:

$$\begin{aligned}
[x\sigma, \alpha] &= [\sigma(x), \alpha] = [x, \beta] \\
[f(t_1, \ldots, t_n)\sigma, \alpha] &= [f]([t_1\sigma, \alpha], \ldots, [t_n\sigma, \alpha]) \\
&\overset{\text{IH}}{=} [f]([t_1, \beta], \ldots, [t_n, \beta]) \\
&= [f(t_1, \ldots, t_n), \beta]
\end{aligned}$$

Hence $[t\sigma, \alpha] = [t, \beta] > [s, \beta] = [s\sigma, \alpha]$. ∎

Lemma

Let $\mathcal{A}$ well-founded monotone Σ-algebra. Then $>_{\mathcal{A}}$ is closed under contexts.

Proof.

Let $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$ such that $t >_{\mathcal{A}} s$. We show $C[s] >_{\mathcal{A}} C[t]$ for every $C[\,]$.

That is: $[C[t], \alpha] > [C[s], \alpha]$ for every $\alpha : \mathcal{X} \to A$.

Let $\alpha : \mathcal{X} \to A$. We show $[C[t], \alpha] > [C[s], \alpha]$ by induction over structure of $C[\,]$:

$$C[\,] \equiv [\,] \Rightarrow [C[t]] = [t] > [s] = [C[s]]$$

$C[\,] \equiv f(t_1, \ldots, t_n) \Rightarrow$ Let i such that t_i contains the hole $\Box$.

Then $C[u] = f(t_1, \ldots, t_i[u], \ldots, t_n)$. Hence

$$[C[s], \alpha] = [f]([t_1, \alpha], \ldots, [t_i[s], \alpha], \ldots, [t_n, \alpha]).$$

$$[C[t], \alpha] = [f]([t_1, \alpha], \ldots, [t_i[t], \alpha], \ldots, [t_n, \alpha]).$$

By IH $[t_i[t], \alpha] > [t_i[s], \alpha]$.

By monotonicity $[C[t], \alpha] > [C[s], \alpha]$.

■

Lemma

Let $\mathcal{A}$ well-founded monotone Σ-algebra. Then $>_{\mathcal{A}}$ is well-founded.

Proof.

Assume there is an infinite sequence $t_1 >_{\mathcal{A}} t_2 >_{\mathcal{A}} t_3 >_{\mathcal{A}} \ldots$.

Let $\alpha : \mathcal{X} \to A$. Then by definition of $>_{\mathcal{A}}$: $[t_1, \alpha] > [t_2, \alpha] > [t_3, \alpha] > \ldots$.

This infinite decreasing $>$ chain contradicts well-foundedness of $>$. ∎

We have shown that:

- $>_{\mathcal{A}}$ is closed under substitutions
- $>_{\mathcal{A}}$ is closed under contexts
- $>_{\mathcal{A}}$ is well-founded

Hence we have proven that:

Lemma

$>_{\mathcal{A}}$ is a reduction order for every well-founded monotone Σ-algebra $(A, [\cdot], >)$

Theorem

TRS R is terminating $\iff$ $R \subseteq >_{\mathcal{A}}$ *for well-founded monotone algebra* $(\mathcal{A}, >)$

Proof.

$\Leftarrow$ Follows since we have shown that $>_{\mathcal{A}}$ is a reduction order.

$\Rightarrow$ As Σ-algebra we take $(A, [\cdot], >)$ with

- $A = \mathcal{T}(\Sigma, \mathcal{X})$,
- $[f](t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$,
- $> := \to$.

Then $>$ is well-founded since R is terminating.

Monotonicity of $>$ follows from closure of $\to$ under contexts:

$$s > t \Rightarrow [f](\ldots, s, \ldots) = f(\ldots, s, \ldots) > f(\ldots, t, \ldots) = [f](\ldots, t, \ldots)$$

$R \subseteq >_{\mathcal{A}}$ since for all $\ell \to r \in R$ and $\alpha : \mathcal{X} \to A$:

$$[\ell, \alpha] = \ell\alpha > r\alpha = [r, \alpha].$$

Well-Founded Monotone Algebras

used in termination proofs/tools :

- polynomial interpretations over $\mathbb{N}$
- polynomial interpretations over $\mathbb{Q}$ and $\mathbb{R}$
- matrix interpretations over $\mathbb{N}$
- matrix interpretations over $\mathbb{N} \cup \{-\infty\}$
- $\dots$

Outline

Polynomial interpretations

Definition

A polynomial interpretations over $\mathbb{N}$ consists of:

- Σ-algebra is $(\mathbb{N}, [\cdot])$,
- $>$ defined as usual on $\mathbb{N}$,
- the interpretations $[f]$ are polynomials.

The first two conditions of well-founded monotone Σ-algebras are fulfilled:

- $(\mathbb{N}, [\cdot])$ is a Σ-algebra
- $>$ is well-founded

However, to prove termination we need to check:

- monotonicity of the polynoms $[f]$, and
- $R \subseteq >$, that is, $[\ell, \alpha] > [r, \alpha]$ for all $\ell \to r \in R$ and $\alpha : \mathcal{X} \to A$

Example 1

Example

$$\mathrm{f}(\mathrm{g}(x)) \to \mathrm{f}(\mathrm{f}(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[\mathrm{f}](x) = ??? \qquad\qquad [\mathrm{g}](x) = ???$$

Example 1

Example

$$\mathrm{f}(\mathrm{g}(x)) \rightarrow \mathrm{f}(\mathrm{f}(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[\mathrm{f}](x) = x \qquad\qquad [\mathrm{g}](x) = x + 1$$

- Are the functions $[f]$ monotone?

  Yes, since whenever $a > b$, then
  $[\mathrm{f}](a) = a > b = [\mathrm{f}](b)$,
  $[\mathrm{g}](a) = a + 1 > b + 1 = [\mathrm{g}](b)$,

- Does $[\ell, \alpha] > [r, \alpha]$ hold?

  Yes since,

  $$[\mathrm{f}(\mathrm{g}(x)), \alpha] = [\mathrm{f}]([\mathrm{g}](\alpha(x))) = \alpha(x) + 1 > \alpha(x) = [\mathrm{f}(\mathrm{f}(x)), \alpha]$$

Hence we have proven termination.

Example 2

Example

$$f(g(x)) \to g(f(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[f](x) = ??? \qquad [g](x) = ???$$

Example 2

Example

$$\mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{f}(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[\mathrm{f}](x) = 2 \cdot x \qquad [\mathrm{g}](x) = x + 1$$

- Are the functions $[f]$ monotone?

