Term Rewriting Systems

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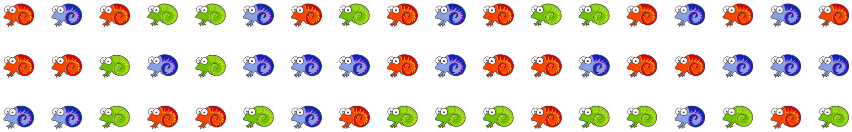
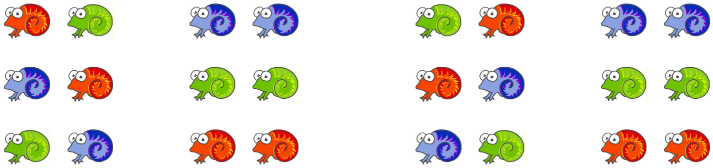
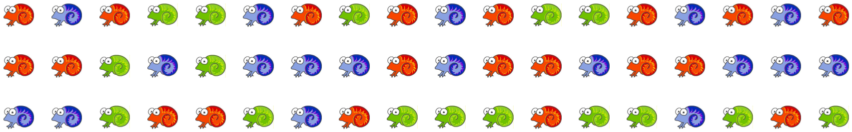
Outline

Examples



A colony of chameleons includes 20 red, 18 blue, and 16 green individuals. Whenever two chameleons of different colors meet, each changes to the third color. Some time passes during which no chameleons are born or die nor do any enter or leave the colony. Is it possible that at the end of this period, all 54 chameleons are the same color?





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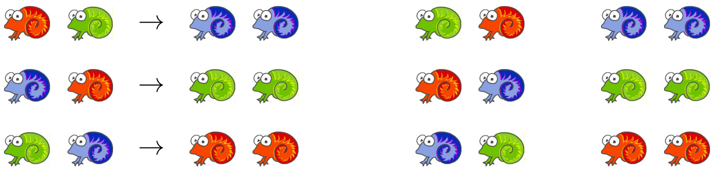

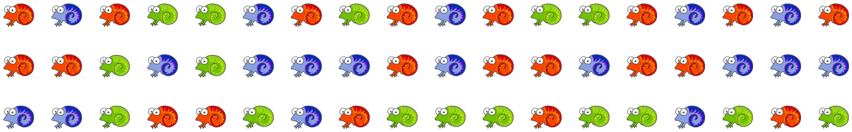

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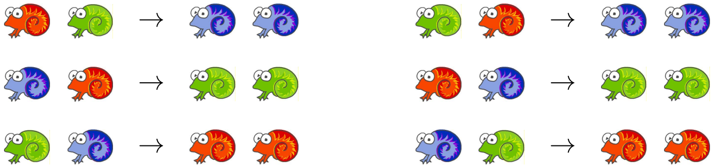

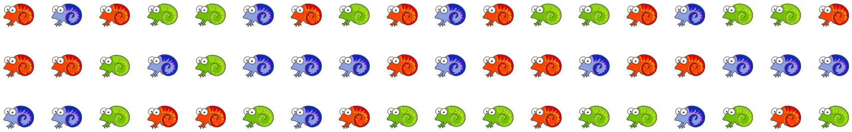

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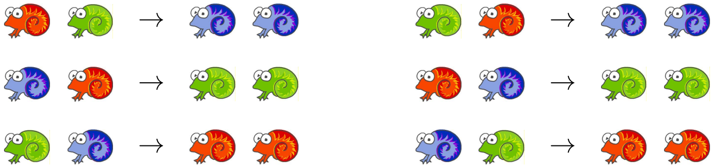

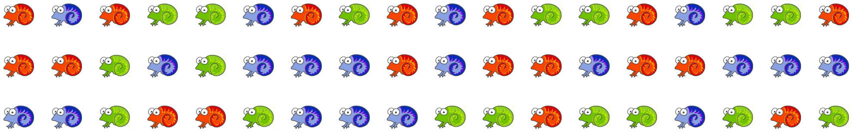

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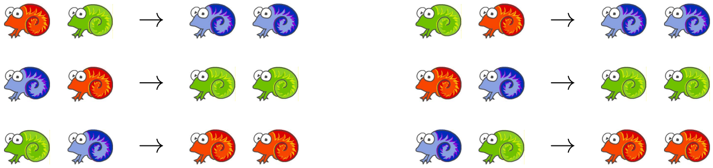

