Outline

Examples of Term Rewriting Systems

Example (Addition on Natural Numbers in Unary Notation)

signature $\quad 0$ (constants) $\quad s$ (unary) $\quad +$ (binary, infix)

Example (Addition on Natural Numbers in Unary Notation)

signature	0	(constants)	s	(unary)	$+$	(binary, infix)

terms $\mathsf{s}(\mathsf{s}(0))$ $\quad$ $\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\quad$ $\mathsf{s}(x) + y$

Example (Addition on Natural Numbers in Unary Notation)

signature 0 (constants) s (unary) $+$ (binary, infix)

terms $\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(x)+y$

rewrite rules

$$0 + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(x) + y$
rewrite rules	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(x)+y$
rewrite rules	$0+y \to y$
	$\mathsf{s}(x)+y \to \mathsf{s}(x+y)$
rewriting	$\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(x) + y$
rewrite rules	$0 + y \to y$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\qquad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0))$ $\quad \mathsf{s}(x)+y$
rewrite rules	$0+y \to y$ $\mathsf{s}(x)+y \to \mathsf{s}(x+y)$
rewriting	$\mathsf{s}(0)+\mathsf{s}(\mathsf{s}(0)) \to \mathsf{s}(0+\mathsf{s}(\mathsf{s}(0)))$ $\quad x \mapsto 0 \quad y \mapsto \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

signature	0 (constants) s (unary) $+$ (binary, infix)
terms	$\mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0))$ $\mathsf{s}(x) + y$
rewrite rules	$0 + y \rightarrow y$
	$\mathsf{s}(x) + y \rightarrow \mathsf{s}(x + y)$
rewriting	$\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \rightarrow \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0)))$ $x \mapsto 0$ $y \mapsto \mathsf{s}(\mathsf{s}(0))$

Example (Addition on Natural Numbers in Unary Notation)

signature $\quad 0$ (constants) $\quad \mathsf{s}$ (unary) $\quad +$ (binary, infix)

terms $\quad \mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) \quad \mathsf{s}(x) + y$

rewrite rules

$$
\begin{aligned}
0 + y &\to y \\
\mathsf{s}(x) + y &\to \mathsf{s}(x + y)
\end{aligned}
$$

rewriting

$$
\begin{aligned}
\mathsf{s}(0) + \mathsf{s}(\mathsf{s}(0)) &\to \mathsf{s}(0 + \mathsf{s}(\mathsf{s}(0))) \qquad y \mapsto \mathsf{s}(\mathsf{s}(0)) \\
&\to \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))
\end{aligned}
$$

Example (Combinatory Logic)

signature S K I (constants)

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

rewriting $((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $\quad ((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $\quad (x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$\mathsf{I} \cdot x \to x$$
$$(\mathsf{K} \cdot x) \cdot y \to x$$
$$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

rewriting $((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x \ \to \ (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x)$

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $(x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

rewriting

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

Example (Combinatory Logic)

signature: S K I (constants) $\cdot$ (application, binary, infix)

terms: S $\quad ((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S} \quad (x \cdot z) \cdot (y \cdot z)$

rewrite rules:

$$\begin{aligned} \mathsf{I} \cdot x &\to x \\ (\mathsf{K} \cdot x) \cdot y &\to x \\ ((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

rewriting:

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

inventor: Moses Schönfinkel (1924)

Example (Addition on Natural Numbers in Decimal Notation)

signature $\quad 0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 \\
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

Example (Addition on Natural Numbers in Decimal Notation)

signature $\quad 0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $\quad 1 + 3 \quad 2 + (7 : 3) \quad (2 : (3 : x)) + ((1 + 7) : 2)$

rewrite rules

$$
\begin{array}{llll}
0 + 0 \to 0 & 1 + 0 \to 1 & \cdots & 9 + 0 \to 9 \\
0 + 1 \to 1 & 1 + 1 \to 2 & \cdots & 9 + 1 \to 1 : 0 \\
0 + 2 \to 2 & 1 + 2 \to 3 & \cdots & 9 + 2 \to 1 : 1 \\
0 + 3 \to 3 & 1 + 3 \to 4 & \cdots & 9 + 3 \to 1 : 2 \\
0 + 4 \to 4 & 1 + 4 \to 5 & \cdots & 9 + 4 \to 1 : 3 \\
0 + 5 \to 5 & 1 + 5 \to 6 & \cdots & 9 + 5 \to 1 : 4 \\
0 + 6 \to 6 & 1 + 6 \to 7 & \cdots & 9 + 6 \to 1 : 5 \\
0 + 7 \to 7 & 1 + 7 \to 8 & \cdots & 9 + 7 \to 1 : 6 \\
0 + 8 \to 8 & 1 + 8 \to 9 & \cdots & 9 + 8 \to 1 : 7 \\
0 + 9 \to 9 & 1 + 9 \to 1 : 0 & \cdots & 9 + 9 \to 1 : 8
\end{array}
$$