  Yes, since whenever $a > b$, then
  $[\mathrm{f}](a) = 2 \cdot a > 2 \cdot b = [\mathrm{f}](b)$,
  $[\mathrm{g}](a) = a + 1 > b + 1 = [\mathrm{g}](b)$,

- Does $[\ell, \alpha] > [r, \alpha]$ hold?

  Yes since,

$$[\mathrm{f}(\mathrm{g}(x)), \alpha] = 2 \cdot (\alpha(x) + 1) > 2 \cdot \alpha(x) + 1 = [\mathrm{g}(\mathrm{f}(x)), \alpha]$$

Hence we have proven termination.

Example 3

Example

$$\mathrm{A}(x, 0) \to x$$
$$\mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$[0] = ???$ $\qquad [\mathrm{s}](x) = ???$ $\qquad [\mathrm{A}](x, y) = ???$

Example 3

Example

$$\mathrm{A}(x, 0) \to x$$
$$\mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[0] = 1 \qquad [\mathrm{s}](x) = x + 1 \qquad [\mathrm{A}](x, y) = x + 2 \cdot y$$

- Are the functions $[f]$ monotone?

  Yes, since whenever $a > b$, then
  $[\mathrm{s}](a) = a + 1 > b + 1 = [\mathrm{s}](b)$,
  $[\mathrm{A}](a, y) = a + 2 \cdot y > b + 2 \cdot y = [\mathrm{A}](b, y)$, and
  $[\mathrm{A}](x, a) = x + 2 \cdot a > x + 2 \cdot b = [\mathrm{A}](x, b)$,

Example 3

Example

$$A(x, 0) \to x$$
$$A(x, s(y)) \to s(A(x, y))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[0] = 1 \qquad [s](x) = x + 1 \qquad [A](x, y) = x + 2 \cdot y$$

- Does $[\ell, \alpha] > [r, \alpha]$ hold?

  Yes since,
$$\begin{aligned}[A(x, s(y)), \alpha] &= [A](\alpha(x), [s](\alpha(y))) \\ &= \alpha(x) + 2 \cdot \alpha(y) + 2 \\ &> \alpha(x) + 2 \cdot \alpha(y) + 1 \\ &= [s(+(x, y)), \alpha] \\ [A(x, 0), \alpha] &= \alpha(x) + 2 \cdot 1 > \alpha(x) = [x, \alpha]\end{aligned}$$

Hence we have proven termination.

Why do we need the conditions?

Example ($>$ not well-founded)

Let $R = \{ \mathrm{f}(x) \to \mathrm{f}(\mathrm{f}(x)) \}$ with the Σ-algebra $(\mathbb{Z}, [\cdot])$ and

$$[\mathrm{f}](x) = x - 1$$

and $>$ as usual on $\mathbb{Z}$.

Then R is not terminating

$$\mathrm{f}(x) \to \mathrm{f}(\mathrm{f}(x)) \to \mathrm{f}(\mathrm{f}(\mathrm{f}(x))) \to \dots$$

but

- $[f]$ is monotone, and
- $[\mathrm{f}(x), \alpha] = \alpha(x) - 1 > \alpha(x) - 2 = [\mathrm{f}(\mathrm{f}(x)), \alpha]$.

Hence $>$ needs to be well-founded!

Why do we need the conditions?

Example ([f] not monotone)

Let $R = \{ \mathrm{f}(x) \to \mathrm{g}(\mathrm{f}(x)) \}$ with the Σ-algebra $(\mathbb{N}, [\cdot])$ and

$$[\mathrm{f}](x) = x + 1 \qquad\qquad [\mathrm{g}](x) = 0$$

and $>$ as usual on $\mathbb{N}$.

Then R is not terminating

$$\mathrm{f}(x) \to \mathrm{g}(\mathrm{f}(x)) \to \mathrm{g}(\mathrm{g}(\mathrm{f}(x))) \to \dots$$

but

- $>$ is well-founded, and
- $[\mathrm{f}(x), \alpha] = \alpha(x) + 1 > 0 = [\mathrm{g}(\mathrm{f}(x)), \alpha]$.

Hence the functions $[f]$ need to be monotone!

Question

How to find suitable polynomials ?

Modern Approach

(a) choose abstract polynomial interpretations (linear, quadratic, . . .)

(b) transform rewrite rules into polynomial ordering constraints

(c) add monotonicity and well-definedness constraints

(d) eliminate universally quantified variables

(e) translate resulting constraints to SAT or SMT problem

Example

- rewrite system

$$0 + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- interpretations

$$0_{\mathcal{A}} = a$$
$$\mathsf{s}_{\mathcal{A}}(x) = bx + c$$
$$+_{\mathcal{A}}(x, y) = dx + ey + f$$

- polynomial constraints $\forall x, y \in \mathbb{N}$

$$da + ey + f > y$$
$$d\,(bx + c) + ey + f > b\,(dx + ey + f) + c$$
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

Example

- rewrite system

$$0 + y \to y$$
$$s(x) + y \to s(x + y)$$

- interpretations

$$0_{\mathcal{A}} = a$$
$$s_{\mathcal{A}}(x) = bx + c$$
$$+_{\mathcal{A}}(x, y) = dx + ey + f$$

- polynomial constraints $\forall x, y \in \mathbb{N}$

$$(e - 1)y + da + f > 0$$
$$(e - be)y + dc + f - bf - c > 0$$
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

Example

- rewrite system

$$0 + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- interpretations

$$0_{\mathcal{A}} = a$$
$$\mathsf{s}_{\mathcal{A}}(x) = bx + c$$
$$+_{\mathcal{A}}(x, y) = dx + ey + f$$

- diophantine constraints

$$e - 1 \geqslant 0 \quad da + f > 0$$
$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Example

- rewrite system

$$0 + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

- interpretations

$$0_{\mathcal{A}} = 0$$
$$\mathsf{s}_{\mathcal{A}}(x) = x + 1$$
$$+_{\mathcal{A}}(x, y) = 2x + y + 1$$

- diophantine constraints

$$e - 1 \geqslant 0 \quad da + f > 0$$
$$e - be \geqslant 0 \quad dc + f - bf - c > 0$$
$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Example

- TRS

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to y + (x \times y)$$

- interpretations in $\mathbb{N}$

$$0_{\mathcal{A}} = 1 \qquad\qquad +_{\mathcal{A}}(x, y) = 2x + y$$
$$\mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad\qquad \times_{\mathcal{A}}(x, y) = 2xy + x + y + 1$$

