- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
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Outline

- Overview
- Examples
- Terms
- Term Rewriting

Examples of Term Rewriting Systems

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) $\quad \mathsf{s}$ (unary) $\quad +$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(x)+y$
rewrite rules	$0+y \to y$ $\mathsf{s}(x)+y \to \mathsf{s}(x+y)$
rewriting	$\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0+\mathsf{s}(\mathsf{s}(0))) \qquad y \mapsto \mathsf{s}(\mathsf{s}(0))$ $\to \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$

Example (Combinatory Logic)

signature	S K I (constants) $\cdot$ (application, binary, infix)
terms	S $((\mathsf{K}\cdot\mathsf{I})\cdot\mathsf{I})\cdot\mathsf{S}$ $(x\cdot z)\cdot(y\cdot z)$
rewrite rules	$\mathsf{I}\cdot x \to x$ $(\mathsf{K}\cdot x)\cdot y \to x$ $((\mathsf{S}\cdot x)\cdot y)\cdot z \to (x\cdot z)\cdot(y\cdot z)$
rewriting	$((\mathsf{S}\cdot\mathsf{K})\cdot\mathsf{K})\cdot x \to (\mathsf{K}\cdot x)\cdot(\mathsf{K}\cdot x)$ $\to x$
inventor	Moses Schönfinkel (1924)

Example (Addition on Natural Numbers in Decimal Notation)

signature	$0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)
terms	$1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$
rewrite rules	$0+0 \to 0 \quad 1+0 \to 1 \quad \cdots \quad 9+0 \to 9$
	$0+1 \to 1 \quad 1+1 \to 2 \quad \cdots \quad 9+1 \to 1:0$
	$0+2 \to 2 \quad 1+2 \to 3 \quad \cdots \quad 9+2 \to 1:1$
	$0+3 \to 3 \quad 1+3 \to 4 \quad \cdots \quad 9+3 \to 1:2$
	$0+4 \to 4 \quad 1+4 \to 5 \quad \cdots \quad 9+4 \to 1:3$
	$0+5 \to 5 \quad 1+5 \to 6 \quad \cdots \quad 9+5 \to 1:4$
	$0+6 \to 6 \quad 1+6 \to 7 \quad \cdots \quad 9+6 \to 1:5$
	$0+7 \to 7 \quad 1+7 \to 8 \quad \cdots \quad 9+7 \to 1:6$
	$0+8 \to 8 \quad 1+8 \to 9 \quad \cdots \quad 9+8 \to 1:7$
	$0+9 \to 9 \quad 1+9 \to 1:0 \quad \cdots \quad 9+9 \to 1:8$
	$x+(y:z) \to y:(x+z) \quad 0:x \to x$
	$(x:y)+z \to x:(y+z) \quad x:(y:z) \to (x+y):z$
rewriting	$(2:3)+(7:7) \to^* 7:(2:(3+7))$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llllll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \to^* 7:((2+1):0)$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \to^* (7+3):0$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \qquad 2+(7:3) \qquad (2:(3:x))+((1+7):2)$

rewrite rules

$0+0 \to 0$	$1+0 \to 1$	$\cdots$	$9+0 \to 9$
$0+1 \to 1$	$1+1 \to 2$	$\cdots$	$9+1 \to 1:0$
$0+2 \to 2$	$1+2 \to 3$	$\cdots$	$9+2 \to 1:1$
$0+3 \to 3$	$1+3 \to 4$	$\cdots$	$9+3 \to 1:2$
$0+4 \to 4$	$1+4 \to 5$	$\cdots$	$9+4 \to 1:3$
$0+5 \to 5$	$1+5 \to 6$	$\cdots$	$9+5 \to 1:4$
$0+6 \to 6$	$1+6 \to 7$	$\cdots$	$9+6 \to 1:5$
$0+7 \to 7$	$1+7 \to 8$	$\cdots$	$9+7 \to 1:6$
$0+8 \to 8$	$1+8 \to 9$	$\cdots$	$9+8 \to 1:7$
$0+9 \to 9$	$1+9 \to 1:0$	$\cdots$	$9+9 \to 1:8$

