- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

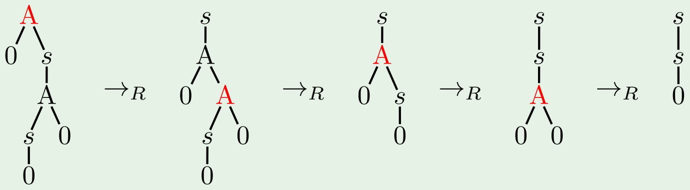
Outline

Termination

Termination, Example 1

Example

$$A(x, s(y)) \to s(A(x, y))$$
$$A(x, 0) \to x$$



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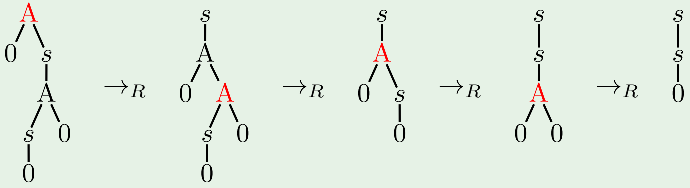

Looks terminating:

- second rule makes terms smaller
- first rule makes 's' move upwards

Termination, Example 2

Example

$$f(g(x)) \to g(f(x))$$

Termination, Example 2

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Looks terminating:

- f's move to the right
- g's move to the left

Termination, Example 3

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Termination, Example 3

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Termination, Example 3

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Looks terminating:

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But we have an infinite rewrite sequence:

$$\begin{aligned} & f(g(g(x))) \\ \to\ & g(g(f(f(g(x))))) \\ \to\ & g(g(f(g(g(f(f(x))))))) \\ \to\ & \ldots \end{aligned}$$

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We need proofs of termination!

Termination, Example 4

Example

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Definition (Termination SN(R))

A rewrite system R is terminating if there are no infinite rewrite sequences.

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Termination Methods 1967

Knuth-Bendix order

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Knuth-Bendix order, polynomial interpretations

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Knuth-Bendix order, polynomial interpretations, multiset order, simple path order, lexicographic path order, semantic path order, recursive decomposition order, multiset path order, recursive path order, transformation order, elementary interpretations, type introduction, well-founded monotone algebras, general path order, semantic labeling, dummy elimination, dependency pairs, freezing, top-down labeling

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Termination Research

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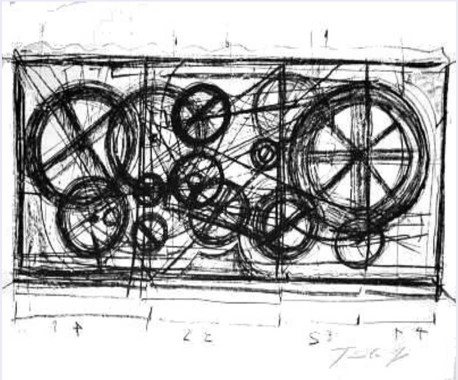

Termination Tools

AProVE, Cariboo, CiME, Jambox, Termptation, Matchbox, MuTerm, NTI, Torpa, TPA, $\mathsf{T_TT_2}$, VMTL, . . .

Outline

Lemma

TRS R is terminating

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$$s > t \quad\iff\quad \varphi(s) >_{\mathbb{N}} \varphi(t) \text{ with } \varphi(u) = \begin{cases} 1 & \text{if } u = 0 \\ \varphi(v) + 1 & \text{if } u = s(v) \\ 2\varphi(v) + \varphi(w) & \text{if } u = v + w \\ 0 & \text{otherwise} \end{cases}$$

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Remark

(very) inconvenient to check all rewrite steps

Definitions

A reduction order is well-founded order $>$ on terms which is

- closed under contexts $\quad s > t \implies C[s] > C[t]$
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Let $>$ be reduction order such that $R \subseteq >$.

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However, this contradicts well-foundedness of $>$. ∎

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Example (We use the Σ-algebra $(\mathbb{N}, [\cdot])$ with)

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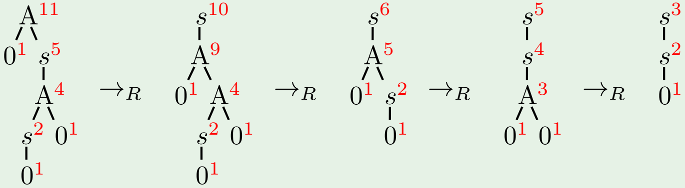

Interpretation of terms

Definition (Interpretation of Terms)

Let $\alpha : \mathcal{X} \to A$ be an interpretation of the variables.
We define the evaluation of terms

$$[\cdot, \alpha] : \mathcal{T}(\Sigma, \mathcal{X}) \to A$$

inductively:

$$[x, \alpha] = \alpha(x) \qquad \text{if } x \in \mathcal{X}$$
$$[f(t_1, \ldots, t_n), \alpha] = [f]([t_1, \alpha], \ldots, [t_n, \alpha])$$

- $[\mathrm{A}(0, \mathrm{s}(0)), \alpha] = 5$,
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Monotone Σ-algebras

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A well-founded monotone Σ-algebra $(A, [\cdot], >)$:

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Lemma

$>_{\mathcal{A}}$ is a reduction order for every well-founded monotone Σ-algebra $(A, [\cdot], >)$

Lemma

Let $\mathcal{A}$ well-founded monotone Σ-algebra. Then $>_{\mathcal{A}}$ is closed under substitutions.

Let σ be a substitution. We show $t\sigma >_{\mathcal{A}} s\sigma$.

That is: $[t\sigma, \alpha] > [s\sigma, \alpha]$ for every $\alpha : \mathcal{X} \to A$.

Define $\beta : \mathcal{X} \to A$ by $\beta(x) = [\sigma(x), \alpha]$. Claim: $[u\sigma, \alpha] = [u, \beta]$ for all u.

- Proof of the claim by induction over the term structure of u:

$$
\begin{aligned}
[x\sigma, \alpha] &= [\sigma(x), \alpha] = [x, \beta] \\
[f(t_1, \ldots, t_n)\sigma, \alpha] &= [f]([t_1\sigma, \alpha], \ldots, [t_n\sigma, \alpha]) \\
&\overset{\text{IH}}{=} [f]([t_1, \beta], \ldots, [t_n, \beta]) \\
&= [f(t_1, \ldots, t_n), \beta]
\end{aligned}
$$

Hence $[t\sigma, \alpha] = [t, \beta] > [s, \beta] = [s\sigma, \alpha]$. ∎

Lemma

Let $\mathcal{A}$ well-founded monotone Σ-algebra. Then $>_{\mathcal{A}}$ is closed under substitutions.

Proof.

Let $s, t \in \mathcal{T}(\Sigma, \mathcal{X})$ such that $t >_{\mathcal{A}} s$.

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Hence $[t\sigma, \alpha] = [t, \beta] > [s, \beta] = [s\sigma, \alpha]$. ■

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This infinite decreasing $>$ chain contradicts well-foundedness of $>$. ∎

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Hence we have proven that:

Lemma

$>_{\mathcal{A}}$ is a reduction order for every well-founded monotone Σ-algebra $(A, [\cdot], >)$

Theorem

TRS R is terminating $\iff$ $R \subseteq >_{\mathcal{A}}$ *for well-founded monotone algebra* $(\mathcal{A}, >)$

$\Leftarrow$ Follows since we have shown that $>_{\mathcal{A}}$ is a reduction order.