$$
\begin{array}{ll}
x + (y : z) \to y : (x + z) & 0 : x \to x \\
(x : y) + z \to x : (y + z) & x : (y : z) \to (x + y) : z
\end{array}
$$

rewriting $\quad (2 : 3) + (7 : 7)$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \qquad 2+(7:3) \qquad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \qquad x \mapsto 2 \quad y \mapsto 3 \quad z \mapsto 7:7$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

terms $1+3 \qquad 2+(7:3) \qquad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llllll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \qquad x \mapsto 2:3 \quad y \mapsto 7 \quad z \mapsto 7$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$\begin{array}{lllll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}$$

$$x+(y:z) \to y:(x+z) \qquad 0:x \to x$$

$$(x:y)+z \to x:(y+z) \qquad x:(y:z) \to (x+y):z$$

rewriting $(2:3)+(7:7) \to 7:((2:3)+7)$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$x+(y:z) \to y:(x+z) \qquad 0:x \to x$$

$$(x:y)+z \to x:(y+z) \qquad x:(y:z) \to (x+y):z$$

rewriting $(2:3)+(7:7) \;\to\; 7:((2:3)+7)$

Example (Addition on Natural Numbers in Decimal Notation)

signature	$0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)
terms	$1 + 3 \quad 2 + (7 : 3) \quad (2 : (3 : x)) + ((1 + 7) : 2)$
rewrite rules	$0+0 \to 0 \quad 1+0 \to 1 \quad \cdots \quad 9+0 \to 9$
	$0+1 \to 1 \quad 1+1 \to 2 \quad \cdots \quad 9+1 \to 1:0$
	$0+2 \to 2 \quad 1+2 \to 3 \quad \cdots \quad 9+2 \to 1:1$
	$0+3 \to 3 \quad 1+3 \to 4 \quad \cdots \quad 9+3 \to 1:2$
	$0+4 \to 4 \quad 1+4 \to 5 \quad \cdots \quad 9+4 \to 1:3$
	$0+5 \to 5 \quad 1+5 \to 6 \quad \cdots \quad 9+5 \to 1:4$
	$0+6 \to 6 \quad 1+6 \to 7 \quad \cdots \quad 9+6 \to 1:5$
	$0+7 \to 7 \quad 1+7 \to 8 \quad \cdots \quad 9+7 \to 1:6$
	$0+8 \to 8 \quad 1+8 \to 9 \quad \cdots \quad 9+8 \to 1:7$
	$0+9 \to 9 \quad 1+9 \to 1:0 \quad \cdots \quad 9+9 \to 1:8$
	$x + (y : z) \to y : (x + z) \quad 0 : x \to x$
	$(x : y) + z \to x : (y + z) \quad x : (y : z) \to (x + y) : z$
rewriting	$(2 : 3) + (7 : 7) \to^* 7 : (2 : (3 + 7))$

Example (Addition on Natural Numbers in Decimal Notation)

signature $\quad 0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $\quad 1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
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0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
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0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8 \\
\multicolumn{2}{l}{x+(y:z) \to y:(x+z)} & & 0:x \to x \\
\multicolumn{2}{l}{(x:y)+z \to x:(y+z)} & & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $\quad (2:3)+(7:7) \to^* 7:(2:(3+7))$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$0+0 \to 0$	$1+0 \to 1$	$\cdots$	$9+0 \to 9$
$0+1 \to 1$	$1+1 \to 2$	$\cdots$	$9+1 \to 1:0$
$0+2 \to 2$	$1+2 \to 3$	$\cdots$	$9+2 \to 1:1$
$0+3 \to 3$	$1+3 \to 4$	$\cdots$	$9+3 \to 1:2$
$0+4 \to 4$	$1+4 \to 5$	$\cdots$	$9+4 \to 1:3$
$0+5 \to 5$	$1+5 \to 6$	$\cdots$	$9+5 \to 1:4$
$0+6 \to 6$	$1+6 \to 7$	$\cdots$	$9+6 \to 1:5$
$0+7 \to 7$	$1+7 \to 8$	$\cdots$	$9+7 \to 1:6$
$0+8 \to 8$	$1+8 \to 9$	$\cdots$	$9+8 \to 1:7$
$0+9 \to 9$	$1+9 \to 1:0$	$\cdots$	$9+9 \to 1:8$

