

Term Rewriting Systems

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- Lecture 1: Introduction, Abstract Rewriting
- Lecture 2: Term Rewriting
- Lecture 3: Combinatory Logic
- Lecture 4: Termination
- Lecture 5: Matching, Unification
- Lecture 6: Equational Reasoning, Completion
- Lecture 7: Confluence
- Lecture 8: Modularity
- Lecture 9: Strategies
- Lecture 10: Decidability
- Lecture 11: Infinitary Rewriting

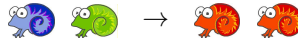
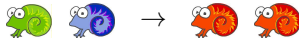
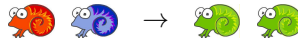
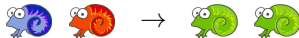
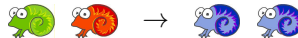
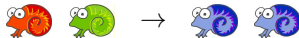
Outline

- Overview
- Examples
- Abstract Rewrite Systems
- Newman's Lemma
- Properties of Elements
- ARSs with Multiple Relations

Examples



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?



A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT



Techniques exist to perform the following DNA substitutions

TCAT \leftrightarrow T GAG \leftrightarrow AG CTC \leftrightarrow TC AGTA \leftrightarrow A TAT \leftrightarrow CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature 0 (constants) s (unary) + (binary, infix)

terms	$s(s(0))$	$s(0) + s(s(0))$	$s(x) + y$
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rewrite rules

$$0 + y \rightarrow y$$
$$s(x) + y \rightarrow s(x + y)$$

rewriting $s(0) + s(s(0)) \rightarrow s(0 + s(s(0))) \quad y \mapsto s(s(0))$
 $\rightarrow s(s(s(0)))$

Example (Group Theory)

signature	e (constant) $^-$ (unary, postfix) \cdot (binary, infix)	
equations	$e \cdot x \approx x$ $x^- \cdot x \approx e$ $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$	\mathcal{E}
theorems	$e^- \approx_{\mathcal{E}} e$ $(x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot x^-$	
rewrite rules	$ \begin{array}{ll} e \cdot x \rightarrow x & x \cdot e \rightarrow x \\ x^- \cdot x \rightarrow e & x \cdot x^- \rightarrow e \\ (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) & x^{--} \rightarrow x \\ e^- \rightarrow e & (x \cdot y)^- \rightarrow y^- \cdot x^- \\ x^- \cdot (x \cdot y) \rightarrow y & x \cdot (x^- \cdot y) \rightarrow y \end{array} $	R

- ① $s \approx t$ is valid in \mathcal{E} ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R -normal form
- ② R admits no infinite computations
- ① & ② $\implies \mathcal{E}$ has decidable validity problem

Example (Combinatory Logic)

signature $S \quad K \quad I$ (constants) \cdot (application, binary, infix)

terms $S \quad ((K \cdot I) \cdot I) \cdot S \quad (x \cdot z) \cdot (y \cdot z)$

rewrite rules

$$I \cdot x \rightarrow x$$

$$(K \cdot x) \cdot y \rightarrow x$$

$$((S \cdot x) \cdot y) \cdot z \rightarrow (x \cdot z) \cdot (y \cdot z)$$

rewriting

$$((S \cdot K) \cdot K) \cdot x \rightarrow (K \cdot x) \cdot (K \cdot x)$$

$$\rightarrow x$$

inventor **Moses Schönfinkel** (1924)



Example (Lambda Calculus)

signature λ (binds variables) \cdot (application, binary, infix)

terms $M ::= x \mid (\lambda x. M) \mid (M \cdot M)$

α conversion $\lambda x. x \cdot y =_{\alpha} \lambda z. z \cdot y$

β reduction $(\lambda x. M) \cdot N \rightarrow_{\beta} M[x := N]$
 replace free occurrences of x in M by N

rewriting $(\lambda x. x \cdot x) \cdot (\lambda x. x \cdot x) \rightarrow (\lambda x. x \cdot x) \cdot (\lambda x. x \cdot x)$

inventor **Alonzo Church** (1936)



Motivation

Term rewriting is used in:

- functional programming (higher order term rewriting)
- model checking (e.g. mCRL)
- compiler construction (graph rewriting)
- computer algebra systems (e.g. Mathematica, Wolfram Alpha)
- proof assistants / automated theorem provers
- deciding equality in equational systems (axiom systems)
- abstract model of computation
- ...