- constraints $\quad \forall x, y \in \mathbb{N}$

$$y + 2 > y \qquad\qquad 3y + 2 > 1$$
$$2x + y + 2 > 2x + y + 1 \qquad\qquad 2xy + x + 3y + 2 > 2xy + x + 3y + 1$$

Example

- TRS

$$\begin{aligned} 0 + y &\to y & 0 \times y &\to 0 \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) & \mathsf{s}(x) \times y &\to y + (x \times y) \end{aligned}$$

- interpretations in $\mathbb{N}$

$$\begin{aligned} 0_{\mathcal{A}} &= 1 & +_{\mathcal{A}}(x, y) &= 2x + y \\ \mathsf{s}_{\mathcal{A}}(x) &= x + 1 & \times_{\mathcal{A}}(x, y) &= 2xy + x + y + 1 \end{aligned}$$

- constraints $\quad \forall x, y \in \mathbb{N}$

$$\begin{aligned} 2 &> 0 & 3y + 1 &> 0 \\ 1 &> 0 & 1 &> 0 \end{aligned}$$

- $$\begin{array}{ccccccccc} \mathsf{s}(0) \times \mathsf{s}(\mathsf{s}(0)) & \to & \mathsf{s}(\mathsf{s}(0)) + (0 \times \mathsf{s}(\mathsf{s}(0))) & \to & \mathsf{s}(\mathsf{s}(0)) + 0 & \to & \mathsf{s}(\mathsf{s}(0) + 0) \\ 18 & > & 17 & > & 7 & > & 6 \\ & \to & \mathsf{s}(\mathsf{s}(0 + 0)) & \to & \mathsf{s}(\mathsf{s}(0)) & & \\ & > & 5 & > & 3 & & \end{array}$$

Example

- TRS

$$\begin{array}{ll} \partial(x+y) \to \partial(x)+\partial(y) & \partial(\alpha)=1 \\ \partial(x-y) \to \partial(x)-\partial(y) & \partial(\beta)=0 \\ \partial(x\times y) \to (\partial(x)\times y)+(x\times\partial(y)) & \\ \partial(x\div y) \to ((\partial(x)\times y)-(x\times\partial(y)))\div(y\times y) & \end{array}$$

- interpretations in $\mathbb{N}$

$$\alpha_{\mathcal{A}} = \beta_{\mathcal{A}} = 0_{\mathcal{A}} = 1_{\mathcal{A}} = 1$$
$$+_{\mathcal{A}}(x,y) = -_{\mathcal{A}}(x,y) = \times_{\mathcal{A}}(x,y) = \div_{\mathcal{A}}(x,y) = x+y+3$$
$$\partial_{\mathcal{A}}(x) = x^2+6x+6$$

- constraints $\forall x,y \in \mathbb{N}$

$$\begin{array}{ll} x^2+y^2+2xy+12x+12y+33 > x^2+y^2+6x+6y+15 & 13>1 \\ x^2+y^2+2xy+12x+12y+33 > x^2+y^2+6x+6y+15 & 13>1 \\ x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+7y+21 & \\ x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+9y+27 & \end{array}$$

Example

- TRS

$$\partial(x + y) \to \partial(x) + \partial(y) \qquad \partial(\alpha) = 1$$
$$\partial(x - y) \to \partial(x) - \partial(y) \qquad \partial(\beta) = 0$$
$$\partial(x \times y) \to (\partial(x) \times y) + (x \times \partial(y))$$
$$\partial(x \div y) \to ((\partial(x) \times y) - (x \times \partial(y))) \div (y \times y)$$

- interpretations in $\mathbb{N}$

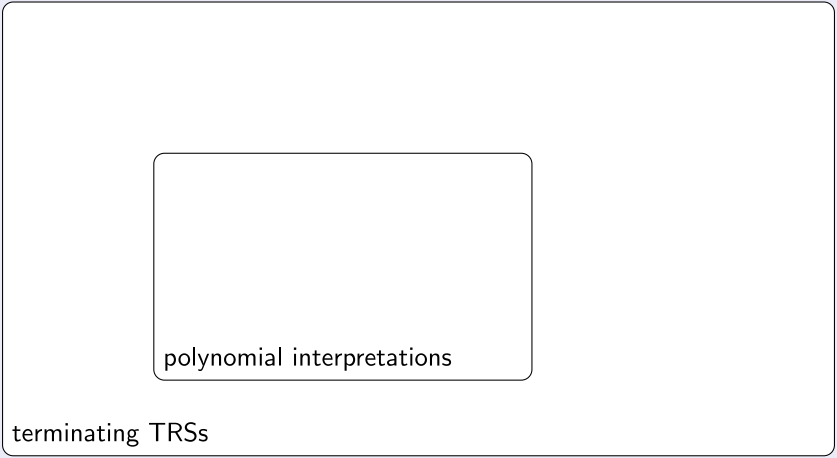
$$\alpha_{\mathcal{A}} = \beta_{\mathcal{A}} = 0_{\mathcal{A}} = 1_{\mathcal{A}} = 1$$
$$+_{\mathcal{A}}(x,y) = -_{\mathcal{A}}(x,y) = \times_{\mathcal{A}}(x,y) = \div_{\mathcal{A}}(x,y) = x + y + 3$$
$$\partial_{\mathcal{A}}(x) = x^2 + 6x + 6$$

- constraints $\quad \forall x, y \in \mathbb{N}$

$$2xy + 6x + 6y + 18 > 0 \qquad 13 > 1$$
$$2xy + 6x + 6y + 18 > 0 \qquad 13 > 1$$
$$2xy + 5x + 5y + 12 > 0$$
$$2xy + 5x + 3y + 6 > 0$$

Remark

Numerous terminating TRSs are not polynomially terminating.



A non-simply terminating TRS: $f(f(x)) \to f(g(f(x)))$

Example ($S = \{f(f(x)) \to f(g(f(x)))\}$)

There exists no monotone Σ-algebra $\mathcal{A}$ with $A = \mathbb{N}$ proving SN(S).

Assume contrary it would exist. Let $\alpha : \mathcal{X} \to A$, then:

$$[f(f(x)), \alpha] > [f(g(f(x))), \alpha] \ .$$

But then also

$$[f(x), \alpha] > [g(f(x)), \alpha]$$

since otherwise:

- $[f(x), \alpha] = [g(f(x)), \alpha]$ implies $[f(f(x)), \alpha] = [f(g(f(x))), \alpha]$,
- $[f(x), \alpha] < [g(f(x)), \alpha]$ implies $[f(f(x)), \alpha] < [f(g(f(x))), \alpha]$

by monotonicity.