$$x+(y:z) \to y:(x+z) \qquad 0:x \to x$$

$$(x:y)+z \to x:(y+z) \qquad x:(y:z) \to (x+y):z$$

rewriting $(2:3)+(7:7) \to^* 7:(2:(1:0))$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

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0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \to^* 7:(2:(1:0))$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

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\begin{array}{llllll}
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0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \to^* 7:((2+1):0)$

Example (Addition on Natural Numbers in Decimal Notation)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)

terms $1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
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0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
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\end{array}
$$

$$
\begin{array}{ll}
x+(y:z) \to y:(x+z) & 0:x \to x \\
(x:y)+z \to x:(y+z) & x:(y:z) \to (x+y):z
\end{array}
$$

rewriting $(2:3)+(7:7) \to^* 7:((2+1):0)$

Example (Addition on Natural Numbers in Decimal Notation)

signature	$0 \quad 1 \quad \cdots \quad 9$ (constants) $\quad + \quad :$ (binary, infix)
terms	$1+3 \quad 2+(7:3) \quad (2:(3:x))+((1+7):2)$

rewrite rules

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0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
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rewrite rules

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$0+1 \to 1$	$1+1 \to 2$	$\cdots$	$9+1 \to 1:0$
$0+2 \to 2$	$1+2 \to 3$	$\cdots$	$9+2 \to 1:1$
$0+3 \to 3$	$1+3 \to 4$	$\cdots$	$9+3 \to 1:2$
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rewriting $(2:3)+(7:7) \to^* (1:0):0$ normal form

Example (Binary Trees)

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leaf $\quad$ sum (unary) $\quad$ node (binary)

terms $\quad \mathsf{leaf}((1 : 0) : 0) \quad \mathsf{node}(\mathsf{leaf}(1), \mathsf{leaf}(2)) \quad \mathsf{leaf}(\mathsf{node}(1, \mathsf{leaf}(2)))$

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terms $\mathrm{leaf}((1:0):0)$ $\mathrm{node}(\mathrm{leaf}(1), \mathrm{leaf}(2))$ $\mathrm{leaf}(\mathrm{node}(1, \mathrm{leaf}(2)))$

rewrite rules

$$\cdots$$

$$\mathrm{sum}(\mathrm{leaf}(x)) \to x$$

$$\mathrm{sum}(\mathrm{node}(x, y)) \to \mathrm{sum}(x) + \mathrm{sum}(y)$$

Example (Binary Trees)

signature $0 \quad 1 \quad \cdots \quad 9$ (constants) $+ \quad :$ (binary, infix)
$\mathsf{leaf} \quad \mathsf{sum}$ (unary) node (binary)

terms $\mathsf{leaf}((1:0):0) \quad \mathsf{node}(\mathsf{leaf}(1), \mathsf{leaf}(2)) \quad \mathsf{leaf}(\mathsf{node}(1, \mathsf{leaf}(2)))$

rewrite rules

$$\begin{aligned} &\cdots \\ \mathsf{sum}(\mathsf{leaf}(x)) &\to x \\ \mathsf{sum}(\mathsf{node}(x, y)) &\to \mathsf{sum}(x) + \mathsf{sum}(y) \end{aligned}$$

rewriting $\mathsf{sum}(\mathsf{node}(\mathsf{leaf}(2:3), \mathsf{leaf}(7:7)))$

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$$\mathsf{sum}(\mathsf{node}(\mathsf{leaf}(2:3), \mathsf{leaf}(7:7)))$$

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Outline

Term Rewriting Systems

Definition

- signature Σ function symbols $f \in \Sigma$ with arities $\#(f)$

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  - if $f \in \Sigma$ has arity $n \geqslant 1$ and $t_1, \ldots, t_n \in \mathcal{T}(\Sigma, \mathcal{X})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\Sigma, \mathcal{X})$

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Definitions (Operations on Terms)