$$x+(y:z) \to y:(x+z) \qquad 0:x \to x$$

$$(x:y)+z \to x:(y+z) \qquad x:(y:z) \to (x+y):z$$

rewriting $(2:3)+(7:7) \to^* (1:0):0$ normal form

Example (Binary Trees)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

leaf sum (unary) node (binary)

terms $\mathsf{leaf}((1:0):0) \quad \mathsf{node}(\mathsf{leaf}(1), \mathsf{leaf}(2)) \quad \mathsf{leaf}(\mathsf{node}(1, \mathsf{leaf}(2)))$

rewrite rules

$$\cdots$$
$$\mathsf{sum}(\mathsf{leaf}(x)) \to x$$
$$\mathsf{sum}(\mathsf{node}(x, y)) \to \mathsf{sum}(x) + \mathsf{sum}(y)$$

rewriting

$$\begin{aligned} & \mathsf{sum}(\mathsf{node}(\mathsf{leaf}(2:3), \mathsf{leaf}(7:7))) \\ & \quad \to \quad \mathsf{sum}(\mathsf{leaf}(2:3)) + \mathsf{sum}(\mathsf{leaf}(7:7)) \\ & \quad \to^* \quad (2:3) + (7:7) \\ & \quad \to^* \quad (1:0):0 \end{aligned}$$

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Term Rewriting Systems

Definition

- signature Σ function symbols $f \in \Sigma$ with arities $\#(f)$
- variables $\mathcal{X}$ $\Sigma \cap \mathcal{X} = \varnothing$ infinitely many
- terms $\mathcal{T}(\Sigma, \mathcal{X})$ smallest set such that
  - $\mathcal{X} \subseteq \mathcal{T}(\Sigma, \mathcal{X})$
  - if $f \in \Sigma$ has arity 0 then $f \in \mathcal{T}(\Sigma, \mathcal{X})$
  - if $f \in \Sigma$ has arity $n \geqslant 1$ and $t_1, \ldots, t_n \in \mathcal{T}(\Sigma, \mathcal{X})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, \mathcal{X})$

Definition

- signature Σ — function symbols $f \in \Sigma$ with arities $\#(f)$
- variables $\mathcal{X}$ — $\Sigma \cap \mathcal{X} = \varnothing$ — infinitely many
- terms $\mathcal{T}(\Sigma, \mathcal{X})$ — smallest set such that
  - $\mathcal{X} \subseteq \mathcal{T}(\Sigma, \mathcal{X})$
  - if $f \in \Sigma$ has arity 0 then $f \in \mathcal{T}(\Sigma, \mathcal{X})$
  - if $f \in \Sigma$ has arity $n \geqslant 0$ and $t_1, \ldots, t_n \in \mathcal{T}(\Sigma, \mathcal{X})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, \mathcal{X})$ assuming that $f() = f$
- ground terms $\mathcal{T}(\Sigma)$ — smallest set such that
  - if $f \in \Sigma$ has arity $n \geqslant 0$ and $t_1, \ldots, t_n \in \mathcal{T}(\Sigma)$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma)$

Definitions (Operations on Terms)

- $\mathcal{V}\text{ar}(\,\cdot\,)$, the variables of a term $\qquad \mathcal{V}\text{ar}((2:x)+((1:x):y)) = \{x,y\}$

$$\mathcal{V}\text{ar}(t) = \begin{cases} \{t\} & \text{if } t \in \mathcal{X} \\ \bigcup_{i=1}^{n} \mathcal{V}\text{ar}(t_i) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

- $\mathcal{F}\text{un}(\,\cdot\,)$, the function symbols $\qquad \mathcal{F}\text{un}((2:x)+((1:x):y)) = \{1,2,:\}$

$$\mathcal{F}\text{un}(t) = \begin{cases} \varnothing & \text{if } t \in \mathcal{X} \\ \{f\} \cup \bigcup_{i=1}^{n} \mathcal{F}\text{un}(t_i) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

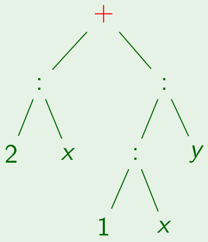
Definition (Operations on Terms)

- root(·), the root of a term

$$\mathrm{root}(t) = \begin{cases} t & \text{if } t \in \mathcal{X} \\ f & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Example

$(2 : x) + ((1 : x) : y)$



Definitions (Operations on Terms)