But then $\mathcal{A}$ would also prove termination of $f(x) \to g(f(x))$.

Thus we need another Σ-algebra, for example $A = \mathbb{N}^2$.

A non-simply terminating TRS: $f(f(x)) \to f(g(f(x)))$

Example ($S = \{f(f(x)) \to f(g(f(x)))\}$)

We choose the Σ-algebra $(\mathbb{N}^2, [\cdot])$ with:

$$[\mathrm{f}](\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad [\mathrm{g}](\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $>$ on $\mathbb{N}^2$ is defined as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \iff x_1 > y_1 \text{ and } x_2 \geq y_2$$

Let $\alpha\colon \mathcal{X} \to A$ be arbitrary; write $\vec{x} = \alpha(x)$. We obtain

$$[f(f(x))] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} > \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [f(g(f(x)))]$$

Hence S is terminating.

Termination via Dependency Pairs

Minimal Terms

We call a term is minimal if all its strict subterms are terminating.

Theorem

Let R be a non-terminating TRS.
Then there exists a minimal term which is non-terminating.

Proof.

Since R is non-terminating, there exists a non-terminating term t.

We prove: every non-terminating term t contains a minimal subterm.

We use induction on the term structure of t:

- Base case: all strict subterms of t are terminating. Then t is minimal itself.
- Induction step: t has a strict subterm t' which is not terminating.

  Then by IH the term t' contains a minimal, non-terminating subterm t''. The term t'' is a minimal non-terminating subterm of t.

Steps below the root do not change minimality:

Lemma

Let t be minimal and $t \to s$ a rewrite step below the root. Then s is minimal.

Proof.

We have $t = f(t_1, \ldots, t_n)$ and $s = f(s_1, \ldots, s_n)$ and $i \in \mathbb{N}$ such that:

- $t_i \to s_i$, and
- $t_j = s_j$ for all $j \neq i$.

Then since t is minimal, it follows that all t_k are terminating.

Since $t_k \to^* s_k$, we obtain s_k is terminating for all $1 \leq k \leq n$.

Hence s is minimal. ∎

Dependency Pairs, Introduction

Assume R is non-terminating. There exists a minimal, non-terminating term t_0.

$$t_0 \to_R t_1 \to_R t_2 \to_R \dots$$

At some point there must be a root step $t_i \to t_{i+1}$.

Proof.

Assume all steps would be below the root. Then:

- all t_i are minimal,
- all rewrite steps are in terminating terminating.

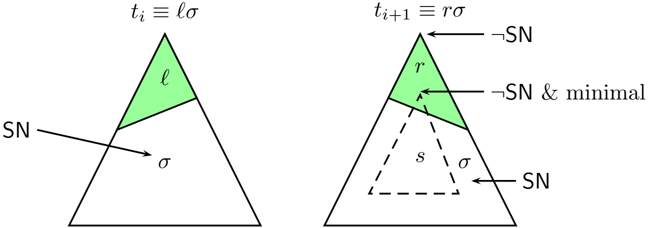
∎

Notation: we use $\xrightarrow{\text{top}}_R$ to denote root rewrite steps.

Dependency Pairs, Introduction

Let t_0 be a minimal and $t_0 \to_R t_1 \to_R t_2 \to_R \dots$ an infinite rewrite sequence.

We consider the first root rewrite step $t_i \to t_{i+1}$:



- The temr t_{i+1} contains a minimal, non-terminating subterm s.
- The root of s lies in the pattern of r.

Hence there exists a non-variable subterm r' of r such that $s = r'\sigma$.

Idea: add a rule $\ell \to r'$ then $t_i \to s$.

Dependency Pairs

For every $f \in \Sigma$ let $f_\#$ be a fresh symbol with the same arity as f.
By $t_\#$ we denote $f_\#(t_1, \ldots, t_n)$ for $t = f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, \mathcal{X})$.

Definition (Dependency Pairs)

$$\mathsf{DP}(R) = \{\ell_\# \to r'_\# \mid \ell \to r \in R,\ r' \trianglelefteq r \text{ with } r' \notin \mathcal{X}\}$$

Example

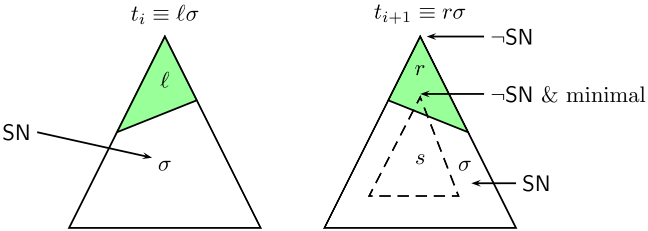
$$R = \{\ \mathrm{f}(x) \to \mathrm{g}(\mathrm{f}(x))\ \}$$

$$\begin{aligned}\mathsf{DP}(R) = \{\ & \mathrm{f}_\#(x) \to \mathrm{g}_\#(\mathrm{f}(x)), \\ & \mathrm{f}_\#(x) \to \mathrm{f}_\#(x)\ \}\end{aligned}$$

Dependency Pairs

Let t_0 be a minimal and $t_0 \to_R t_1 \to_R t_2 \to_R \ldots$ an infinite rewrite sequence.

We consider the first root rewrite step $t_i \to t_{i+1}$:



Then

$$t_{0,\#} \to_R \ldots \to_R t_{i,\#} \xrightarrow{\text{top}}_{\mathsf{DP}(R)} s_\#$$

Repeating the construction with s yields:

$$t_{0,\#} \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdots$$

an infinite rewrite sequence containing infinitely many DP(R) steps.