- $\mathcal{V}\mathrm{ar}(\,\cdot\,)$, the variables of a term

$$\mathcal{V}\mathrm{ar}(t) = \begin{cases} \{t\} & \text{if } t \in \mathcal{X} \\ \displaystyle\bigcup_{i=1}^{n} \mathcal{V}\mathrm{ar}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Definitions (Operations on Terms)

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- $\mathcal{F}\text{un}(\,\cdot\,)$, the function symbols

$$\mathcal{F}\text{un}(t) = \begin{cases} \varnothing & \text{if } t \in \mathcal{X} \\ \{f\} \cup \bigcup_{i=1}^{n} \mathcal{F}\text{un}(t_i) & \text{if } t = f(t_1,\ldots,t_n) \end{cases}$$

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- $\mathcal{F}\text{un}(\,\cdot\,)$, the function symbols $\quad \mathcal{F}\text{un}((2 : x) + ((1 : x) : y)) = \{1, 2, :\}$

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Definition (Operations on Terms)

- root(·), the root of a term

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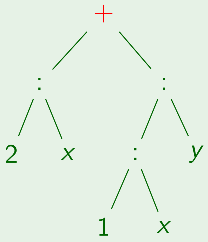
Example

$(2 : x) + ((1 : x) : y)$



Definitions (Operations on Terms)

- $|\cdot|$, the size of a term

$$|t| = \begin{cases} 1 & \text{if } t \in \mathcal{X} \\ 1 + \sum_{i=1}^{n} |t_i| & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Definitions (Operations on Terms)

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- $\|\cdot\|$, the number of function symbols

$$\|t\| = \begin{cases} 0 & \text{if } t \in \mathcal{X} \\ 1 + \sum_{i=1}^{n} \|t_i\| & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Definitions (Operations on Terms)

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- $\|\cdot\|$, the number of function symbols $\quad \|(2:x)+((1:x):y)\| = 6$

$$\|t\| = \begin{cases} 0 & \text{if } t \in \mathcal{X} \\ 1 + \sum_{i=1}^{n} \|t_i\| & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Definition (Operations on Terms)

- height(·), the height (or depth) of a term

$$\text{height}(t) = \begin{cases} 0 & \text{if } t \in \mathcal{X} \text{ or } t \text{ is a constant} \\ 1 + \max\limits_{1 \leqslant i \leqslant n} \text{height}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ with } n \geqslant 1 \end{cases}$$

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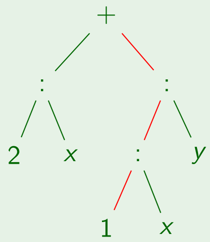
Example

$\mathrm{height}((2 : x) + ((1 : x) : y)) = 3$



Definitions

- $s \trianglelefteq t$ $\quad s$ is subterm of t
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Definition (Positions)

- $\mathcal{P}os(\cdot)$, the set of positions of a term

$$\mathcal{P}os(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{X} \\ \{\epsilon\} \cup \{ip \mid p \in \mathcal{P}os(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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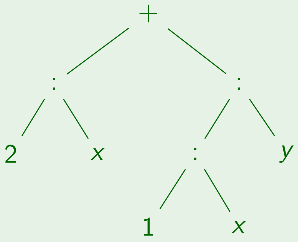

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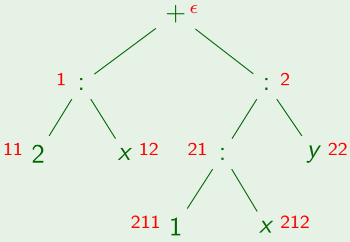

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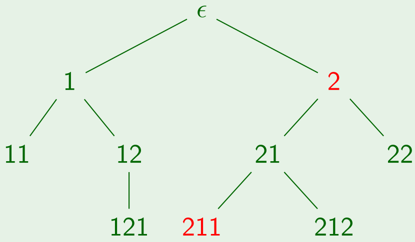

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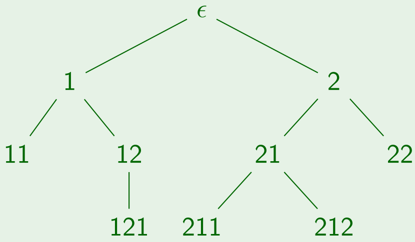

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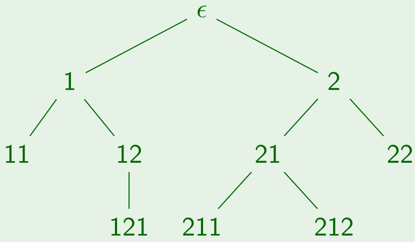