$\Rightarrow$ As Σ-algebra we take $(A, [\cdot], >)$ with

- $A = \mathcal{T}(\Sigma, \mathcal{X})$,
- $[f](t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$,
- $> := \to$.

Then $>$ is well-founded since R is terminating.

Monotonicity of $>$ follows from closure of $\to$ under contexts:

$$s > t \Rightarrow [f](\ldots, s, \ldots) = f(\ldots, s, \ldots) > f(\ldots, t, \ldots) = [f](\ldots, t, \ldots)$$

$R \subseteq >_{\mathcal{A}}$ since for all $\ell \to r \in R$ and $\alpha : \mathcal{X} \to A$:

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$\Rightarrow$ As Σ-algebra we take $(A, [\cdot], >)$ with

- $A = \mathcal{T}(\Sigma, \mathcal{X})$,
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∎

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used in termination proofs/tools:

- polynomial interpretations over $\mathbb{N}$

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Outline

Polynomial interpretations

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A polynomial interpretations over $\mathbb{N}$ consists of:

- Σ-algebra is $(\mathbb{N}, [\cdot])$,
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- the interpretations $[f]$ are polynomials.

- monotonicity of the polynoms $[f]$, and
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$$\mathrm{f}(\mathrm{g}(x)) \to \mathrm{f}(\mathrm{f}(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

$$[\mathrm{f}](x) = ??? \qquad [\mathrm{g}](x) = ???$$

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$$f(g(x)) \to f(f(x))$$

Find a polynomial interpretation over $\mathbb{N}$ which proves termination:

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$$\mathrm{A}(x, 0) \to x$$
$$\mathrm{A}(x, \mathrm{s}(y)) \to \mathrm{s}(\mathrm{A}(x, y))$$

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Why do we need the conditions?

Example ($>$ not well-founded)

Let $R = \{ \mathrm{f}(x) \to \mathrm{f}(\mathrm{f}(x)) \}$ with the Σ-algebra $(\mathbb{Z}, [\cdot])$ and

$$[\mathrm{f}](x) = x - 1$$

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but

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(a) choose abstract polynomial interpretations (linear, quadratic, ...)

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(c) add monotonicity and well-definedness constraints

(d) eliminate universally quantified variables

(e) translate resulting constraints to SAT or SMT problem

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- rewrite system

$$\begin{aligned} 0 + y &\to y \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \end{aligned}$$

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$$\begin{aligned} (e - 1)y + da + f &> 0 \\ (e - be)y + dc + f - bf - c &> 0 \end{aligned}$$

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$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

- possible solution

$$a = 0 \quad b = 1 \quad c = 1 \quad d = 2 \quad e = 1 \quad f = 1$$

Example

- rewrite system

$$\begin{aligned} 0 + y &\to y \\ \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \end{aligned}$$

- interpretations

$$\begin{aligned} 0_{\mathcal{A}} &= 0 \\ \mathsf{s}_{\mathcal{A}}(x) &= x + 1 \\ +_{\mathcal{A}}(x, y) &= 2x + y + 1 \end{aligned}$$

- diophantine constraints

$$\begin{aligned} e - 1 \geqslant 0 \quad & da + f > 0 \\ e - be \geqslant 0 \quad & dc + f - bf - c > 0 \end{aligned}$$

$$a \geqslant 0 \quad b \geqslant 1 \quad c \geqslant 0 \quad d \geqslant 1 \quad e \geqslant 1 \quad f \geqslant 0$$

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Example

- TRS

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad\qquad \mathsf{s}(x) \times y \to y + (x \times y)$$

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$$0 + y \to y \qquad 0 \times y \to 0$$
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- interpretations in $\mathbb{N}$

$$0_{\mathcal{A}} = 1 \qquad +_{\mathcal{A}}(x, y) = 2x + y$$
$$\mathsf{s}_{\mathcal{A}}(x) = x + 1 \qquad \times_{\mathcal{A}}(x, y) = 2xy + x + y + 1$$

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- constraints $\quad \forall x, y \in \mathbb{N}$

$$y + 2 > y \qquad\qquad 3y + 2 > 1$$
$$2x + y + 2 > 2x + y + 1 \qquad\qquad 2xy + x + 3y + 2 > 2xy + x + 3y + 1$$

Example

- TRS

$$0 + y \to y \qquad\qquad 0 \times y \to 0$$

$$s(x) + y \to s(x + y) \qquad\qquad s(x) \times y \to y + (x \times y)$$

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- constraints $\forall x, y \in \mathbb{N}$

$$2 > 0 \qquad\qquad 3y + 1 > 0$$

$$1 > 0 \qquad\qquad 1 > 0$$

Example

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$$0 + y \to y \qquad\qquad 0 \times y \to 0$$
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- $\mathsf{s}(0) \times \mathsf{s}(\mathsf{s}(0)) \;\to\; \mathsf{s}(\mathsf{s}(0)) + (0 \times \mathsf{s}(\mathsf{s}(0))) \;\to\; \mathsf{s}(\mathsf{s}(0)) + 0 \;\to\; \mathsf{s}(\mathsf{s}(0) + 0)$

  $\to\; \mathsf{s}(\mathsf{s}(0 + 0)) \;\to\; \mathsf{s}(\mathsf{s}(0))$

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- $\underset{18}{\mathsf{s}(0) \times \mathsf{s}(\mathsf{s}(0))} \underset{>}{\to} \underset{17}{\mathsf{s}(\mathsf{s}(0)) + (0 \times \mathsf{s}(\mathsf{s}(0)))} \underset{>}{\to} \underset{7}{\mathsf{s}(\mathsf{s}(0)) + 0} \underset{>}{\to} \underset{6}{\mathsf{s}(\mathsf{s}(0) + 0)}$
  $\underset{>}{\to} \underset{5}{\mathsf{s}(\mathsf{s}(0 + 0))} \underset{>}{\to} \underset{3}{\mathsf{s}(\mathsf{s}(0))}$

Example

- TRS

$$\partial(x+y) \to \partial(x)+\partial(y) \qquad \partial(\alpha)=1$$
$$\partial(x-y) \to \partial(x)-\partial(y) \qquad \partial(\beta)=0$$
$$\partial(x\times y) \to (\partial(x)\times y)+(x\times\partial(y))$$
$$\partial(x\div y) \to ((\partial(x)\times y)-(x\times\partial(y)))\div(y\times y)$$

Example

- TRS

$$\begin{aligned}
\partial(x + y) &\to \partial(x) + \partial(y) & \partial(\alpha) &= 1 \\
\partial(x - y) &\to \partial(x) - \partial(y) & \partial(\beta) &= 0 \\
\partial(x \times y) &\to (\partial(x) \times y) + (x \times \partial(y)) \\
\partial(x \div y) &\to ((\partial(x) \times y) - (x \times \partial(y))) \div (y \times y)
\end{aligned}$$

- interpretations in $\mathbb{N}$

$$\begin{aligned}
\alpha_{\mathcal{A}} &= \beta_{\mathcal{A}} = 0_{\mathcal{A}} = 1_{\mathcal{A}} = 1 \\
+_{\mathcal{A}}(x, y) &= -_{\mathcal{A}}(x, y) = \times_{\mathcal{A}}(x, y) = \div_{\mathcal{A}}(x, y) = x + y + 3 \\
\partial_{\mathcal{A}}(x) &= x^2 + 6x + 6
\end{aligned}$$

Example

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\begin{array}{ll}
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\end{aligned}
$$