Dependency Pairs

Lemma

Let R be a non-terminating TRS. Then there exists a rewrite sequence:

$$t_{0,\#} \rightarrow_R^* \cdot \xrightarrow{top}_{\mathrm{DP}(R)} \cdot \rightarrow_R^* \cdot \xrightarrow{top}_{\mathrm{DP}(R)} \cdot \rightarrow_R^* \cdot \xrightarrow{top}_{\mathrm{DP}(R)} \cdots$$

such that:

- *the sequence contains infinitely many* $\mathrm{DP}(R)$ *steps,*
- *all R steps are below the root, and*
- *all* $\mathrm{DP}(R)$ *steps are at the root.*

Dependency Pairs, Examples

Example

$$R = \{\ \mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{f}(x))))\ \}$$

$$\begin{aligned}\mathsf{DP}(R) = \{\ & \mathrm{f}_\#(\mathrm{g}(x)) \to \mathrm{g}_\#(\mathrm{g}(\mathrm{f}(\mathrm{f}(x)))),\\ & \mathrm{f}_\#(\mathrm{g}(x)) \to \mathrm{g}_\#(\mathrm{f}(\mathrm{f}(x))),\\ & \mathrm{f}_\#(\mathrm{g}(x)) \to \mathrm{f}_\#(\mathrm{f}(x)),\\ & \mathrm{f}_\#(\mathrm{g}(x)) \to \mathrm{f}_\#(x)\ \}\end{aligned}$$

R is non-terminating:

$$\mathrm{f}(\mathrm{g}(\mathrm{g}(x))) \to \mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{f}(\mathrm{g}(x))))) \to \mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{f}(x))))))) \to \dots$$

How does an infinite $\to_R \cup \xrightarrow{\text{top}}_{\mathsf{DP}(R)}$ rewrite sequence look like?

$$\mathrm{f}_\#(\mathrm{g}(\mathrm{g}(x))) \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \mathrm{f}_\#(\mathrm{f}(\mathrm{g}(x))) \to_R \mathrm{f}_\#(\mathrm{g}(\mathrm{g}(\mathrm{f}(\mathrm{f}(x))))) \to \dots$$

Dependency Pairs, Examples

Example

$$R = \{\ \mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y)),$$
$$\mathrm{A}(x, 0) \to x\ \}$$

$$\mathsf{DP}(R) = \{\ \mathrm{A}_{\#}(x, \mathrm{s}(y)) \to \mathrm{s}_{\#}(\mathrm{A}(x, y)),$$
$$\mathrm{A}_{\#}(x, \mathrm{s}(y)) \to \mathrm{A}_{\#}(x, y)\ \}$$

Dependency Pairs, Main Theorem

Definition (Relative Termination)

A relation $\to_1$ is called terminating relative to $\to_2$, denoted $\mathsf{SN}(\to_1 / \to_2)$, if every $\to_1 \cup \to_2$ rewrite sequence contains only finitely many $\to_1$ steps.

Lemma

$$\mathsf{SN}(\to_1 / \to_2) \iff \mathsf{SN}(\to_2^* \cdot \to_1 \cdot \to_2^*)$$

The main theorem from dependency pairs is:

Theorem

$$\mathsf{SN}(R) \iff \mathsf{SN}(\mathsf{DP}(R)_{top}/R)$$

That is, a TRS R is terminating if and only if $\xrightarrow{\text{top}}_{\mathsf{DP}(R)}$ terminates relative to $\to_R$.

Dependency Pairs, Termination Proofs

Definition

A well-founded weakly monotone Σ-algebra $(A, [\cdot], >, \succeq)$ consists of:

- a Σ-algebra $(A, [\cdot])$ with relations $>$, $\succeq$ on A
- $>$ is well-founded,
- $> \cdot \succeq \subseteq >$ (compatibility),
- for all $f \in \Sigma$ the function $[f]$ is monotone w.r.t. $\succeq$

Theorem

$\mathrm{SN}(\mathrm{DP}(R)_{top}/R)$ *if there exists a weakly monotone* Σ*-algebra s.t.*

- $\mathrm{DP}(R) \subseteq >$ *that is,* $[\ell, \alpha] > [r, \alpha] \quad \forall \alpha,\ \ell \to r \in \mathrm{DP}(R)$
- $R \subseteq \succeq$ *that is,* $[\ell, \alpha] \succeq [r, \alpha] \quad \forall \alpha,\ \ell \to r \in R$

Advantages: no monotonicity for $>$, and $\succeq$ not well-founded.

Frequently used are polynomial interpretations over $\mathbb{N}$:

- $>$ as usual on $\mathbb{N}$ and $\succeq := \geq$
- the interpretations $[f]$ are polynomials

We will see some examples...

Example: $\mathrm{f}(\mathrm{f}(x)) \to \mathrm{f}(\mathrm{g}(\mathrm{f}(x)))$

Example

$$\begin{aligned}\mathsf{DP}(R) = \{\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(\mathrm{g}(\mathrm{f}(x))),\\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{g}_\#(\mathrm{f}(x)),\\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(x)\ \}\end{aligned}$$

$[\mathrm{f}](x) = ???$ $\quad [\mathrm{f}_\#](x) = ???$ $\quad [\mathrm{g}](x) = ???$ $\quad [\mathrm{g}_\#](x) = ???$

Example: $\mathrm{f}(\mathrm{f}(x)) \to \mathrm{f}(\mathrm{g}(\mathrm{f}(x)))$

Example

$$\begin{aligned} \mathsf{DP}(R) = \{\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(\mathrm{g}(\mathrm{f}(x))), \\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{g}_\#(\mathrm{f}(x)), \\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(x)\ \} \end{aligned}$$

$$[\mathrm{f}](x) = x + 1 \qquad [\mathrm{f}_\#](x) = x + 1 \qquad [\mathrm{g}](x) = 0 \qquad [\mathrm{g}_\#](x) = 0$$

- Are the functions $[f]$ monotone w.r.t. $\geq$?
  Yes, since whenever $a \geq b$, then
  $[\mathrm{f}](a) = a \geq b = [\mathrm{f}](b)$,
  $[\mathrm{g}](a) = 0 \geq 0 = [\mathrm{g}](b)$,

Example: $\mathrm{f}(\mathrm{f}(x)) \to \mathrm{f}(\mathrm{g}(\mathrm{f}(x)))$

Example

$$\begin{aligned} \mathsf{DP}(R) = \{\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(\mathrm{g}(\mathrm{f}(x))), \\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{g}_\#(\mathrm{f}(x)), \\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(x)\ \} \end{aligned}$$

$$[\mathrm{f}](x) = x + 1 \qquad [\mathrm{f}_\#](x) = x + 1 \qquad [\mathrm{g}](x) = 0 \qquad [\mathrm{g}_\#](x) = 0$$

- Does $[\ell, \alpha] > [r, \alpha]$ for all $\ell \to r \in \mathsf{DP}(R)$ hold?

$$[\mathrm{f}_\#(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 > 1 = [\mathrm{f}_\#(\mathrm{g}(\mathrm{f}(x))), \alpha]$$
$$[\mathrm{f}_\#(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 > 0 = [\mathrm{g}_\#(\mathrm{f}(x)), \alpha]$$
$$[\mathrm{f}_\#(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 > \alpha(x) + 1 = [\mathrm{f}_\#(x), \alpha]$$

- Does $[\ell, \alpha] \geq [r, \alpha]$ for all $\ell \to r \in R$ hold?

$$[\mathrm{f}(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 \geq 1 = [\mathrm{f}(\mathrm{g}(\mathrm{f}(x))), \alpha]$$

Hence we have proven termination.