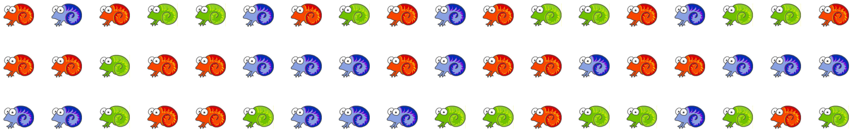

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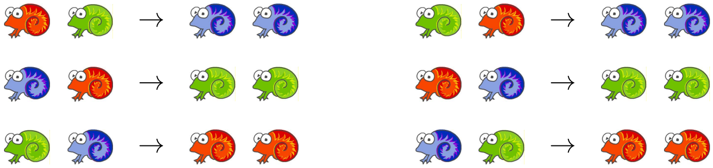

A team of genetic engineers decides to create cows that produce cola instead of milk. To that end they have to transform the DNA of the milk gene

TAGCTAGCTAGCT

in every fertilized egg into the cola gene

CTGACTGACT

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Techniques exist to perform the following DNA substitutions

TCAT $\leftrightarrow$ T   GAG $\leftrightarrow$ AG   CTC $\leftrightarrow$ TC   AGTA $\leftrightarrow$ A   TAT $\leftrightarrow$ CT

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Techniques exist to perform the following DNA substitutions

TCAT $\leftrightarrow$ T GAG $\leftrightarrow$ AG CTC $\leftrightarrow$ TC AGTA $\leftrightarrow$ A TAT $\leftrightarrow$ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if accidentally cows with this virus are created? According to the engineers there is little risk because this never happened in their experiments, but various action groups demand absolute assurances.

Example (Addition on Natural Numbers in Unary Notation)

signature $\quad 0$ (constants) $\quad s$ (unary) $\quad +$ (binary, infix)

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rewrite rules

$$0 + y \to y$$
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Example (Group Theory)

signature $\quad \mathsf{e}$ (constant) $\quad {}^{-}$ (unary, postfix) $\quad \cdot$ (binary, infix)

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equations $\quad \mathsf{e} \cdot x \approx x \qquad x^{-} \cdot x \approx \mathsf{e} \qquad (x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \mathcal{E}$

Example (Group Theory)

signature	e	(constant)	$^-$	(unary, postfix)	$\cdot$	(binary, infix)	
equations	$\mathsf{e} \cdot x \approx x$		$x^- \cdot x \approx \mathsf{e}$		$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$		$\mathcal{E}$
theorems			$\mathsf{e}^- \approx_{\mathcal{E}} \mathsf{e}$		$(x \cdot y)^- \approx_{\mathcal{E}} y^- \cdot x^-$		

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rewrite rules	$\begin{aligned} \mathsf{e} \cdot x &\to x \\ x^- \cdot x &\to \mathsf{e} \\ (x \cdot y) \cdot z &\to x \cdot (y \cdot z) \end{aligned}$	R

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rewrite rules

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\begin{array}{rclrcl}
\mathsf{e} \cdot x & \to & x & x \cdot \mathsf{e} & \to & x \\
x^- \cdot x & \to & \mathsf{e} & x \cdot x^- & \to & \mathsf{e} \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) & x^{--} & \to & x \\
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x^- \cdot (x \cdot y) & \to & y & x \cdot (x^- \cdot y) & \to & y
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rewrite rules	see below	R

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x^{-} \cdot (x \cdot y) & \to & y & x \cdot (x^{-} \cdot y) & \to & y
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① $s \approx t$ is valid in $\mathcal{E}$ ($s \approx_{\mathcal{E}} t$) if and only if s and t have same R-normal form

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② R admits no infinite computations

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② R admits no infinite computations

① & ② $\Longrightarrow$ $\mathcal{E}$ has decidable validity problem

Example (Combinatory Logic)

signature S K I (constants)

Example (Combinatory Logic)

signature S K I (constants) $\cdot$ (application, binary, infix)

Example (Combinatory Logic)

signature $\quad \mathsf{S} \quad \mathsf{K} \quad \mathsf{I}$ (constants) $\quad \cdot \quad$ (application, binary, infix)

terms $\quad \mathsf{S} \quad ((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S} \quad (x \cdot z) \cdot (y \cdot z)$

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rewrite rules

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rewriting $((\mathsf{S}\cdot\mathsf{K})\cdot\mathsf{K})\cdot x$