- $|\cdot|$, the size of a term $\qquad |(2 : x) + ((1 : x) : y)| = 9$

$$|t| = \begin{cases} 1 & \text{if } t \in \mathcal{X} \\ 1 + \sum_{i=1}^{n} |t_i| & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- $\|\cdot\|$, the number of function symbols $\qquad \|(2 : x) + ((1 : x) : y)\| = 6$

$$\|t\| = \begin{cases} 0 & \text{if } t \in \mathcal{X} \\ 1 + \sum_{i=1}^{n} \|t_i\| & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

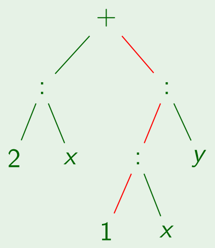
Definition (Operations on Terms)

- height(·), the height (or depth) of a term

$$\text{height}(t) = \begin{cases} 0 & \text{if } t \in \mathcal{X} \text{ or } t \text{ is a constant} \\ 1 + \max_{1 \leqslant i \leqslant n} \text{height}(t_i) & \text{if } t = f(t_1, \dots, t_n) \text{ with } n \geqslant 1 \end{cases}$$

Example

$\text{height}((2 : x) + ((1 : x) : y)) = 3$



Definitions

- $s \trianglelefteq t$ s is subterm of t
  - $s = t$ or
  - $t = f(t_1, \ldots, t_n)$ and $s \trianglelefteq t_i$ for some $1 \leqslant i \leqslant n$
- $s \lhd t$ s is proper subterm of t
  - $s \trianglelefteq t$ and $s \neq t$

Example

term $(2 : x) + ((1 : x) : y)$ has subterms

2 x $2 : x$ 1 $1 : x$ y $(1 : x) : y$ $(2 : x) + ((1 : x) : y)$

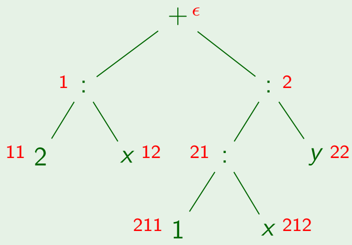
Definition (Positions)

- $\mathcal{P}os(\cdot)$, the set of positions of a term

$$\mathcal{P}os(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{X} \\ \{\epsilon\} \cup \{ip \mid p \in \mathcal{P}os(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Example

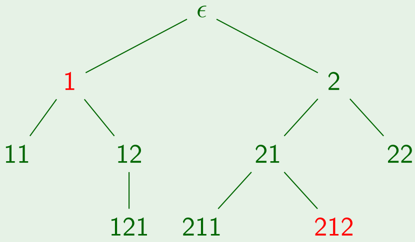
$(2 : x) + ((1 : x) : y)$



Definition

- $p < q$ if $\exists\, r \neq \epsilon$: $pr = q$ "p is strictly above q" "q is strictly below q"
- $p \leqslant q$ if $\exists\, r$: $pr = q$ "p is above q" "q is below q"
- $p \parallel q$ if $p \nleqslant q$ and $q \nleqslant p$ "p and q are parallel"

Example



- $2 < 211$
- $1 \parallel 2$
- $1 \parallel 212$

Definitions

- $t|_p$ subterm of t at position p

$$t|_p = \begin{cases} t & \text{if } p = \epsilon \\ t_i|_q & \text{if } t = f(t_1, \dots, t_n) \text{ and } p = iq \end{cases}$$

- $t(p)$ symbol in t at position p

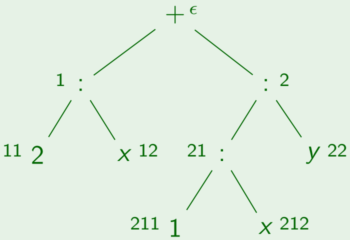
$$t(p) = \begin{cases} \text{root}(t) & \text{if } p = \epsilon \\ t_i(q) & \text{if } t = f(t_1, \dots, t_n) \text{ and } p = iq \end{cases}$$

- $t[s]_p$ replace subterm in t at position p by s

$$t[s]_p = \begin{cases} s & \text{if } p = \epsilon \\ f(t_1, \dots, t_i[s]_q, \dots, t_n) & \text{if } t = f(t_1, \dots, t_n) \text{ and } p = iq \end{cases}$$