Stepwise Termination Proofs

Stepwise termination proofs with monotone Σ-algebras:

Theorem

If there exists a monotone Σ-algebra $(A, [\cdot], >)$ s.t.

- $R \subseteq \geq$, *and*
- $R' \subseteq >$, *and*

where $\geq := > \cup =$. Then

$$\mathsf{SN}(R) \implies \mathsf{SN}(R \cup R')$$

This theorem allows us to stepwise remove rules until none are left.

Remark

Instead of $\geq$ we can more generally use a monotone relation $\succeq$ with $> \cdot \succeq \subseteq >$

Example

Example

$$\mathrm{f}(\mathrm{f}(x)) \to \mathrm{g}(x) \qquad\qquad \mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{f}(x))$$

We use the following interpretation:

$$[\mathrm{f}](x) = x + 1 \qquad\qquad [\mathrm{g}](x) = x + 1$$

We get the following interpretation of rules:

$$[\mathrm{f}(\mathrm{g}(x)), \alpha] = \alpha(x) + 2 > \alpha(x) + 1 = [\mathrm{g}(x), \alpha]$$
$$[\mathrm{f}(\mathrm{g}(x)), \alpha] = \alpha(x) + 2 \geq \alpha(x) + 2 = [\mathrm{g}(\mathrm{f}(x)), \alpha]$$

The first rule is strictly decreasing, hence we can remove it.

Thus for termination it sufficies to show $\mathsf{SN}(\mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{f}(x)))$.

We have already shown this a few slides ago.
Hence we have proven termination.

Stepwise Termination Proofs

Stepwise termination proofs with dependency pairs:

Theorem

If there exists a weakly monotone Σ-algebra s.t.

- $T_1 \subseteq >$
- $T_2 \cup R \subseteq \succeq$

Then

$$\mathsf{SN}(T_{2,top}/R) \implies \mathsf{SN}((T_1 \cup T_2)_{top}/R)$$

That is, we may remove the strictly decreasing top-rules.

Typically, $T_1, T_2 \subseteq \mathsf{DP}(R)$. But we can also tackle other top-termination problems.

We are not allowed to remove strictly decreasing rules in R!
(for removing from R we need monotonic interpretations)

Example

$$\begin{aligned} \mathrm{minus}(x, 0) &\to x \\ \mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}(x, y) \\ \mathrm{quot}(0, \mathrm{s}(y)) &\to 0 \\ \mathrm{quot}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y))) \end{aligned}$$

$$\begin{aligned} \mathsf{DP}(R) = \{\ \mathrm{minus}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}_{\#}(x, y) \\ \mathrm{quot}_{\#}(0, \mathrm{s}(y)) &\to 0_{\#} \\ \mathrm{quot}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}_{\#}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y))) \\ \mathrm{quot}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{quot}_{\#}(\mathrm{minus}(x, y), \mathrm{s}(y)) \\ \mathrm{quot}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}_{\#}(x, y) \\ \mathrm{quot}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}_{\#}(y)\ \} \end{aligned}$$

We use the interpretation:

$$[\mathrm{minus}_{\#}](x, y) = 1 \quad [\mathrm{quot}_{\#}](x, y) = 1 \quad [\mathrm{minus}](x, y) = x \quad [\mathrm{s}](x) = x$$

$$[\mathrm{f}](\vec{x}) = 0 \text{ for all other symbols } \mathrm{f}$$

Example

$$\begin{aligned}
\mathrm{minus}(x, 0) &\to x \\
\mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}(x, y) \\
\mathrm{quot}(0, \mathrm{s}(y)) &\to 0 \\
\mathrm{quot}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y)))
\end{aligned}$$

$$\begin{aligned}
\mathrm{DP}(R) = \{\ \mathrm{minus}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}_\#(x, y) \\
\mathrm{quot}_\#(0, \mathrm{s}(y)) &\to 0_\# \\
\mathrm{quot}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}_\#(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y))) \\
\mathrm{quot}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{quot}_\#(\mathrm{minus}(x, y), \mathrm{s}(y)) \\
\mathrm{quot}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}_\#(x, y) \\
\mathrm{quot}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}_\#(y)\ \}
\end{aligned}$$

The following interpretation removes the remaining DP rules and proves:

$$[\mathrm{minus}_\#](x, y) = [\mathrm{minus}](x, y) = [\mathrm{quot}_\#](x, y) = [\mathrm{quot}](x, y) = x$$

$$[\mathrm{s}](x) = x + 1 \qquad [0] = 0$$

Dependency Graphs

Example

$$\begin{aligned} \mathrm{minus}(x, 0) &\to x \\ \mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}(x, y) \\ \mathrm{quot}(0, \mathrm{s}(y)) &\to 0 \\ \mathrm{quot}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y))) \end{aligned}$$

Dependency graph: analysis which DP-rules may follow each other

$$(1)\ \mathrm{minus}_\#(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{minus}_\#(x, y) \circlearrowleft$$

$$\uparrow$$

$$(2)\ \mathrm{quot}_\#(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{minus}_\#(x, y)$$

$$\uparrow$$

$$(3)\ \mathrm{quot}_\#(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{quot}_\#(\mathrm{minus}(x, y), \mathrm{s}(y)) \circlearrowleft$$

Idea: consider only strongly connected components $\mathsf{SN}(\{1\}_{top}/R)$, $\mathsf{SN}(\{3\}_{top}/R)$.

Subterm Criterion

Theorem (Subterm Criterion)

Let R be a TRS, $T_1, T_2 \subseteq \mathsf{DP}(R)$, *and* $\pi : \Sigma_\# \to \mathbb{N}$ *such that:*

- $s_{\pi(f_\#)} \rhd t_{\pi(g_\#)}$ *for every rule* $f_\#(s_1, \ldots, s_n) \to g_\#(t_1, \ldots, t_m) \in T_1$
- $s_{\pi(f_\#)} = t_{\pi(g_\#)}$ *for every rule* $f_\#(s_1, \ldots, s_n) \to g_\#(t_1, \ldots, t_m) \in T_2$

Then:

$$\mathsf{SN}(T_{2,top}/R) \implies \mathsf{SN}((T_1 \cup T_2)_{top}/R)$$

Proof.