- constraints $\quad \forall x,y \in \mathbb{N}$

$$
\begin{array}{ll}
x^2+y^2+2xy+12x+12y+33 > x^2+y^2+6x+6y+15 & 13>1 \\
x^2+y^2+2xy+12x+12y+33 > x^2+y^2+6x+6y+15 & 13>1 \\
x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+7y+21 & \\
x^2+y^2+2xy+12x+12y+33 > x^2+y^2+7x+9y+27 &
\end{array}
$$

Example

- TRS

$$\begin{aligned} \partial(x+y) &\to \partial(x)+\partial(y) & \partial(\alpha) &= 1 \\ \partial(x-y) &\to \partial(x)-\partial(y) & \partial(\beta) &= 0 \\ \partial(x\times y) &\to (\partial(x)\times y)+(x\times\partial(y)) \\ \partial(x\div y) &\to ((\partial(x)\times y)-(x\times\partial(y)))\div(y\times y) \end{aligned}$$

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- constraints $\forall x, y \in \mathbb{N}$

$$\begin{aligned} 2xy+6x+6y+18 &> 0 & 13 &> 1 \\ 2xy+6x+6y+18 &> 0 & 13 &> 1 \\ 2xy+5x+5y+12 &> 0 \\ 2xy+5x+3y+\ \ 6 &> 0 \end{aligned}$$

Remark

Numerous terminating TRSs are not polynomially terminating.

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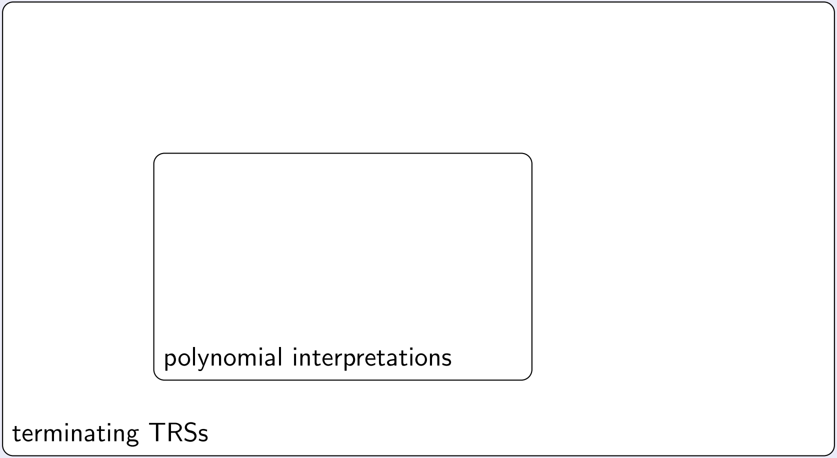

A non-simply terminating TRS: $f(f(x)) \to f(g(f(x)))$

Example ($S = \{f(f(x)) \to f(g(f(x)))\}$)

There exists no monotone Σ-algebra $\mathcal{A}$ with $A = \mathbb{N}$ proving $\mathsf{SN}(S)$.

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There exists no monotone Σ-algebra $\mathcal{A}$ with $A = \mathbb{N}$ proving $\mathsf{SN}(S)$.

Assume contrary it would exist. Let $\alpha : \mathcal{X} \to A$, then:

$$[f(f(x)), \alpha] > [f(g(f(x))), \alpha] \ .$$

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since otherwise:

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But then $\mathcal{A}$ would also prove termination of $f(x) \to g(f(x))$.

Thus we need another Σ-algebra, for example $A = \mathbb{N}^2$.

A non-simply terminating TRS: $f(f(x)) \to f(g(f(x)))$

Example ($S = \{f(f(x)) \to f(g(f(x)))\}$)

We choose the Σ-algebra $(\mathbb{N}^2, [\cdot])$ with:

$$[\mathrm{f}](\vec{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad [\mathrm{g}](\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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where $>$ on $\mathbb{N}^2$ is defined as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \iff x_1 > y_1 \text{ and } x_2 \geq y_2$$

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We choose the Σ-algebra $(\mathbb{N}^2, [\cdot])$ with:

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Let $\alpha\colon \mathcal{X} \to A$ be arbitrary; write $\vec{x} = \alpha(x)$. We obtain

$$[f(f(x))] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} > \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [f(g(f(x)))]$$

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Hence S is terminating.

Termination via Dependency Pairs

Minimal Terms

We call a term is minimal if all its strict subterms are terminating.

We prove: every non-terminating term t contains a minimal subterm.

We use induction on the term structure of t:

- Base case: all strict subterms of t are terminating. Then t is minimal itself.
- Induction step: t has a strict subterm t' which is not terminating.

  Then by IH the term t' contains a minimal, non-terminating subterm t''.
  The term t'' is a minimal non-terminating subterm of t.

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Steps below the root do not change minimality:

Then since t is minimal, it follows that all t_k are terminating.

Since $t_k \to^* s_k$, we obtain s_k is terminating for all $1 \leq k \leq n$.

Hence s is minimal. ∎

Steps below the root do not change minimality:

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Let t be minimal and $t \to s$ a rewrite step below the root. Then s is minimal.

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We have $t = f(t_1, \ldots, t_n)$ and $s = f(s_1, \ldots, s_n)$ and $i \in \mathbb{N}$ such that:

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Notation: we use $\xrightarrow{\text{top}}_R$ to denote root rewrite steps.

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Let t_0 be a minimal and $t_0 \to_R t_1 \to_R t_2 \to_R \ldots$ an infinite rewrite sequence.

We consider the first root rewrite step $t_i \to t_{i+1}$:

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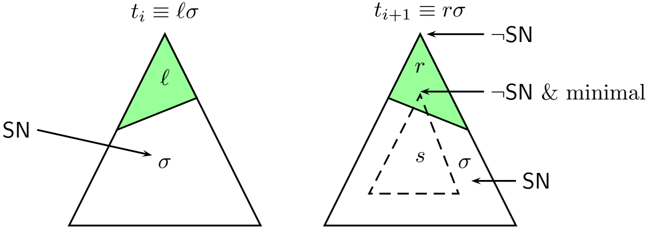

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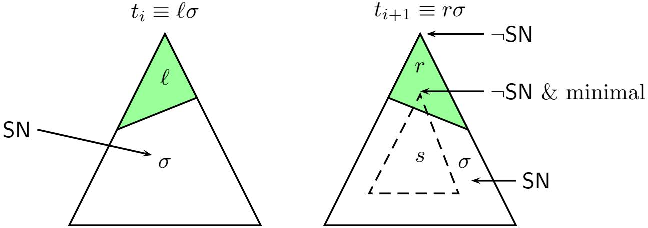

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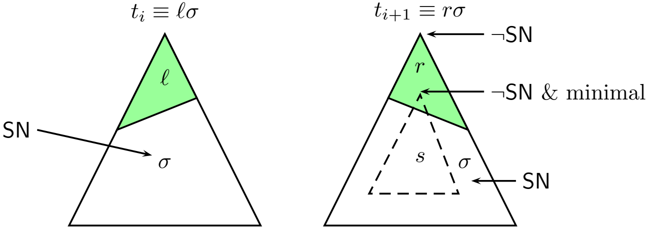

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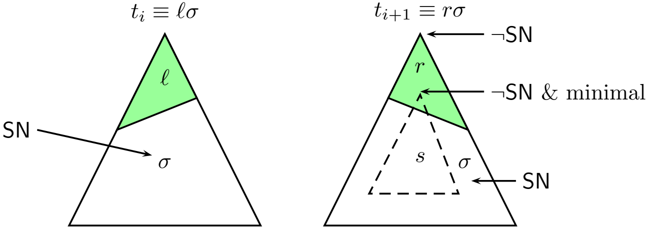

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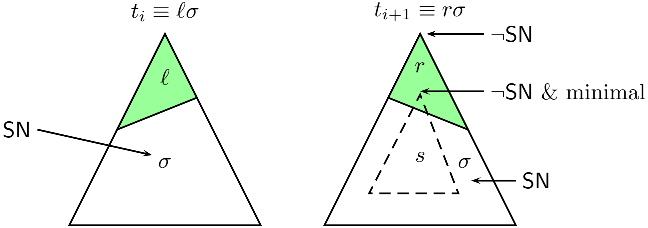

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Idea: add a rule $\ell \to r'$ then $t_i \to s$.