Example

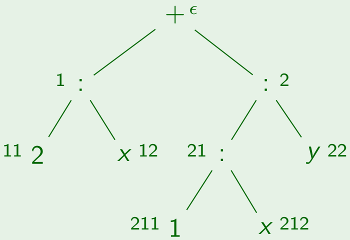
$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = ?$

Example

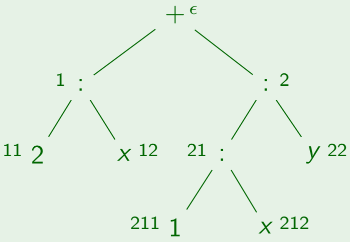
$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = 1 : x$
- $t(212) = ?$

Example

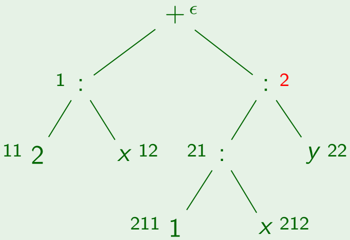
$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = 1 : x$
- $t(212) = x$
- $t[x + 3]_2 = ?$

Example

$t = (2 : x) + ((1 : x) : y)$



- $t|_{21} = 1 : x$
- $t(212) = x$
- $t[x + 3]_2 = (2 : x) + (x + 3)$

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Definitions

- context is term with one hole $\square$,
  i.e., element of $\mathcal{T}(\Sigma \cup \{\square\}, \mathcal{X})$ that contains exactly one occurrence of $\square$
- $C[t]$ denotes result of replacing hole in context C by term t
- relation R on terms is closed under contexts if $\forall$ terms s, t

$$s \; R \; t \quad \Longrightarrow \quad \forall \text{ contexts } C\text{:} \quad C[s] \; R \; C[t]$$

Examples

- $\square \qquad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(\square)) \qquad \square + x$
- $\square[\mathsf{s}(0)] = \mathsf{s}(0) \qquad (\square + x)[0 + x] = (0 + x) + x$

Lemmata

- $s \trianglelefteq t \quad \Longleftrightarrow \quad \exists \; context \; C: \quad t = C[s]$
- $s \triangleleft t \quad \Longleftrightarrow \quad \exists \; context \; C \neq \square: \quad t = C[s]$

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Definitions

- substitution is mapping $\sigma\colon \mathcal{X} \to \mathcal{T}(\Sigma, \mathcal{X})$ such that its domain
$$\mathcal{D}\mathsf{om}(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$$
is finite
- empty substitution $\varepsilon \quad (\mathcal{D}\mathsf{om}(\varepsilon) = \varnothing)$
- application of substitution σ to term t
$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{X} \\ f(t_1\sigma, \ldots, t_n\sigma) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$
- relation R on terms is closed under substitutions if $\forall$ terms s, t
$$s \mathrel{R} t \quad \Longrightarrow \quad \forall \text{ substitutions } \sigma: \quad s\sigma \mathrel{R} t\sigma$$

Example

$t = x + \mathsf{s}(y + z) \quad \sigma = \{x \mapsto \mathsf{s}(y), y \mapsto x + \mathsf{s}(0)\} \quad t\sigma = \mathsf{s}(y) + \mathsf{s}((x + \mathsf{s}(0)) + z)$

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Definitions

- A rewrite rule $(\ell \to r)$ is a pair (ℓ, r) of terms such that
  - $\ell \notin \mathcal{X}$ (the lhs is not a variable)
  - $\mathcal{V}\text{ar}(r) \subseteq \mathcal{V}\text{ar}(\ell)$ (all variables in the rhs occur in the lhs)

  The terms ℓ and r are called left-hand side (lhs) and right-hand side (rhs).
- A term rewrite system (TRS) is pair (Σ, R) consisting of
  - Σ signature
  - R set of rewrite rules between terms in $\mathcal{T}(\Sigma, \mathcal{X})$

Example

TRS (Σ, R) with signature Σ

0 (constant) s (unary) add (binary)

$\#(0) = 0$, $\#(\mathsf{s}) = 1$, $\#(\mathsf{add}) = 2$, and rewrite rules R

$$\mathsf{add}(0, y) \to y$$
$$\mathsf{add}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{add}(x, y))$$

Definition (Term Rewriting)

Let $\mathcal{R} = (\Sigma, R)$ be a TRS. The rewrite relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\Sigma, \mathcal{X})$ is defined as:

$$C[\ell\sigma] \to_{\mathcal{R}} C[r\sigma]$$

for every:

- rule $\ell \to r \in R$,
- context C, and
- substitution σ.