After the dependency pairs transformation, we consider only minimal terms.

We can only finitely often make a terminating term smaller ($\rhd$). ■

Example: Ackermann function

Example

$$\begin{aligned}
\mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\
\mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\
\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

$$\begin{aligned}
\mathsf{DP}(R) = \{\ \mathrm{Ack}_\#(0, y) &\to \mathrm{s}_\#(y) \\
\mathrm{Ack}_\#(\mathrm{s}(x), 0) &\to \mathrm{Ack}_\#(x, \mathrm{s}(0)) \\
\mathrm{Ack}_\#(\mathrm{s}(x), 0) &\to \mathrm{s}_\#(0) \\
\mathrm{Ack}_\#(\mathrm{s}(x), 0) &\to 0_\# \\
\mathrm{Ack}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}_\#(x, \mathrm{Ack}(\mathrm{s}(x), y)) \\
\mathrm{Ack}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}_\#(\mathrm{s}(x), y) \\
\mathrm{Ack}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}_\#(x)\ \}
\end{aligned}$$

We use the interpretation:

$$[\mathrm{Ack}_\#](x, y) = 1 \quad [\mathrm{Ack}](x, y) = 0 \quad [\mathrm{s}](x) = 0 \quad [0] = 0 \quad \mathrm{s}_\#(x) = 0$$

Example: Ackermann function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

$$\begin{aligned} \mathsf{DP}(R) = \{\ & \mathrm{Ack}_{\#}(\mathrm{s}(x), 0) \to \mathrm{Ack}_{\#}(x, \mathrm{s}(0)) \\ & \mathrm{Ack}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_{\#}(x, \mathrm{Ack}(\mathrm{s}(x), y)) \\ & \mathrm{Ack}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_{\#}(\mathrm{s}(x), y)\ \} \end{aligned}$$

We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = ?$$

Example: Ackermann function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

$$\begin{aligned}\mathsf{DP}(R) = \{\ & \mathrm{Ack}_\#(\mathrm{s}(x), 0) \to \mathrm{Ack}_\#(x, \mathrm{s}(0)) \\ & \mathrm{Ack}_\#(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_\#(x, \mathrm{Ack}(\mathrm{s}(x), y)) \\ & \mathrm{Ack}_\#(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_\#(\mathrm{s}(x), y)\ \}\end{aligned}$$

We use the subterm criterion:

$$\pi(\mathrm{Ack}_\#) = 1$$

We can remove the first two DP-rules.

Example: Ackermann function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

$$\mathsf{DP}(R) = \{\ \mathrm{Ack}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_{\#}(\mathrm{s}(x), y)\ \}$$

We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = ?$$

Example: Ackermann function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

$$\mathsf{DP}(R) = \{\ \mathrm{Ack}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_{\#}(\mathrm{s}(x), y)\ \}$$

We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = 2$$

We can remove the remaining DP-rule.

Hence we have proven termination.

Iterative Lexicographic Path Order (ILPO)

ILPO. . . Historical overview

Kamin and Lévy [1980] (lexicographic path order, LPO):

- Kruskal's Tree Theorem was used in the original proofs
- Buchholz [1995] simplified the proof: Kruskal not needed

Bergsta and Klop [1985]:

- Iterative version of RPO: 'star method'
- Operational definition of reduction order
  via an auxiliary term rewriting system (with stars)

Klop, van Oostrom and de Vrijer [2005]:

- Extension of the star method to LPO
- Iterative lexicographic path order (ILPO)

ILPO... The star TRS $\mathcal{L}\mathrm{ex}_{\succ}$

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}_{\succ}$

- Signature: $\Sigma \uplus \Sigma^*$, where $\Sigma^* = \{f^* \mid f \in \Sigma\}$

  f^* is fresh and has the same arity as f

- Reduction rules (four types, for arbitrary $f, g \in \Sigma$):

$$
\begin{aligned}
f(\vec{x}) &\to_{\text{put}} f^*(\vec{x}) \\
f^*(\vec{x}) &\to_{\text{select}} x_i \\
f^*(\vec{x}) &\to_{\text{copy}} g(f^*(\vec{x}), \ldots, f^*(\vec{x})) && \text{if } f \succ g \\
f^*(\vec{x}, g(\vec{y}), \vec{z}) &\to_{\text{lex}} f(\vec{x}, g^*(\vec{y}), l, \ldots, l) && \text{where } l = f^*(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
$$

Definition (ILPO)

$\succ_{ilpo}$ is the restriction of $\to^+_{\mathcal{L}\mathrm{ex}_{\succ}}$ to terms over Σ, i.e.

$$t \succ_{ilpo} s \iff t \to^+_{\mathcal{L}\mathrm{ex}_{\succ}} s \ \wedge\ t, s \in T(\Sigma \uplus V)$$

ILPO

Claim

$\succ_{ilpo}$ is a reduction order, that is:

- $\succ_{ilpo}$ is well-founded, and
- closed under substitution and contexts

The closure under substitutions and contexts is immediate since $\rightarrow^{+}_{\mathcal{L}\mathrm{ex}_{\succ}}$ is.

Only required: proof of termination of $\succ_{ilpo}$.

Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

Proof. $\succ_{ilpo}$ is a reduction order with $\mathcal{R} \subseteq \succ_{ilpo}$.

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \rightarrow & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \rightarrow & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \rightarrow & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \rightarrow & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{array}{rl} \mathtt{A}(x,\mathtt{0}) & \rightarrow_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \rightarrow_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) & \rightarrow_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ & \rightarrow_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \rightarrow_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \rightarrow_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \rightarrow_{\text{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \rightarrow_{\text{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y))) \\ & \rightarrow_{\text{select}} \mathtt{A}(x,\mathtt{M}^*(x,\mathtt{S}(y))) \rightarrow_{\text{lex}} \mathtt{A}(x,\mathtt{M}(x,\mathtt{S}^*(y))) \\ & \rightarrow_{\text{select}} \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}\mathrm{ex}^{\omega}$

But $\mathcal{L}$ex is in general not terminating, e.g. if $\mathtt{A} > \mathtt{S}$, then

$$\begin{aligned} \mathtt{A}(x,y) &\rightarrow_{\mathsf{put}} \mathtt{A}^*(x,y) \\ &\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,y)) \\ &\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y))) \\ &\dots \end{aligned}$$

Starred symbol $\mathtt{A}^*$ is 'used' infinitely often.