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For every $f \in \Sigma$ let $f_{\#}$ be a fresh symbol with the same arity as f.

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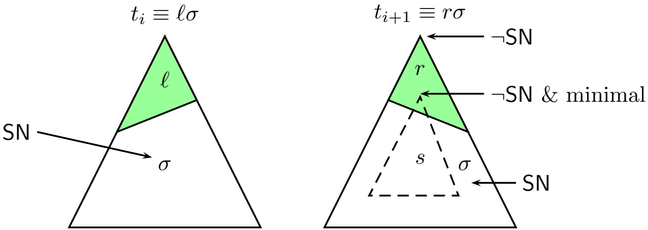

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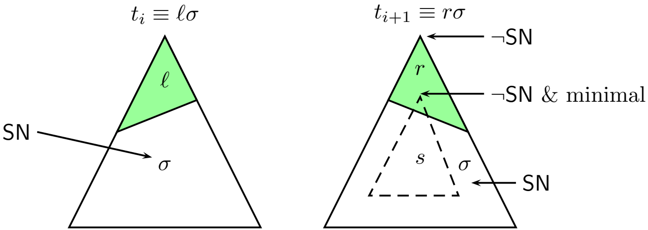

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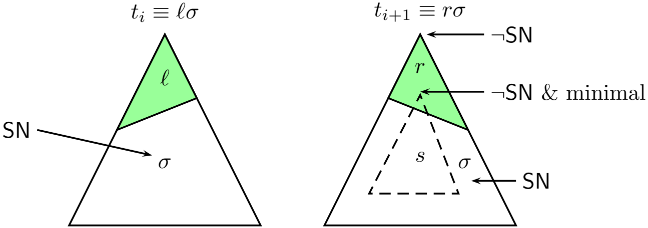

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Repeating the construction with s yields:

$$t_{0,\#} \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdot \to_R^* \cdot \xrightarrow{\text{top}}_{\mathsf{DP}(R)} \cdots$$

an infinite rewrite sequence containing infinitely many DP(R) steps.

Dependency Pairs

Lemma

Let R be a non-terminating TRS. Then there exists a rewrite sequence:

$$t_{0,\#} \rightarrow_R^* \cdot \xrightarrow{top}_{\mathsf{DP}(R)} \cdot \rightarrow_R^* \cdot \xrightarrow{top}_{\mathsf{DP}(R)} \cdot \rightarrow_R^* \cdot \xrightarrow{top}_{\mathsf{DP}(R)} \cdots$$

such that:

- *the sequence contains infinitely many* $\mathsf{DP}(R)$ *steps,*
- *all R steps are below the root, and*
- *all* $\mathsf{DP}(R)$ *steps are at the root.*

Dependency Pairs, Examples

Example

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Dependency Pairs, Main Theorem

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A relation $\to_1$ is called terminating relative to $\to_2$, denoted $\mathsf{SN}(\to_1 / \to_2)$, if every $\to_1 \cup \to_2$ rewrite sequence contains only finitely many $\to_1$ steps.

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That is, a TRS R is terminating if and only if $\xrightarrow{\text{top}}_{\mathsf{DP}(R)}$ terminates relative to $\rightarrow_R$.

Dependency Pairs, Termination Proofs

Definition

A well-founded weakly monotone Σ-algebra $(A, [\cdot], >, \succeq)$ consists of:

- a Σ-algebra $(A, [\cdot])$ with relations $>$, $\succeq$ on A
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Advantages: no monotonicity for $>$, and $\succeq$ not well-founded.

Frequently used are polynomial interpretations over $\mathbb{N}$:

- $>$ as usual on $\mathbb{N}$ and $\succeq := \geq$
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We will see some examples...

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  Yes, since whenever $a \geq b$, then
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$$[\mathrm{f}(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 \geq 1 = [\mathrm{f}(\mathrm{g}(\mathrm{f}(x))), \alpha]$$

Example: $\mathrm{f}(\mathrm{f}(x)) \to \mathrm{f}(\mathrm{g}(\mathrm{f}(x)))$

Example

$$\begin{aligned} \mathsf{DP}(R) = \{\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(\mathrm{g}(\mathrm{f}(x))), \\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{g}_\#(\mathrm{f}(x)), \\ & \mathrm{f}_\#(\mathrm{f}(x)) \to \mathrm{f}_\#(x)\ \} \end{aligned}$$

$$[\mathrm{f}](x) = x + 1 \qquad [\mathrm{f}_\#](x) = x + 1 \qquad [\mathrm{g}](x) = 0 \qquad [\mathrm{g}_\#](x) = 0$$

- Does $[\ell, \alpha] > [r, \alpha]$ for all $\ell \to r \in \mathsf{DP}(R)$ hold?

$$\begin{aligned} [\mathrm{f}_\#(\mathrm{f}(x)), \alpha] &= \alpha(x) + 2 > 1 = [\mathrm{f}_\#(\mathrm{g}(\mathrm{f}(x))), \alpha] \\ [\mathrm{f}_\#(\mathrm{f}(x)), \alpha] &= \alpha(x) + 2 > 0 = [\mathrm{g}_\#(\mathrm{f}(x)), \alpha] \\ [\mathrm{f}_\#(\mathrm{f}(x)), \alpha] &= \alpha(x) + 2 > \alpha(x) + 1 = [\mathrm{f}_\#(x), \alpha] \end{aligned}$$

- Does $[\ell, \alpha] \geq [r, \alpha]$ for all $\ell \to r \in R$ hold?

$$[\mathrm{f}(\mathrm{f}(x)), \alpha] = \alpha(x) + 2 \geq 1 = [\mathrm{f}(\mathrm{g}(\mathrm{f}(x))), \alpha]$$

Hence we have proven termination.

Stepwise Termination Proofs

Stepwise termination proofs with monotone Σ-algebras:

Theorem

If there exists a monotone Σ-algebra $(A, [\cdot], >)$ s.t.

- $R \subseteq \geq$, *and*
- $R' \subseteq >$, *and*

where $\geq := > \cup =$. *Then*

$$\mathsf{SN}(R) \implies \mathsf{SN}(R \cup R')$$

This theorem allows us to stepwise remove rules until none are left.

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Remark

Instead of $\geq$ *we can more generally use a monotone relation* $\succeq$ *with* $> \cdot \succeq \subseteq >$

Example

Example

$$f(f(x)) \to g(x) \qquad f(g(x)) \to g(f(x))$$

Example

Example

$$\mathrm{f}(\mathrm{f}(x)) \to \mathrm{g}(x) \qquad \mathrm{f}(\mathrm{g}(x)) \to \mathrm{g}(\mathrm{f}(x))$$

We use the following interpretation:

$$[\mathrm{f}](x) = x + 1 \qquad [\mathrm{g}](x) = x + 1$$

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$$[\mathrm{f}(\mathrm{g}(x)), \alpha] = \alpha(x) + 2 \geq \alpha(x) + 2 = [\mathrm{g}(\mathrm{f}(x)), \alpha]$$

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The first rule is strictly decreasing, hence we can remove it.