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Abstract Rewrite Systems

Motivation

concrete rewrite formalisms

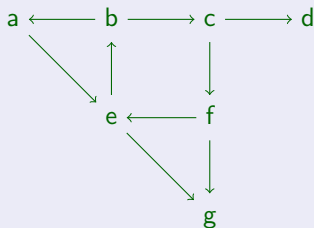
- string rewriting
- term rewriting
- graph rewriting
- λ -calculus
- interaction nets
- ...

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation \rightarrow



$$\text{ARS } \mathcal{A} = \langle A, \rightarrow \rangle$$

- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow = \left\{ (a, e), (b, a), (b, c), (c, d), (c, f), (e, b), (e, g), (f, e), (f, g) \right\}$

- rewrite sequence**

rewrite step

- finite $a \rightarrow e \rightarrow b \rightarrow c \rightarrow f$ length 4 $a \rightarrow^* f$
- empty a length 0 $a \rightarrow^* a$
- infinite $a \rightarrow e \rightarrow b \rightarrow a \rightarrow e \rightarrow b \rightarrow \dots$ length ω

The length of a rewrite sequence is the number of rewrite steps.
 We write $x \rightarrow^* y$ if x rewrites to y in 0 or more steps.

Definition (Derived Relations of \rightarrow)

- \leftarrow or \rightarrow^{-1} inverse of \rightarrow
- $\rightarrow^=$ reflexive closure of \rightarrow
- \rightarrow^+ transitive closure of \rightarrow
- \rightarrow^* or \twoheadrightarrow transitive and reflexive closure of \rightarrow
- $^*\leftarrow$ or $\overleftarrow{\leftarrow}$ inverse of \rightarrow^* (transitive and reflexive closure of \leftarrow)
- \leftrightarrow symmetric closure of \rightarrow , that is, $\leftrightarrow = \rightarrow \cup \leftarrow$
- \leftrightarrow^* conversion (equivalence relation generated by \rightarrow)
- \downarrow joinability $\downarrow = \rightarrow^* \cdot ^*\leftarrow$
- \uparrow **meetability** $\uparrow = ^*\leftarrow \cdot \rightarrow^*$

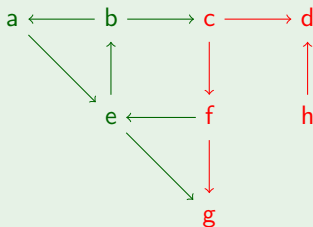
a relation R is

- reflexive if $a R a$ for all $a \in A$,
- transitive if $a R c$ whenever $a R b$ and $b R c$,
- symmetric if $a R b$ whenever $b R a$.

Terminology

- if $x \rightarrow^* y$ then x rewrites to y and y is reduct of x
- if $x \rightarrow^* z$ $z \leftarrow^* y$ then z is common reduct of x and y
- if $x \leftrightarrow^* y$ then x and y are **convertible**

Example

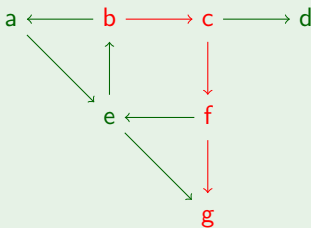


- $a \rightarrow^* f$
- $e \downarrow f$ $f \downarrow h$ not $g \downarrow h$
- $g \leftrightarrow^* h$

Definition (Normal Forms)

- normal form is element x such that $x \not\rightarrow y$ for all y
- $\text{NF}(\mathcal{A})$ denotes set of normal forms of ARS \mathcal{A}
- $x \rightarrow^! y$ if $x \rightarrow^* y$ for normal form y (x has normal form y)

Example



ARS $\mathcal{A} = \langle A, \rightarrow \rangle$

- d is normal form
- $\text{NF}(\mathcal{A}) = \{d, g\}$
- $b \rightarrow^! g$

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Definitions

- SN strong normalization or termination
 - no infinite rewrite sequences
- WN weak normalization
 - every element has (rewrites to) at least one normal form
 - $\forall a \exists b \ a \rightarrow^! b$