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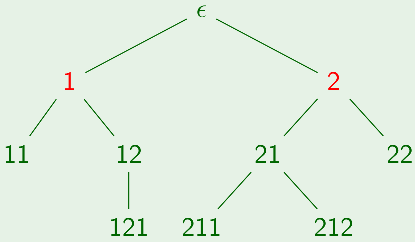

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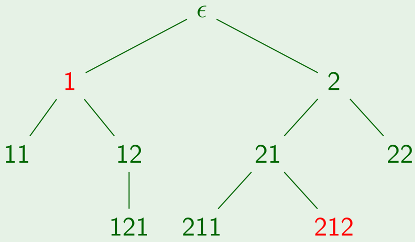

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Definitions

- $t|_p$ subterm of t at position p

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- $t[s]_p$ replace subterm in t at position p by s

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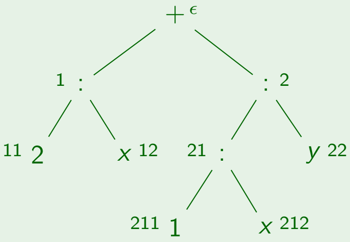

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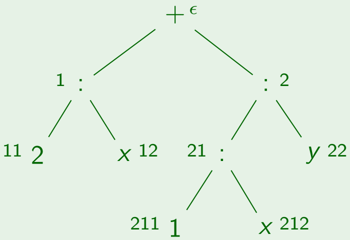

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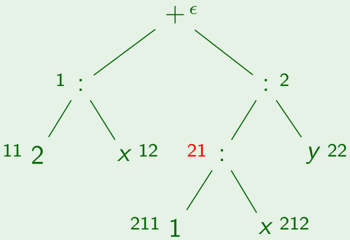

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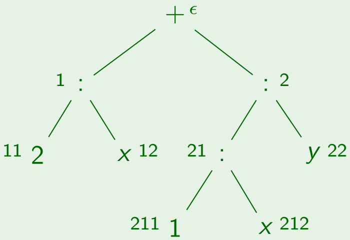

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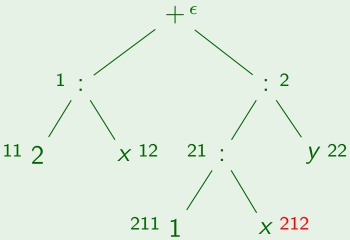

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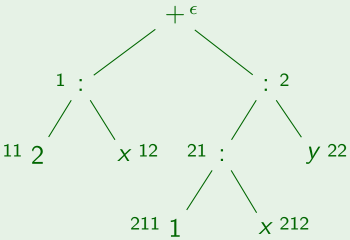

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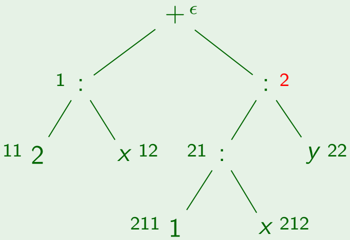

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- substitution is mapping $\sigma\colon \mathcal{X} \to \mathcal{T}(\Sigma, \mathcal{X})$ such that its domain

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Outline

Definitions

- A rewrite rule ($\ell \to r$) is a pair (ℓ, r) of terms such that
  - $\ell \notin \mathcal{X}$ (the lhs is not a variable)
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Example

TRS (Σ, R) with signature Σ

0 (constant) s (unary) add (binary)

$\#(0) = 0$, $\#(\mathrm{s}) = 1$, $\#(\mathrm{add}) = 2$, and rewrite rules R

$$\mathrm{add}(0, y) \to y$$
$$\mathrm{add}(\mathrm{s}(x), y) \to \mathrm{s}(\mathrm{add}(x, y))$$

Definition (Term Rewriting)

Let $\mathcal{R} = (\Sigma, R)$ be a TRS. The rewrite relation $\rightarrow_{\mathcal{R}}$ on $\mathcal{T}(\Sigma, \mathcal{X})$ is defined as:

$$C[\ell\sigma] \rightarrow_{\mathcal{R}} C[r\sigma]$$

for every:

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Example

The rule $\mathsf{add}(0, y) \to y$ with $y \mapsto \mathsf{s}(0)$ and $C = \mathsf{add}(0, \Box)$ gives rise to the step:

$$\mathsf{add}(0, \mathsf{add}(0, \mathsf{s}(0))) \to \mathsf{add}(0, \mathsf{s}(0))$$

at position 2.