Let p be the position of $\square$ in C. Then we say that the rewrite step is at position p.

Example

The rule $\mathsf{add}(0, y) \to y$ with $y \mapsto \mathsf{s}(0)$ and $C = \mathsf{add}(0, \square)$ gives rise to the step:

$$\mathsf{add}(0, \mathsf{add}(0, \mathsf{s}(0))) \to \mathsf{add}(0, \mathsf{s}(0))$$

at position 2.

Lemma

The rewrite relation $\rightarrow_{\mathcal{R}}$ is the smallest relation on $\mathcal{T}(\Sigma, \mathcal{X})$ that:

- *contains the rules R,*
- *is closed under context, and*
- *is closed under substitutions.*

That is, $\rightarrow_{\mathcal{R}}$ is the closure of R under contexts and substitutions.

Properties of Term Rewriting Systems

We consider TRSs $\mathcal{R} = (\Sigma, R)$ as

$$\text{ARSs } \langle \mathcal{T}(\Sigma, \mathcal{X}), \to_{\mathcal{R}} \rangle$$

TRSs inherit the properties: CR, WCR, SN, WN, NF, UN, $\text{UN}^{\to}$, ...

Example

- A term $t \in \mathcal{T}(\Sigma, \mathcal{X})$ is called terminating if t admits no infinite rewrite sequence $t = t_1 \to t_2 \to t_3 \to \dots$
- A TRS (Σ, R) is called terminating if all terms in $\mathcal{T}(\Sigma, \mathcal{X})$ are.
- ...

Example

TRS R modeling Sieve of Eratostheness for generating list of prime numbers

$$\begin{aligned} \mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y) \\ \mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x)) \\ \mathsf{head}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w) \\ \mathsf{tail}(x : y) &\to y & \mathsf{filter}(\mathsf{s}(x), y : z, w) &\to y : \mathsf{filter}(x, z, w) \end{aligned}$$

- R is confluent but not terminating

$$\mathsf{from}(0) \to 0 : \mathsf{from}(\mathsf{s}(0)) \to 0 : (\mathsf{s}(0) : \mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \to \cdots$$

- how to prove confluence of R ? orthogonality (lecture 8)
- $\exists$ non-terminating terms with (unique) normal form

$$\mathsf{head}(\mathsf{tail}(\mathsf{tail}(\mathsf{primes}))) \to^{!} \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))))$$

- how to compute normal forms in R ? strategy (lecture 9)

Example (Combinatory Logic)

$$
\begin{array}{rlrlrl}
\mathsf{I} \cdot x & \to x & \mathsf{I}\, x & \to x & \mathsf{I}\, x & \to x \\
(\mathsf{K} \cdot x) \cdot y & \to x & (\mathsf{K}\, x)\, y & \to x & \mathsf{K}\, x\, y & \to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z & \to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\, x)\, y)\, z & \to (x\, z)\,(y\, z) & \mathsf{S}\, x\, y\, z & \to x\, z\,(y\, z)
\end{array}
$$

- applicative notation: surpress $\cdot$ and adopt left-association
- CL is confluent but not terminating

$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

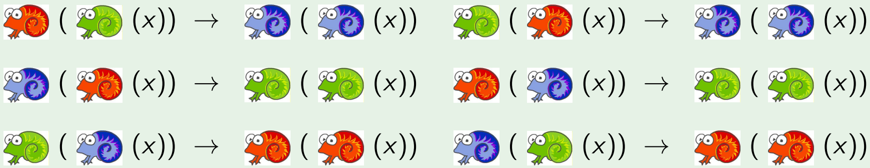
- CL is consistent

$$\mathsf{S} \not\leftrightarrow^* \mathsf{K}$$

Definition

TRS R over signature Σ is string rewrite system (SRS) if Σ consists of unary function symbols

Example



Definition

TRS R over signature Σ is string rewrite system (SRS) if Σ consists of unary function symbols

Example