Lemmata

$$1 \quad \text{SN} \implies \text{WN}$$

$$2 \quad \text{SN} \not\Leftarrow \text{WN}$$

$$\text{C} \quad a \longrightarrow b$$

Definitions

- NF normal form property
 - if an element a is convertible with a normal form b , then a rewrites to b
 - $\forall a, b$ if $a \leftrightarrow^* b$ and b is a normal form then $a \rightarrow^* b$
- UN unique normal forms
 - convertible normal forms are equal
 - $\forall a, b$ if $a \leftrightarrow^* b$ and a, b are normal forms then $a = b$
- $\text{UN} \rightarrow$ unique normal forms with respect to reduction
 - no element has more than one normal form
 - $\forall a, b, c$ if $a \rightarrow^! b$ and $a \rightarrow^! c$ then $b = c$
 - $\leftarrow^! \cdot \rightarrow^! \subseteq =$

Lemmata

$$1 \quad \text{SN} \implies \text{WN}$$

$$2 \quad \text{SN} \not\Leftarrow \text{WN}$$

$$3 \quad \text{NF} \implies \text{UN} \implies \text{UN}^\rightarrow$$

$$4 \quad \text{UN} \not\Leftarrow \text{UN}^\rightarrow$$

$$5 \quad \text{NF} \not\Leftarrow \text{UN}$$

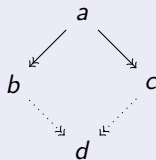


Definition

- CR confluence or Church-Rosser property

- $\uparrow \subseteq \downarrow$

- $\forall a, b, c$



in diagrams: \Rightarrow for \rightarrow^*

$\exists d$

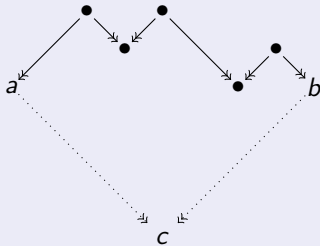
- $\forall a, b, c. \quad a \rightarrow^* b \wedge a \rightarrow^* c \quad \Rightarrow \quad \exists d. \quad b \rightarrow^* d \wedge c \rightarrow^* d$

Lemma (An Equivalent Formulation of Confluence)

Confluence $\uparrow \subseteq \downarrow$ is equivalent to:

- $\leftrightarrow^* \subseteq \downarrow$

- $\forall a, b$



- $\forall a, b. \quad a \leftrightarrow^* b \quad \Rightarrow \quad \exists c. \quad a \rightarrow^* c \wedge b \rightarrow^* c$

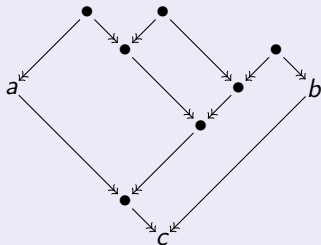
Lemma

$$\uparrow \subseteq \downarrow \iff \leftrightarrow^* \subseteq \downarrow$$

First proof: diagram tiling.

\Leftarrow Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.

\Rightarrow Assume $\uparrow \subseteq \downarrow$. We show $\leftrightarrow^* \subseteq \downarrow$. Therefore let $a, b \in A$ such that $a \leftrightarrow^* b$. Then $a (\leftarrow^* \cdot \rightarrow^*)^* b$ since $\leftrightarrow \subseteq \leftarrow^* \cdot \rightarrow^*$. Hence:



- we stepwise replace
peaks $e \xleftarrow{*} d \xrightarrow{*} f$
by valleys $e \xrightarrow{*} \cdot \xleftarrow{*} f$
- after each step one peak less
- hence finally no peaks left
 $\implies a \downarrow b$

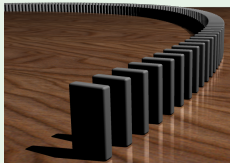
Induction

Lemma (Induction)

To prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$ do:

- 1 The base case:
show that the statement holds for $n = 0$.
- 2 The inductive step:
show for all n that if the $P(n)$ holds, then also $P(n + 1)$ holds.

Example



Wikipedia

- Base case: proof that the first domino falls
- Induction step: proof that if the n -th domino falls then the $(n + 1)$ -st domino falls

Then you have proven that all dominoes will fall.