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signature   S   K   I   (constants) $\cdot$ (application, binary, infix)

terms S $((\mathsf{K}\cdot\mathsf{I})\cdot\mathsf{I})\cdot\mathsf{S}$ $(x\cdot z)\cdot(y\cdot z)$

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rewriting $((\mathsf{S}\cdot\mathsf{K})\cdot\mathsf{K})\cdot x \;\to\; (\mathsf{K}\cdot x)\cdot(\mathsf{K}\cdot x)$

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signature: S K I (constants) $\cdot$ (application, binary, infix)

terms: S $\quad ((\mathsf{K} \cdot \mathsf{I}) \cdot \mathsf{I}) \cdot \mathsf{S}$ $\quad (x \cdot z) \cdot (y \cdot z)$

rewrite rules:

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rewriting:

$$\begin{aligned} ((\mathsf{S} \cdot \mathsf{K}) \cdot \mathsf{K}) \cdot x &\to (\mathsf{K} \cdot x) \cdot (\mathsf{K} \cdot x) \\ &\to x \end{aligned}$$

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rewriting

$$\begin{aligned} ((\mathsf{S}\cdot\mathsf{K})\cdot\mathsf{K})\cdot x &\to (\mathsf{K}\cdot x)\cdot(\mathsf{K}\cdot x) \\ &\to x \end{aligned}$$

inventor Moses Schönfinkel (1924)

Example (Lambda Calculus)

signature λ (binds variables)

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

Example (Lambda Calculus)

signature	λ (binds variables) $\quad\cdot\quad$ (application, binary, infix)
terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
α conversion	$\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

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terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
α conversion	$\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$
β reduction	$(\lambda x.\, M) \cdot N \to_\beta M[x := N]$

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terms $M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $(\lambda x.\, M) \cdot N \rightarrow_\beta M[x := N]$

replace free occurrences of x in M by N

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β reduction	$(\lambda x.\, M) \cdot N \to_\beta M[x := N]$
	replace free occurrences of x in M by N
rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

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terms	$M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$
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rewriting	$(\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \;\to\; (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

Example (Lambda Calculus)

signature $\quad \lambda$ (binds variables) $\quad \cdot$ (application, binary, infix)

terms $\quad M ::= x \mid (\lambda x.\, M) \mid (M \cdot M)$

α conversion $\quad \lambda x.\, x \cdot y =_\alpha \lambda z.\, z \cdot y$

β reduction $\quad (\lambda x.\, M) \cdot N \to_\beta M[x := N]$

replace free occurrences of x in M by N

rewriting $\quad (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x) \ \to\ (\lambda x.\, x \cdot x) \cdot (\lambda x.\, x \cdot x)$

inventor $\quad$ Alonzo Church (1936)

Motivation

Term rewriting is used in:

- functional programming (higher order term rewriting)
- model checking (e.g. mCRL)
- compiler construction (graph rewriting)
- computer algebra systems (e.g. Mathematica, Wolfram Alpha)
- proof assistants / automated theorem provers
- deciding equality in equational systems (axiom systems)
- abstract model of computation
- ...

Outline

Abstract Rewrite Systems

Motivation

concrete rewrite formalisms

- string rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- $\ldots$

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- ...

abstract rewriting

- no structure on objects that are rewritten

Motivation

concrete rewrite formalisms

- string rewriting
- term rewriting
- graph rewriting
- λ-calculus
- interaction nets
- ...

abstract rewriting

- no structure on objects that are rewritten
- uniform presentation of properties and proofs

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$
\begin{tikzcd}
a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & d \\
 & e \arrow[u] \arrow[dr] & f \arrow[l] \arrow[d] & \\
 & & g &
\end{tikzcd}
$$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
& \mathsf{e} \arrow[u] \arrow[dr] & \mathsf{f} \arrow[l] \arrow[d] & \\
& & \mathsf{g} &
\end{tikzcd}$$

ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$
- $\to = \left\{ \begin{matrix} (\mathsf{a},\mathsf{e}), (\mathsf{b},\mathsf{a}), (\mathsf{b},\mathsf{c}), (\mathsf{c},\mathsf{d}), (\mathsf{c},\mathsf{f}) \\ (\mathsf{e},\mathsf{b}), (\mathsf{e},\mathsf{g}), (\mathsf{f},\mathsf{e}), (\mathsf{f},\mathsf{g}) \end{matrix} \right\}$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
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- rewrite sequence
  - finite $\quad \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$