Example

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Thus for termination it sufficies to show $\mathsf{SN}(f(g(x)) \to g(f(x)))$.

Example

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The first rule is strictly decreasing, hence we can remove it.

Thus for termination it sufficies to show SN($f(g(x)) \to g(f(x))$).

We have already shown this a few slides ago.
Hence we have proven termination.

Stepwise Termination Proofs

Stepwise termination proofs with dependency pairs:

Theorem

If there exists a weakly monotone Σ-algebra s.t.

- $T_1 \subseteq >$
- $T_2 \cup R \subseteq \succeq$

Then

$$\mathsf{SN}(T_{2,top}/R) \implies \mathsf{SN}((T_1 \cup T_2)_{top}/R)$$

That is, we may remove the strictly decreasing top-rules.

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Typically, $T_1, T_2 \subseteq \mathsf{DP}(R)$. But we can also tackle other top-termination problems.

We are not allowed to remove strictly decreasing rules in R!
(for removing from R we need monotonic interpretations)

Example

$$
\begin{aligned}
\mathrm{minus}(x, 0) &\to x \\
\mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}(x, y) \\
\mathrm{quot}(0, \mathrm{s}(y)) &\to 0 \\
\mathrm{quot}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y)))
\end{aligned}
$$

Example

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$$

$$
\begin{aligned}
\mathsf{DP}(R) = \{\ \mathrm{minus}_\#(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}_\#(x, y) \\
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\end{aligned}$$

We use the interpretation:

$$[\mathrm{minus}_\#](x, y) = 1 \quad [\mathrm{quot}_\#](x, y) = 1 \quad [\mathrm{minus}](x, y) = x \quad [\mathrm{s}](x) = x$$

$$[\mathrm{f}](\vec{x}) = 0 \text{ for all other symbols } \mathrm{f}$$

Example

$$\begin{aligned} \mathrm{minus}(x, 0) &\to x \\ \mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{minus}(x, y) \\ \mathrm{quot}(0, \mathrm{s}(y)) &\to 0 \\ \mathrm{quot}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{s}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y))) \end{aligned}$$

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The following interpretation removes the remaining DP rules and proves:

$$[\mathrm{minus}_\#](x, y) = [\mathrm{minus}](x, y) = [\mathrm{quot}_\#](x, y) = [\mathrm{quot}](x, y) = x$$

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Dependency Graphs

Example

$$\mathrm{minus}(x, 0) \to x$$
$$\mathrm{minus}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{minus}(x, y)$$
$$\mathrm{quot}(0, \mathrm{s}(y)) \to 0$$
$$\mathrm{quot}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{s}(\mathrm{quot}(\mathrm{minus}(x, y), \mathrm{s}(y)))$$

Dependency Graphs

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Dependency graph: analysis which DP-rules may follow each other

$$(1)\ \mathrm{minus}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{minus}_{\#}(x, y) \circlearrowleft$$

$$\uparrow$$

$$(2)\ \mathrm{quot}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{minus}_{\#}(x, y)$$

$$\uparrow$$

$$(3)\ \mathrm{quot}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{quot}_{\#}(\mathrm{minus}(x, y), \mathrm{s}(y)) \circlearrowleft$$

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$$\uparrow$$

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Idea: consider only strongly connected components $\mathsf{SN}(\{1\}_{top}/R)$, $\mathsf{SN}(\{3\}_{top}/R)$.

Subterm Criterion

Theorem (Subterm Criterion)

Let R be a TRS, $T_1, T_2 \subseteq \mathsf{DP}(R)$, *and* $\pi : \Sigma_\# \to \mathbb{N}$ *such that:*

- $s_{\pi(f_\#)} \rhd t_{\pi(g_\#)}$ *for every rule* $f_\#(s_1, \ldots, s_n) \to g_\#(t_1, \ldots, t_m) \in T_1$
- $s_{\pi(f_\#)} = t_{\pi(g_\#)}$ *for every rule* $f_\#(s_1, \ldots, s_n) \to g_\#(t_1, \ldots, t_m) \in T_2$

Then:

$$\mathsf{SN}(T_{2,top}/R) \implies \mathsf{SN}((T_1 \cup T_2)_{top}/R)$$

We can only finitely often make a terminating term smaller ($\rhd$).

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Proof.

After the dependency pairs transformation, we consider only minimal terms.

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Example: Ackermann function

Example

$$\begin{aligned} \mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\ \mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\ \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y)) \end{aligned}$$

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We use the interpretation:

$$[\mathrm{Ack}_\#](x, y) = 1 \quad [\mathrm{Ack}](x, y) = 0 \quad [\mathrm{s}](x) = 0 \quad [0] = 0 \quad \mathrm{s}_\#(x) = 0$$

Example: Ackermann function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = ?$$

Example: Ackermann function

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We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = 1$$

We can remove the first two DP-rules.

Example: Ackermann function

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$$\mathsf{DP}(R) = \{ \ \mathrm{Ack}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_{\#}(\mathrm{s}(x), y) \ \}$$

We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = ?$$

Example: Ackermann function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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We can remove the remaining DP-rule.

Example: Ackermann function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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$$\mathsf{DP}(R) = \{\ \mathrm{Ack}_{\#}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}_{\#}(\mathrm{s}(x), y)\ \}$$

We use the subterm criterion:

$$\pi(\mathrm{Ack}_{\#}) = 2$$

We can remove the remaining DP-rule.

Hence we have proven termination.

Iterative Lexicographic Path Order (ILPO)

ILPO. . . Historical overview

Kamin and Lévy [1980] (lexicographic path order, LPO):

- Kruskal's Tree Theorem was used in the original proofs
- Buchholz [1995] simplified the proof: Kruskal not needed

Bergsta and Klop [1985]:

- Iterative version of RPO: 'star method'
- Operational definition of reduction order
  via an auxiliary term rewriting system (with stars)

Klop, van Oostrom and de Vrijer [2005]:

- Extension of the star method to LPO
- Iterative lexicographic path order (ILPO)

ILPO. . . The star TRS $\mathcal{L}\mathrm{ex}_{\succ}$

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}_{\succ}$

ILPO... The star TRS $\mathcal{L}\mathrm{ex}_{\succ}$

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}_{\succ}$

- Signature: $\Sigma \uplus \Sigma^*$, where $\Sigma^* = \{f^* \mid f \in \Sigma\}$

  f^* is fresh and has the same arity as f

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$$f(\vec{x}) \rightarrow_{\text{put}} f^*(\vec{x})$$

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$$f^*(\vec{x}) \rightarrow_{\text{select}} x_i$$

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- Reduction rules (four types, for arbitrary $f, g \in \Sigma$):

$$\begin{aligned} f(\vec{x}) &\to_{\text{put}} f^*(\vec{x}) \\ f^*(\vec{x}) &\to_{\text{select}} x_i \\ f^*(\vec{x}) &\to_{\text{copy}} g(f^*(\vec{x}), \ldots, f^*(\vec{x})) \qquad \text{if } f \succ g \end{aligned}$$

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$$
\begin{aligned}
f(\vec{x}) &\to_{\text{put}} f^*(\vec{x}) \\
f^*(\vec{x}) &\to_{\text{select}} x_i \\
f^*(\vec{x}) &\to_{\text{copy}} g(f^*(\vec{x}), \ldots, f^*(\vec{x})) && \text{if } f \succ g \\
f^*(\vec{x}, g(\vec{y}), \vec{z}) &\to_{\text{lex}} f(\vec{x}, g^*(\vec{y}), l, \ldots, l) && \text{where } l = f^*(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
$$

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\end{aligned}
$$

Definition (ILPO)

$\succ_{ilpo}$ is the restriction of $\rightarrow^+_{\mathcal{L}ex_{\succ}}$ to terms over Σ, i.e.

$$t \succ_{ilpo} s \iff t \rightarrow^+_{\mathcal{L}ex_{\succ}} s \ \wedge\ t, s \in T(\Sigma \uplus V)$$

ILPO

Claim

$\succ_{ilpo}$ is a reduction order, that is:

- $\succ_{ilpo}$ is well-founded, and
- closed under substitution and contexts

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The closure under substitutions and contexts is immediate since $\to^{+}_{\mathcal{L}\mathsf{ex}_{\succ}}$ is.