Example

We use induction to prove that:

$$1 + 2 + \dots + n = \frac{n \cdot (n + 1)}{2}$$

1 Base case $n = 0$: Then

$$1 + 2 + \dots + 0 = 0 \quad \text{and} \quad \frac{0 \cdot (0 + 1)}{2} = 0$$

Thus the statement holds for $n = 0$.

2 Induction step:

Induction hypothesis (IH): Assume the statement hold for n .

We show it for $n + 1$:

$$\begin{aligned} 1 + 2 + \dots + n + (n + 1) &= \frac{n \cdot (n + 1)}{2} + (n + 1) \quad \text{by IH} \\ &= \frac{n \cdot (n + 1) + 2 \cdot (n + 1)}{2} = \frac{(n + 1) \cdot ((n + 1) + 1)}{2} \end{aligned}$$

Hence the formula holds for all $n \in \mathbb{N}$.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

\Leftarrow Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.

\Rightarrow Assume $\uparrow \subseteq \downarrow$. We show $\leftrightarrow^* \subseteq \downarrow$.

We proof by induction on n that $a (\leftarrow^* \cdot \rightarrow^*)^n b$ implies $a \rightarrow^* \cdot \leftarrow^* b$.

- Base case $n = 0$: $a (\leftarrow^* \cdot \rightarrow^*)^0 b$. Then $a = b$ and hence $a \rightarrow^* \cdot \leftarrow^* b$.
- Induction step $n + 1$: (assume it holds for n , show it for $n + 1$)

Let $a (\leftarrow^* \cdot \rightarrow^*)^{n+1} b$. Then $a (\leftarrow^* \cdot \rightarrow^*)^n d \leftarrow^* e \rightarrow^* b$ for some d, e .

- Hence $a \rightarrow^* f \leftarrow^* d$ for some f by induction hypothesis.
- Now $f \leftarrow^* e \rightarrow^* b$ and thus $f \rightarrow^* \cdot \leftarrow^* b$ since by assumption $\uparrow \subseteq \downarrow$.

We conclude $a \rightarrow^* \cdot \leftarrow^* b$, that is, $a \downarrow b$.

Hence we have shown $\uparrow^* \subseteq \downarrow$. It follows $\leftrightarrow^* \subseteq \downarrow$ since $\leftrightarrow^* \subseteq \uparrow^*$. ■

Lemma

Confluence $\uparrow \subseteq \downarrow$ is equivalent to:

- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$

- $\forall a, b, c$

$\exists d$



- $\forall a, b, c. \quad a \rightarrow^* b \wedge a \rightarrow c \quad \Rightarrow \quad \exists d. \quad b \rightarrow^* d \wedge c \rightarrow^* d$

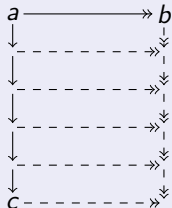
Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

\Rightarrow Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

\Leftarrow Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



By induction on n we show: ${}^n\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n .

- Base case $n = 0$: ${}^0\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$:

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

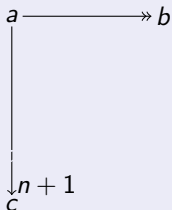
Proof.

\Rightarrow Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

\Leftarrow Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

By induction on n we show: ${}^n\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n .

- Base case $n = 0$: ${}^0\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$: let $c \xrightarrow{{}^{n+1}\leftarrow} a \rightarrow^* b$.



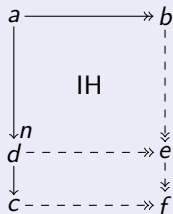
Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

\Rightarrow Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

\Leftarrow Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.



By induction on n we show: ${}^n\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n .

- Base case $n = 0$: ${}^0\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$: let $c \xrightarrow{n+1}\leftarrow a \rightarrow^* b$.
Then $c \leftarrow d \xrightarrow{n}\leftarrow a \rightarrow^* b$ for some d , and:
 - By induction hypothesis $d \rightarrow^* e \xleftarrow{*} b$.
 - Then $c \rightarrow^* f \xleftarrow{*} e$ since $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

Hence $c \rightarrow^* \cdot \xleftarrow{*} b$.