Lemma

The rewrite relation $\rightarrow_{\mathcal{R}}$ is the smallest relation on $\mathcal{T}(\Sigma, \mathcal{X})$ that:

- *contains the rules R,*
- *is closed under context, and*
- *is closed under substitutions.*

That is, $\rightarrow_{\mathcal{R}}$ is the closure of R under contexts and substitutions.

Properties of Term Rewriting Systems

We consider TRSs $\mathcal{R} = (\Sigma, R)$ as

$$\text{ARSs } \langle \mathcal{T}(\Sigma, \mathcal{X}),\ \rightarrow_{\mathcal{R}} \rangle$$

TRSs inherit the properties: CR, WCR, SN, WN, NF, UN, $\text{UN}^{\rightarrow}$, ...

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TRSs inherit the properties: CR, WCR, SN, WN, NF, UN, $\mathsf{UN}^{\to}$, ...

Example

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Example

TRS R modeling Sieve of Eratostheness for generating list of prime numbers

$$\begin{aligned} \mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) \\ \mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) \\ \mathsf{head}(x : y) &\to x \\ \mathsf{tail}(x : y) &\to y \end{aligned}$$

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- how to compute normal forms in R ? strategy (lecture 9)

Example (Combinatory Logic)

$$
\begin{aligned}
\mathsf{I} \cdot x &\to x \\
(\mathsf{K} \cdot x) \cdot y &\to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z)
\end{aligned}
$$

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\begin{aligned}
\mathsf{I} \cdot x &\to x & \mathsf{I}\, x &\to x \\
(\mathsf{K} \cdot x) \cdot y &\to x & (\mathsf{K}\, x)\, y &\to x \\
((\mathsf{S} \cdot x) \cdot y) \cdot z &\to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\, x)\, y)\, z &\to (x\, z)\, (y\, z)
\end{aligned}
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- applicative notation: surpress $\cdot$

Example (Combinatory Logic)

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\begin{array}{rlrlrl}
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((\mathsf{S} \cdot x) \cdot y) \cdot z & \to (x \cdot z) \cdot (y \cdot z) & ((\mathsf{S}\,x)\,y)\,z & \to (x\,z)\,(y\,z) & \mathsf{S}\,x\,y\,z & \to x\,z\,(y\,z)
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Example (Combinatory Logic)

$\mathsf{I} \cdot x \to x$	$\mathsf{I}\, x \to x$	$\mathsf{I}\, x \to x$
$(\mathsf{K} \cdot x) \cdot y \to x$	$(\mathsf{K}\, x)\, y \to x$	$\mathsf{K}\, x\, y \to x$
$((\mathsf{S} \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z)$	$((\mathsf{S}\, x)\, y)\, z \to (x\, z)\, (y\, z)$	$\mathsf{S}\, x\, y\, z \to x\, z\, (y\, z)$

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$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

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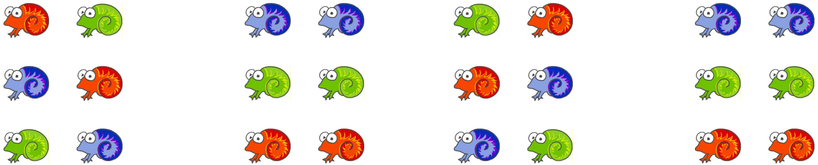
$$\mathsf{SII(SII)} \to \mathsf{I(SII)(I(SII))} \to \mathsf{SII(I(SII))} \to \mathsf{SII(SII)}$$

- CL is consistent

$$\mathsf{S} \not\leftrightarrow^* \mathsf{K}$$

Definition

TRS R over signature Σ is string rewrite system (SRS) if Σ consists of unary function symbols



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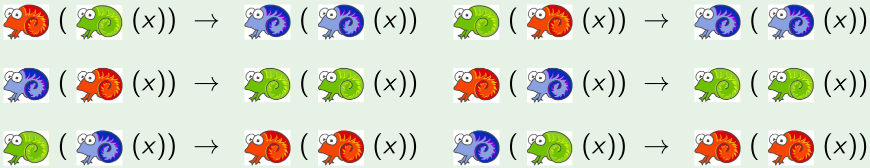

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