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Only required: proof of termination of $\succ_{ilpo}$.

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Only required: proof of termination of $\succ_{ilpo}$.

Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

ILPO

Claim

$\succ_{ilpo}$ is a reduction order, that is:

- $\succ_{ilpo}$ is well-founded, and
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The closure under substitutions and contexts is immediate since $\to^{+}_{\mathcal{L}\mathrm{ex}_{\succ}}$ is.

Only required: proof of termination of $\succ_{ilpo}$.

Corollary

A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ_{ilpo}$.

Proof. $\succ_{ilpo}$ is a reduction order with $\mathcal{R} \subseteq \succ_{ilpo}$.

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

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\mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

Example, Addition and multiplication

$$
\begin{array}{rcl}
\mathtt{A}(x, \mathtt{0}) & \rightarrow & x \\
\mathtt{A}(x, \mathtt{S}(y)) & \rightarrow & \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) & \rightarrow & \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) & \rightarrow & \mathtt{A}(x, \mathtt{M}(x, y))
\end{array}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\rightarrow x \\ \mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0})$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x,\mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x,\mathtt{0})$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\to x \\ \mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0}) \to_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0}) \to_{\mathsf{select}} x$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\rightarrow x \\ \mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0}) \rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\text{select}} x$$

$$\mathtt{A}(x, \mathtt{S}(y))$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\text{select}} x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y))
\end{aligned}
$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\rightarrow x \\ \mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\mathtt{A}(x, \mathtt{0}) \rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\text{select}} x$$

$$\mathtt{A}(x, \mathtt{S}(y)) \rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y)))$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
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Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\mathsf{select}} x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow_{\mathsf{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y))) \\
&\rightarrow_{\mathsf{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y)))
\end{aligned}
$$

Example, Addition and multiplication

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\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow x \\
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Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

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\mathtt{A}(x, \mathtt{0}) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\text{select}} x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y))) \\
&\rightarrow_{\text{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y))) \rightarrow_{\text{select}} \mathtt{S}(\mathtt{A}(x, y))
\end{aligned}
$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\rightarrow x \\ \mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\ \mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x, \mathtt{0}) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\text{select}} x \\ \mathtt{A}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y))) \\ &\rightarrow_{\text{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y))) \rightarrow_{\text{select}} \mathtt{S}(\mathtt{A}(x, y)) \\ \mathtt{M}(x, \mathtt{0}) & \end{aligned}$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \end{aligned}$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\to x \\
\mathtt{A}(x, \mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\to \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\to \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \to_{\text{select}} x \\
\mathtt{A}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y))) \\
&\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x, \mathtt{0}) \to_{\text{select}} \mathtt{0}
\end{aligned}
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Example, Addition and multiplication

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\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to x\\
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\mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\text{select}} x\\
\mathtt{A}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y)))\\
&\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x,y))\\
\mathtt{M}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\text{select}} \mathtt{0}\\
\mathtt{M}(x,\mathtt{S}(y)) &
\end{aligned}
$$

Example, Addition and multiplication

$$\begin{array}{rcl} \mathtt{A}(x,\mathtt{0}) & \to & x \\ \mathtt{A}(x,\mathtt{S}(y)) & \to & \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) & \to & \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) & \to & \mathtt{A}(x,\mathtt{M}(x,y)) \end{array}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

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$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\text{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \end{aligned}$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\text{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y))) \end{aligned}$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to x\\
\mathtt{A}(x,\mathtt{S}(y)) &\to \mathtt{S}(\mathtt{A}(x,y))\\
\mathtt{M}(x,\mathtt{0}) &\to \mathtt{0}\\
\mathtt{M}(x,\mathtt{S}(y)) &\to \mathtt{A}(x,\mathtt{M}(x,y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \to_{\text{select}} x\\
\mathtt{A}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y)))\\
&\to_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \to_{\text{select}} \mathtt{S}(\mathtt{A}(x,y))\\
\mathtt{M}(x,\mathtt{0}) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \to_{\text{select}} \mathtt{0}\\
\mathtt{M}(x,\mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y)))\\
&\to_{\text{select}} \mathtt{A}(x,\mathtt{M}^*(x,\mathtt{S}(y)))
\end{aligned}
$$

Example, Addition and multiplication

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\rightarrow \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\rightarrow \mathtt{A}(x, \mathtt{M}(x, y))
\end{aligned}
$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$
\begin{aligned}
\mathtt{A}(x, \mathtt{0}) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{0}) \rightarrow_{\text{select}} x \\
\mathtt{A}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{A}^*(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x, \mathtt{S}(y))) \\
&\rightarrow_{\text{lex}} \mathtt{S}(\mathtt{A}(x, \mathtt{S}^*(y))) \rightarrow_{\text{select}} \mathtt{S}(\mathtt{A}(x, y)) \\
\mathtt{M}(x, \mathtt{0}) &\rightarrow_{\text{put}} \mathtt{M}^*(x, \mathtt{0}) \rightarrow_{\text{select}} \mathtt{0} \\
\mathtt{M}(x, \mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y))) \\
&\rightarrow_{\text{select}} \mathtt{A}(x, \mathtt{M}^*(x, \mathtt{S}(y))) \rightarrow_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^*(y)))
\end{aligned}
$$

Example, Addition and multiplication

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\rightarrow x \\ \mathtt{A}(x,\mathtt{S}(y)) &\rightarrow \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\rightarrow \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\rightarrow \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Use relation R given by $\mathtt{M} \succ \mathtt{A}$ and $\mathtt{A} \succ \mathtt{S}$.

For each reduction rule a corresponding Lex-reductions:

$$\begin{aligned} \mathtt{A}(x,\mathtt{0}) &\rightarrow_{\text{put}} \mathtt{A}^*(x,\mathtt{0}) \rightarrow_{\text{select}} x \\ \mathtt{A}(x,\mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{A}^*(x,\mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,\mathtt{S}(y))) \\ &\rightarrow_{\text{lex}} \mathtt{S}(\mathtt{A}(x,\mathtt{S}^*(y))) \rightarrow_{\text{select}} \mathtt{S}(\mathtt{A}(x,y)) \\ \mathtt{M}(x,\mathtt{0}) &\rightarrow_{\text{put}} \mathtt{M}^*(x,\mathtt{0}) \rightarrow_{\text{select}} \mathtt{0} \\ \mathtt{M}(x,\mathtt{S}(y)) &\rightarrow_{\text{put}} \mathtt{M}^*(x,\mathtt{S}(y)) \rightarrow_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x,\mathtt{S}(y)),\mathtt{M}^*(x,\mathtt{S}(y))) \\ &\rightarrow_{\text{select}} \mathtt{A}(x,\mathtt{M}^*(x,\mathtt{S}(y))) \rightarrow_{\text{lex}} \mathtt{A}(x,\mathtt{M}(x,\mathtt{S}^*(y))) \\ &\rightarrow_{\text{select}} \mathtt{A}(x,\mathtt{M}(x,y)) \end{aligned}$$

Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}ex^{\omega}$

But $\mathcal{L}$ex is in general not terminating, e.g. if $\mathtt{A} > \mathtt{S}$, then

$$\begin{aligned} \mathtt{A}(x,y) &\rightarrow_{\mathsf{put}} \mathtt{A}^*(x,y) \\ &\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{A}^*(x,y)) \\ &\rightarrow_{\mathsf{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y))) \\ &\ldots \end{aligned}$$

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Starred symbol $\mathtt{A}^*$ is 'used' infinitely often.

Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}\mathsf{ex}^\omega$

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Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

Termination for $\succ_{ilpo}$ via termination of $\mathcal{L}\mathrm{ex}^{\omega}$

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Starred symbol $\mathtt{A}^*$ is 'used' infinitely often.

This is essential in any infinite reduction!

Idea:

- use numbers instead of stars,
- the numbers fix how often a symbol can be used.

$\Rightarrow$ Yields a terminating TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$.

Auxiliary TRS with numerical control symbols

Given a terminating relation $\succ$ on signature Σ, define TRS $\mathcal{L}\mathrm{ex}^{\omega}_{\succ}$

- Signature: $\Sigma \uplus \Sigma^{\omega}$, where $\Sigma^{\omega} = \{f^n \mid f \in \Sigma,\ n \in \mathbb{N}\}$

  f^n is fresh and has same arity as f

- Reduction rules:

$$
\begin{aligned}
f(\vec{x}) &\rightarrow_{\mathrm{put}} f^n(\vec{x}) && \\
f^n(\vec{x}) &\rightarrow_{\mathrm{select}} x_i && \\
f^{n+1}(\vec{x}) &\rightarrow_{\mathrm{copy}} g(f^n(\vec{x}), \ldots, f^n(\vec{x})) && \text{if } f \succ g \\
f^{n+1}(\vec{x}, g(\vec{y}), \vec{z}) &\rightarrow_{\mathrm{lex}} f(\vec{x}, g^n(\vec{y}), l, \ldots, l) && \text{where } l = f^n(\vec{x}, g(\vec{y}), \vec{z})
\end{aligned}
$$

Back and forth between $\mathcal{L}\mathsf{ex}$ and $\mathcal{L}\mathsf{ex}^\omega$

From $\to_{\mathcal{L}\mathsf{ex}^\omega}$ to $\to_{\mathcal{L}\mathsf{ex}}$:

- every reduction can be transformed by replacing f^n by f^*.

Back and forth between $\mathcal{L}$ex and $\mathcal{L}\text{ex}^\omega$

From $\rightarrow_{\mathcal{L}\text{ex}^\omega}$ to $\rightarrow_{\mathcal{L}\text{ex}}$:

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From $\rightarrow_{\mathcal{L}\text{ex}}$ to $\rightarrow_{\mathcal{L}\text{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

Back and forth between $\mathcal{L}$ex and $\mathcal{L}\text{ex}^\omega$

From $\to_{\mathcal{L}\text{ex}^\omega}$ to $\to_{\mathcal{L}\text{ex}}$:

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From $\to_{\mathcal{L}\text{ex}}$ to $\to_{\mathcal{L}\text{ex}^\omega}$:

- every finite reduction can be lifted, in particular
- every reduction between two starless terms can be lifted.

For example:

$$\begin{aligned}\mathtt{M}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^*(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^*(x, \mathtt{S}(y)), \mathtt{M}^*(x, \mathtt{S}(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}^*(x, \mathtt{S}(y))) \to_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^*(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}(x, y))\end{aligned}$$

becomes:

$$\begin{aligned}\mathtt{M}(x, \mathtt{S}(y)) &\to_{\text{put}} \mathtt{M}^2(x, \mathtt{S}(y)) \to_{\text{copy}} \mathtt{A}(\mathtt{M}^1(x, \mathtt{S}(y)), \mathtt{M}^1(x, \mathtt{S}(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}^1(x, \mathtt{S}(y))) \to_{\text{lex}} \mathtt{A}(x, \mathtt{M}(x, \mathtt{S}^0(y))) \\ &\to_{\text{select}} \mathtt{A}(x, \mathtt{M}(x, y))\end{aligned}$$

$\mathcal{L}$ex and $\mathcal{L}$ex$^\omega$

Theorem

$\rightarrow^+_{\mathcal{L}\mathrm{ex}_\succ}$ *and* $\rightarrow^+_{\mathcal{L}\mathrm{ex}^\omega_\succ}$ *coincide on* $T(\Sigma \uplus V)$

$\mathcal{L}$ex and $\mathcal{L}$ex$^\omega$

Theorem

$\rightarrow^+_{\mathcal{L}\text{ex}_\succ}$ *and* $\rightarrow^+_{\mathcal{L}\text{ex}^\omega_\succ}$ *coincide on* $T(\Sigma \uplus V)$

Note that the infinite reduction

$$\begin{aligned} \mathtt{A}(x,y) &\rightarrow_{\text{put}} \mathtt{A}^*(x,y) \\ &\rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^*(x,y)) \\ &\rightarrow_{\text{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^*(x,y))) \\ &\ldots \end{aligned}$$

cannot be lifted:

$$\begin{aligned} \mathtt{A}(x,y) &\rightarrow_{\text{put}} \mathtt{A}^?(x,y) \\ &\rightarrow_{\text{copy}} \mathtt{S}(\mathtt{A}^{?-1}(x,y)) \\ &\rightarrow_{\text{copy}} \mathtt{S}(\mathtt{S}(\mathtt{A}^{?-2}(x,y))) \\ &\ldots \end{aligned}$$

Termination of $\mathcal{L}\mathsf{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f,\ \vec{t},\ \ell \rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\mathsf{ex}^{\omega}_{\succ}})^n, > \rangle$.

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- Here $(\rightarrow_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

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Prove the implication

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- Here $(\to_{\mathcal{L}\mathrm{ex}^\omega_\succ})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

In general a term is SN if all one-step reducts are SN.

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- Here $(\rightarrow_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n$ is the lexicographic order on n-tuples
- No label ℓ counts as ∞ with $\infty > n$.

In general a term is SN if all one-step reducts are SN.

$\Rightarrow$ We check all one step reducts of $f^\ell(t_1, \ldots, t_n)$.

Termination of $\mathcal{L}\mathsf{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\to_{\mathcal{L}\mathsf{ex}^\omega_\succ})^n, > \rangle$.

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by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\mathrm{ex}^\omega_\succ})^n, > \rangle$.

Case 1. Internal step $f^\ell(\ldots, t_i, \ldots) \rightarrow f^\ell(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Termination of $\mathcal{L}\mathrm{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\mathrm{ex}^\omega_\succ})^n, > \rangle$.

Case 1. Internal step $f^\ell(\ldots, t_i, \ldots) \rightarrow f^\ell(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Case 2. $f(t_1, \ldots, t_n) \rightarrow_{\mathrm{put}} f^n(t_1, \ldots, t_n)$.
We have a decrease in the third component.