This is essential in any infinite reduction!

Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

$\Rightarrow$ Yields a terminating TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$.

Auxiliary TRS with numerical control symbols

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathsf{ex}^{\omega}_{\succ}$

- Signature: $\Sigma \uplus \Sigma^{\omega}$, where $\Sigma^{\omega} = \{f^n \mid f \in \Sigma,\ n \in \mathbb{N}\}$

  f^n is fresh and has same arity as f

- Reduction rules:

$$
\begin{aligned}
f(\vec{x}) &\to_{\mathsf{put}} f^n(\vec{x}) \\
f^n(\vec{x}) &\to_{\mathsf{select}} x_i \\
f^{n+1}(\vec{x}) &\to_{\mathsf{copy}} g(f^n(\vec{x}), \ldots, f^n(\vec{x})) && \text{if } f \succ g \\
f^{n+1}(\vec{x}, g(\vec{y}), \vec{z}) &\to_{\mathsf{lex}} f(\vec{x}, g^n(\vec{y}), l, \ldots, l) && \text{where } l = f^n(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
$$

Back and forth between $\mathcal{L}$ex and $\mathcal{L}$ex$^\omega$

From $\rightarrow_{\mathcal{L}\text{ex}^\omega}$ to $\rightarrow_{\mathcal{L}\text{ex}}$:

- every reduction can be transformed by replacing f^n by f^*.

From $\rightarrow_{\mathcal{L}\text{ex}}$ to $\rightarrow_{\mathcal{L}\text{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

For example:

$$\begin{aligned} \mathtt{M}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y))) \\ &\rightarrow_{\text{select}} \mathtt{A}(x, \mathtt{M}^*(x, \mathtt{S}(y))) \rightarrow_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^*(y))) \\ &\rightarrow_{\text{select}} \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

becomes:

$$\begin{aligned} \mathtt{M}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{M}^2(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{A}(\mathtt{M}^1(x, \mathtt{S}(y)), \mathtt{M}^1(x, \mathtt{S}(y))) \\ &\rightarrow_{\text{select}} \mathtt{A}(x, \mathtt{M}^1(x, \mathtt{S}(y))) \rightarrow_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^0(y))) \\ &\rightarrow_{\text{select}} \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

$\mathcal{L}$ex and $\mathcal{L}$ex$^\omega$

Theorem

$\rightarrow^+_{\mathcal{L}\mathrm{ex}_\succ}$ *and* $\rightarrow^+_{\mathcal{L}\mathrm{ex}^\omega_\succ}$ *coincide on* $T(\Sigma \uplus V)$

Note that the infinite reduction

$$\begin{aligned}
\mathtt{A}(x,y) &\rightarrow_{\mathsf{put}} \mathtt{A}^*(x,y) \\
&\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,y)) \\
&\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y))) \\
&\dots
\end{aligned}$$

cannot be lifted:

$$\begin{aligned}
\mathtt{A}(x,y) &\rightarrow_{\mathsf{put}} \mathtt{A}^?(x,y) \\
&\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^{?-1}(x,y)) \\
&\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^{?-2}(x,y))) \\
&\dots
\end{aligned}$$

Termination of $\mathcal{L}\mathsf{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating } \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f,\ \vec{t},\ \ell\rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n, > \rangle$.

- Here $(\rightarrow_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

In general a term is SN if all one-step reducts are SN.

$\Rightarrow$ We check all one step reducts of $f^\ell(t_1, \ldots, t_n)$.

Termination of $\mathcal{L}\mathsf{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n, > \rangle$.

Case 1. Internal step $f^\ell(\ldots, t_i, \ldots) \to f^\ell(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Case 2. $f(t_1, \ldots, t_n) \to_{\mathsf{put}} f^n(t_1, \ldots, t_n)$.
We have a decrease in the third component.

Case 3. $f(t_1, \ldots, t_n) \to_{\mathsf{select}} t_i$.
By assumption t_i is SN.

Termination of $\mathcal{L}\text{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\text{ex}^\omega_\succ})^n, > \rangle$.

Case 4. $f^{n+1}(\vec{t}) \rightarrow_{\text{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.
By IH the arguments $f^n(\vec{t})$ of g are SN since $n + 1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Case 5. $f^{n+1}(\vec{t}, g(\vec{s}), \vec{r}) \rightarrow_{\text{lex}} f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$, $l = f^n(\vec{t}, g(\vec{y}), \vec{r})$.
By assumption the arguments $\vec{t}$ and $g(\vec{s})$ are SN.
By IH we get l is SN.
Since $g(\vec{s}) \rightarrow_{\text{put}} g^n(\vec{s})$ we get

- $g^n(\vec{s})$ is SN, and
- the triple decreases in the second component.

Thus by IH $f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$ is SN.

Termination of $\mathcal{L}\text{ex}$ and $\mathcal{L}\text{ex}^\omega$

Hence we have proven:

Theorem

$\rightarrow_{\mathcal{L}\text{ex}^\omega_\succ}$ *is terminating*

Corollary

$\rightarrow^+_{\mathcal{L}\text{ex}}$ *is terminating on* $T(\Sigma \uplus V)$

Proof. $\rightarrow^+_{\mathcal{L}\text{ex}}$ and $\rightarrow^+_{\mathcal{L}\text{ex}^\omega}$ coincide on $T(\Sigma \uplus V)$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \to_{\mathsf{copy}} \mathrm{Ack}(\mathrm{x}, \mathrm{s}(\mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{s}(0))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\begin{aligned}
\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})) \\
&\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\
&\to_{\mathsf{lex}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}^*(\mathrm{y}))) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

Hence we have proven termination.

Recursive definition of LPO

Let $\succ$ be a strict order on signature Σ

Define $\succ_{lpo}$ on $T(\Sigma, V)$ by: $s \succ_{lpo} t$ iff

(LPO1) $t \in \mathcal{V}ar(s)$ and $s \neq t$, or

(LPO2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (LPO2a) $\exists 1 \leq i \leq m$, with $s_i = t$ or $s_i \succ_{lpo} t$, or
- (LPO2b) $f \succ g$ and $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, or
- (LPO2c) $f = g$, and
  $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, and
  there exists $1 \leq i \leq m$, s.t.
  $s_1 = t_1$, …, $s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

Theorem

$\succ_{ilpo}$ is equivalent with $\succ_{lpo}$