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\end{tikzcd}$$

ARS $\mathcal{A} = \langle A, \to \rangle$

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  - finite $\quad a \to e \to b \to c \to f$

rewrite step

Definitions

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- rewrite sequence
  - finite $\quad a \to e \to b \xrightarrow{\text{rewrite step}} c \to f$
  - empty $\quad a$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\rightarrow$

$$
\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
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- rewrite sequence

rewrite step

  - finite $\quad \mathsf{a} \rightarrow \mathsf{e} \rightarrow \mathsf{b} \rightarrow \mathsf{c} \rightarrow \mathsf{f}$
  - empty $\quad \mathsf{a}$
  - infinite $\quad \mathsf{a} \rightarrow \mathsf{e} \rightarrow \mathsf{b} \rightarrow \mathsf{a} \rightarrow \mathsf{e} \rightarrow \mathsf{b} \rightarrow \cdots$

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

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\end{tikzcd}$$

ARS $\mathcal{A} = \langle A, \to \rangle$

- $A = \{a, b, c, d, e, f, g\}$
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- rewrite sequence

rewrite step

  - finite $\quad a \to e \to b \to c \to f \quad$ length 4
  - empty $\quad a \quad$ length 0
  - infinite $\quad a \to e \to b \to a \to e \to b \to \cdots \quad$ length ω

The length of a rewrite sequence is the number of rewrite steps.

Definitions

- abstract rewrite system (ARS) is set A equipped with binary relation $\to$

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
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rewrite step

  - finite $\quad \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{c} \to \mathsf{f}$ $\quad$ length 4 $\quad a \to^* f$
  - empty $\quad \mathsf{a}$ $\quad$ length 0 $\quad a \to^* a$
  - infinite $\quad \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \mathsf{a} \to \mathsf{e} \to \mathsf{b} \to \cdots$ $\quad$ length ω

The length of a rewrite sequence is the number of rewrite steps.
We write $x \to^* y$ if x rewrites to y in 0 or more steps.

Definition (Derived Relations of $\rightarrow$)

- $\leftarrow$ or $\rightarrow^{-1}$ inverse of $\rightarrow$

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- $\leftarrow$ or $\rightarrow^{-1}$ inverse of $\rightarrow$
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a relation R is

- reflexive if $a \; R \; a$ for all $a \in A$,

Definition (Derived Relations of $\to$)

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a relation R is

- reflexive if $a \mathrel{R} a$ for all $a \in A$,
- transitive if $a \mathrel{R} c$ whenenver $a \mathrel{R} b$ and $b \mathrel{R} c$,

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Definition (Derived Relations of $\to$)

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- ${}^{*}\!\leftarrow$ or $\twoheadleftarrow$ — inverse of $\to^{*}$ (transitive and reflexive closure of $\leftarrow$)
- $\leftrightarrow$ — symmetric closure of $\to$, that is, $\leftrightarrow \, = \, \to \cup \leftarrow$

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- reflexive if $a \; R \; a$ for all $a \in A$,
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Definition (Derived Relations of $\to$)

- $\leftarrow$ or $\to^{-1}$ inverse of $\to$
- $\to^{=}$ reflexive closure of $\to$
- $\to^{+}$ transitive closure of $\to$
- $\to^{*}$ or $\twoheadrightarrow$ transitive and reflexive closure of $\to$
- ${}^{*}\!\leftarrow$ or $\twoheadleftarrow$ inverse of $\to^{*}$ (transitive and reflexive closure of $\leftarrow$)
- $\leftrightarrow$ symmetric closure of $\to$, that is, $\leftrightarrow \,=\, \to \cup \leftarrow$
- $\leftrightarrow^{*}$ conversion (equivalence relation generated by $\to$)
- $\downarrow$ joinability $\downarrow \,=\, \to^{*} \cdot {}^{*}\!\leftarrow$

a relation R is

- reflexive if $a\ R\ a$ for all $a \in A$,
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Definition (Derived Relations of $\to$)