Termination of $\mathcal{L}\text{ex}^\omega$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating} \implies f^\ell(t_1, \ldots, t_n) \text{ is terminating}$$

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Case 1. Internal step $f^\ell(\ldots, t_i, \ldots) \to f^\ell(\ldots, t_i', \ldots)$.
The triple decreases in the second component.

Case 2. $f(t_1, \ldots, t_n) \to_{\text{put}} f^n(t_1, \ldots, t_n)$.
We have a decrease in the third component.

Case 3. $f(t_1, \ldots, t_n) \to_{\text{select}} t_i$.
By assumption t_i is SN.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

$$t_1, \ldots, t_n \text{ are terminating } \implies f^{\ell}(t_1, \ldots, t_n) \text{ is terminating}$$

by induction on triple $\langle f, \vec{t}, \ell \rangle$ in ordering $\langle \succ, (\rightarrow_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}})^n, > \rangle$.

Case 4. $f^{n+1}(\vec{t}) \rightarrow_{\mathrm{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.
By IH the arguments $f^n(\vec{t})$ of g are SN since $n + 1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Termination of $\mathcal{L}\mathrm{ex}^{\omega}$ à la Buchholz

Prove the implication

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Case 4. $f^{n+1}(\vec{t}) \rightarrow_{\mathrm{copy}} g(f^n(\vec{t}), \ldots, f^n(\vec{t}))$.
By IH the arguments $f^n(\vec{t})$ of g are SN since $n + 1 > n$.
Again by IH the term $g(\ldots)$ itself is SN, since $f \succ g$.

Case 5. $f^{n+1}(\vec{t}, g(\vec{s}), \vec{r}) \rightarrow_{\mathrm{lex}} f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$, $l = f^n(\vec{t}, g(\vec{y}), \vec{r})$.
By assumption the arguments $\vec{t}$ and $g(\vec{s})$ are SN.
By IH we get l is SN.
Since $g(\vec{s}) \rightarrow_{\mathrm{put}} g^n(\vec{s})$ we get

- $g^n(\vec{s})$ is SN, and
- the triple decreases in the second component.

Thus by IH $f(\vec{t}, g^n(\vec{s}), l, \ldots, l)$ is SN.

Termination of $\mathcal{L}$ex and $\mathcal{L}$ex$^\omega$

Hence we have proven:

Theorem

$\rightarrow_{\mathcal{L}\mathsf{ex}^\omega_\succ}$ *is terminating*

Termination of $\mathcal{L}\mathrm{ex}$ and $\mathcal{L}\mathrm{ex}^{\omega}$

Hence we have proven:

Theorem

$\rightarrow_{\mathcal{L}\mathrm{ex}^{\omega}_{\succ}}$ *is terminating*

Corollary

$\rightarrow^{+}_{\mathcal{L}\mathrm{ex}}$ *is terminating on* $T(\Sigma \uplus V)$

Proof. $\rightarrow^{+}_{\mathcal{L}\mathrm{ex}}$ and $\rightarrow^{+}_{\mathcal{L}\mathrm{ex}^{\omega}}$ coincide on $T(\Sigma \uplus V)$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

Example, Ackermann Function

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Find an order $\succ$ on Σ which proves termination.

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We get the following derivations:

$$\mathrm{Ack}(0, y)$$

Example, Ackermann Function

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Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y})$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y}))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$

Example, Ackermann Function

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$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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$\mathrm{Ack}(\mathrm{s}(x), 0)$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

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$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
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$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$

Example, Ackermann Function

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Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
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Find an order $\succ$ on Σ which proves termination.

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We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \to_{\mathsf{copy}} \mathrm{Ack}(\mathrm{x}, \mathrm{s}(\mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(0, y) \to_{\mathsf{put}} \mathrm{Ack}^*(0, \mathrm{y}) \to_{\mathsf{copy}} \mathrm{s}(\mathrm{Ack}^*(0, \mathrm{y})) \to_{\mathsf{select}} \mathrm{s}(\mathrm{y})$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0) \to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)) \to_{\mathsf{copy}} \mathrm{Ack}(\mathrm{x}, \mathrm{s}(\mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), 0)))$$
$$\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{s}(0))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y))$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s(x)}, \mathrm{s(y)})$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\begin{aligned}\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})) \\ &\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})))\end{aligned}$$

Example, Ackermann Function

Example

$$\mathrm{Ack}(0, y) \to \mathrm{s}(y)$$
$$\mathrm{Ack}(\mathrm{s}(x), 0) \to \mathrm{Ack}(x, \mathrm{s}(0))$$
$$\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) \to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\begin{aligned}\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})) \\ &\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\ &\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})))\end{aligned}$$

Example, Ackermann Function

Example

$$\begin{aligned} \mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\ \mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\ \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y)) \end{aligned}$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\begin{aligned} \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s(x)}, \mathrm{s(y)}) \\ &\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s(x)}, \mathrm{s(y)})) \\ &\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s(x)}, \mathrm{s(y)})) \\ &\to_{\mathsf{lex}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}^*(\mathrm{y}))) \end{aligned}$$

Example, Ackermann Function

Example

$$\begin{aligned}
\mathrm{Ack}(0, y) &\to \mathrm{s}(y) \\
\mathrm{Ack}(\mathrm{s}(x), 0) &\to \mathrm{Ack}(x, \mathrm{s}(0)) \\
\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

Find an order $\succ$ on Σ which proves termination.

$$\mathrm{Ack} \succ \mathrm{s}$$

We get the following derivations:

$$\begin{aligned}
\mathrm{Ack}(\mathrm{s}(x), \mathrm{s}(y)) &\to_{\mathsf{put}} \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y})) \\
&\to_{\mathsf{lex}} \mathrm{Ack}(\mathrm{s}^*(\mathrm{x}), \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}^*(\mathrm{s}(\mathrm{x}), \mathrm{s}(\mathrm{y}))) \\
&\to_{\mathsf{lex}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), \mathrm{s}^*(\mathrm{y}))) \\
&\to_{\mathsf{select}} \mathrm{Ack}(x, \mathrm{Ack}(\mathrm{s}(x), y))
\end{aligned}$$

Hence we have proven termination.

Recursive definition of LPO

Let $\succ$ be a strict order on signature Σ

Define $\succ_{lpo}$ on $T(\Sigma, V)$ by: $s \succ_{lpo} t$ iff

(LPO1) $t \in \mathcal{V}ar(s)$ and $s \neq t$, or

(LPO2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (LPO2a) $\exists 1 \leq i \leq m$, with $s_i = t$ or $s_i \succ_{lpo} t$, or
- (LPO2b) $f \succ g$ and $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, or
- (LPO2c) $f = g$, and
  - $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, and
  - there exists $1 \leq i \leq m$, s.t.
  - $s_1 = t_1$, …, $s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

Recursive definition of LPO

Let $\succ$ be a strict order on signature Σ

Define $\succ_{lpo}$ on $T(\Sigma, V)$ by: $s \succ_{lpo} t$ iff

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- (LPO2a) $\exists 1 \leq i \leq m$, with $s_i = t$ or $s_i \succ_{lpo} t$, or
- (LPO2b) $f \succ g$ and $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, or
- (LPO2c) $f = g$, and
  $s \succ_{lpo} t_j$ for all $1 \leq j \leq n$, and
  there exists $1 \leq i \leq m$, s.t.
  $s_1 = t_1$, …, $s_{i-1} = t_{i-1}$ and $s_i \succ_{lpo} t_i$.

Theorem

$\succ_{ilpo}$ is equivalent with $\succ_{lpo}$