Lemmata

$$1 \quad \text{SN} \implies \text{WN}$$

$$2 \quad \text{SN} \not\Leftarrow \text{WN}$$

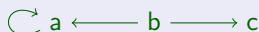


$$3 \quad \text{CR} \implies \text{NF} \implies \text{UN} \implies \text{UN}^\rightarrow$$

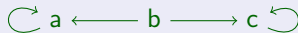
$$4 \quad \text{UN} \not\Leftarrow \text{UN}^\rightarrow$$



$$5 \quad \text{NF} \not\Leftarrow \text{UN}$$



$$6 \quad \text{CR} \not\Leftarrow \text{NF}$$



$$7 \quad \text{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

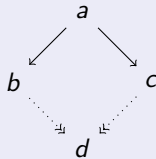
$$8 \quad \text{WN} \ \& \ \text{UN}^\rightarrow \implies \text{CR}$$

Definitions

- **WCR** local confluence or weak Church-Rosser property

- $\leftarrow \cdot \rightarrow \subseteq \downarrow$

- $\forall a, b, c$



Lemmata

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$$3 \quad \text{CR} \implies \text{NF} \implies \text{UN} \implies \text{UN}^\rightarrow$$

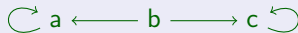
$$4 \quad \text{UN} \not\Leftarrow \text{UN}^\rightarrow$$



$$5 \quad \text{NF} \not\Leftarrow \text{UN}$$



$$6 \quad \text{CR} \not\Leftarrow \text{NF}$$



$$7 \quad \text{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

$$8 \quad \text{WN} \ \& \ \text{UN}^\rightarrow \implies \text{CR}$$

$$9 \quad \text{CR} \implies \text{WCR}$$

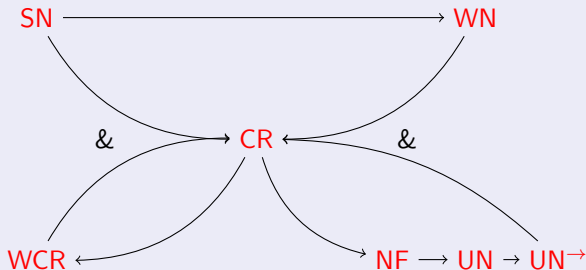
$$10 \quad \text{CR} \not\Leftarrow \text{WCR}$$



$$11 \quad \text{SN} \ \& \ \text{WCR} \implies \text{CR}$$

Newman's Lemma

Summary



Definitions

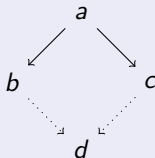
- semi-completeness
 - CR & WN
 - every element has unique normal form
- completeness
 - CR & SN

Definition

- diamond property \diamond

- $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$

- $\forall a, b, c$



$\exists d$

Lemma

An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if \rightarrow has the diamond property.

Proof.

Exercise. ■

Lemma

ARS $\mathcal{A} = \langle A, \rightarrow_1 \rangle$ is confluent if

- $\rightarrow_1 \subseteq \rightarrow_2 \subseteq \rightarrow_1^*$

for a confluent relation \rightarrow_2 on A .

Proof.

Assume \rightarrow_2 is confluent, that is, ${}^*\leftarrow_2 \cdot \rightarrow_2^* \subseteq \rightarrow_2^* \cdot {}^*\leftarrow_2$.

- From $\rightarrow_1 \subseteq \rightarrow_2$ follows $\rightarrow_1^* \subseteq \rightarrow_2^*$.
- Moreover $\rightarrow_2^* \subseteq \rightarrow_1^*$ since \rightarrow_1^* is transitive and contains \rightarrow_2 .

Hence $\rightarrow_1^* = \rightarrow_2^*$.

$$\Rightarrow {}^*\leftarrow_1 \cdot \rightarrow_1^* \subseteq \rightarrow_1^* \cdot {}^*\leftarrow_1$$



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- Properties of Elements
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Well-Founded Induction

given

- strongly normalizing ARS $\mathcal{A} = \langle A, \rightarrow \rangle$
- property P over the elements of \mathcal{A}

to conclude

- $\forall a \in A: P(a)$

it is sufficient to prove

- if $\underbrace{P(b) \text{ for every } b \text{ with } a \rightarrow b}_{\text{induction hypothesis}}$ then $P(a)$

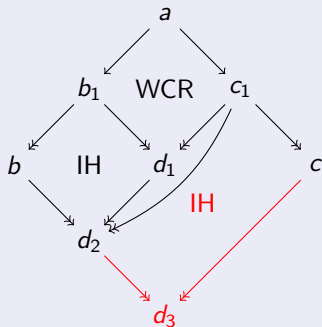
for arbitrary element a

$$\left(\forall a: \left(\forall b: a \rightarrow b \implies P(b) \right) \implies P(a) \right) \implies \forall a: P(a)$$

Newman's Lemma

$$\text{SN}(\mathcal{A}) \ \& \ \text{WCR}(\mathcal{A}) \implies \text{CR}(\mathcal{A})$$

Proof.



induction hypothesis

$\forall a': \text{ if } a \rightarrow a' \text{ then } \text{CR}(a')$

$\text{CR}(c_1)$



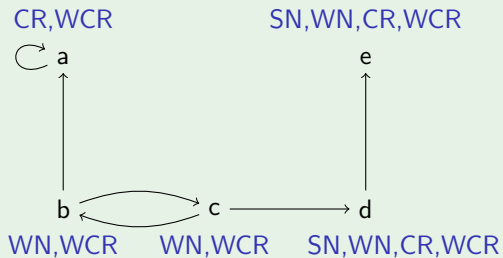
Definitions (Properties of Elements)

Let $\langle A, \rightarrow \rangle$ be an ARS. An element $a \in A$ is called:

- SN strongly normalizing or terminating
if a admits no infinite rewrite sequence $a = a_1 \rightarrow a_2 \rightarrow \dots$
- WN weakly normalizing
if $\exists b. a \rightarrow^! b$
- CR confluent or Church Rosser
if $\forall b, c. (c \xleftarrow{*} a \rightarrow^* b \Rightarrow \exists d. c \rightarrow^* d \xleftarrow{*} b)$
- WCR weakly confluent or weakly Church Rosser
if $\forall b, c. (c \leftarrow a \rightarrow b \Rightarrow \exists d. c \rightarrow^* d \xleftarrow{*} b)$

An ARS has the property if all its elements have the respective property.

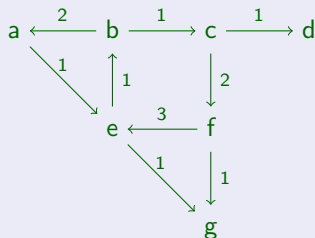
Example



ARSs with Multiple Relations

Definitions

- abstract rewrite system (ARS)** is set A with binary relations \rightarrow_i for $i \in \mathcal{I}$



ARS $\mathcal{A} = \langle A, \rightarrow_1, \rightarrow_2, \rightarrow_3 \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow_1 = \left\{ (a, e), (b, c), (c, d), (e, b), (e, g), (f, g) \right\}$
- $\rightarrow_2 = \{(b, a), (c, f)\}$
- $\rightarrow_3 = \{(f, e)\}$

- $\rightarrow = \rightarrow_1 \cup \rightarrow_2 \cup \rightarrow_3$
- $\rightarrow_{12} = \rightarrow_1 \cup \rightarrow_2, \quad \rightarrow_{13} = \rightarrow_1 \cup \rightarrow_3, \quad \dots$

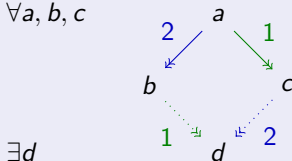
Definition

Let $\mathcal{A} = \langle A, \rightarrow_1, \rightarrow_2 \rangle$ be an ARS.

- \rightarrow_1 commutes with \rightarrow_2

- ${}_2^* \leftarrow \cdot \rightarrow_1^* \subseteq \rightarrow_1^* \cdot {}_2^* \leftarrow$

- $\forall a, b, c$



- \rightarrow_1 commutes weakly with \rightarrow_2

- ${}_2 \leftarrow \cdot \rightarrow_1 \subseteq \rightarrow_1^* \cdot {}_2^* \leftarrow$

- $\forall a, b, c$