- $\leftarrow$ or $\to^{-1}$ — inverse of $\to$
- $\to^{=}$ — reflexive closure of $\to$
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- $\to^{*}$ or $\twoheadrightarrow$ — transitive and reflexive closure of $\to$
- ${}^{*}\!\leftarrow$ or $\twoheadleftarrow$ — inverse of $\to^{*}$ (transitive and reflexive closure of $\leftarrow$)
- $\leftrightarrow$ — symmetric closure of $\to$, that is, $\leftrightarrow \,=\, \to \cup \leftarrow$
- $\leftrightarrow^{*}$ — conversion (equivalence relation generated by $\to$)
- $\downarrow$ — joinability $\downarrow \,=\, \to^{*} \cdot {}^{*}\!\leftarrow$
- $\uparrow$ — meetability $\uparrow \,=\, {}^{*}\!\leftarrow \cdot \to^{*}$

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Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x

Terminology

- if $x \to^* y$ then x **rewrites** to y and y is **reduct** of x

Example

$$
\begin{tikzcd}
a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & d \\
& e \arrow[u] \arrow[dr] & f \arrow[l] \arrow[d] & h \arrow[u] \\
& & g &
\end{tikzcd}
$$

- $a \to^* f$

Terminology

- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \to^* z \;{}^*\!\leftarrow y$ then z is common reduct of x and y

Example

$$\begin{tikzcd}
\mathsf{a} \arrow[dr] & \mathsf{b} \arrow[l] \arrow[r] & \mathsf{c} \arrow[r] \arrow[d] & \mathsf{d} \\
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\end{tikzcd}$$

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 & & g &
\end{tikzcd}
$$

- $a \to^* f$
- $e \downarrow f$

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a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & d \\
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 & & g &
\end{tikzcd}$$

- $a \to^* f$
- $e \downarrow f \qquad f \downarrow h$

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Example

$$\begin{tikzcd}
\mathrm{a} \arrow[dr] & \mathrm{b} \arrow[l] \arrow[r] & \mathrm{c} \arrow[r] \arrow[d] & \mathrm{d} \\
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\end{tikzcd}
$$

- $a \to^* f$
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- $g \leftrightarrow^* h$

Definition (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y

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$$
\begin{tikzcd}
a \arrow[dr] & b \arrow[l] \arrow[r] & c \arrow[r] \arrow[d] & \color{red}{d} \\
& e \arrow[u] \arrow[dr] & f \arrow[l] \arrow[d] & \\
& & g &
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$$

ARS $\mathcal{A} = \langle A, \to \rangle$

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- d is normal form
- $\mathrm{NF}(\mathcal{A}) = \{\mathsf{d}, \mathsf{g}\}$
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Outline

Definitions

- SN strong normalization or termination
  - no infinite rewrite sequences

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  - every element has (rewrites to) at least one normal form
  - $\forall a \; \exists b \quad a \to^{!} b$

Lemmata

1. SN $\implies$ WN

Lemmata

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Lemmata

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2. $\mathsf{SN} \quad \not\Longleftarrow \quad \mathsf{WN}$

$$\circlearrowright a \longrightarrow b$$

Definitions

- NF   normal form property
  - if an element *a* is convertible with a normal form *b*, then *a* rewrites to *b*

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  - if an element a is convertible with a normal form b, then a rewrites to b
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Definitions

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  - if an element a is convertible with a normal form b, then a rewrites to b
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- UN unique normal forms
  - convertible normal forms are equal

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  - ${}^!{\leftarrow} \cdot \rightarrow^! \ \subseteq \ =$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \;\cancel{\impliedby}\; \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\quad\quad \circlearrowright a \longrightarrow b$
3. $\mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$

Lemmata

1. $\mathrm{SN} \implies \mathrm{WN}$
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$$\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$$

3. $\mathrm{NF} \implies \mathrm{UN} \implies \mathrm{UN}^{\rightarrow}$
4. $\mathrm{UN} \;\cancel{\impliedby}\; \mathrm{UN}^{\rightarrow}$

$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$$

Lemmata

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$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$$
5. $\mathsf{NF} \;\not\Longleftarrow\; \mathsf{UN}$

Lemmata

1. $\mathsf{SN} \implies \mathsf{WN}$
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4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\qquad a \longleftarrow b \longrightarrow \overset{\circlearrowleft}{c} \longleftarrow d \longrightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$

Definition

- CR confluence or Church-Rosser property
  - $\uparrow \subseteq \downarrow$

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  - $\uparrow \subseteq \downarrow$
  - $\forall a, b, c$ $\exists d$

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, two heads] \arrow[dr, two heads] & \\ b \arrow[dr, two heads, dotted] & & c \arrow[dl, two heads, dotted] \\ & d & \end{tikzcd}$$

in diagrams: $\twoheadrightarrow$ for $\to^*$

Definition

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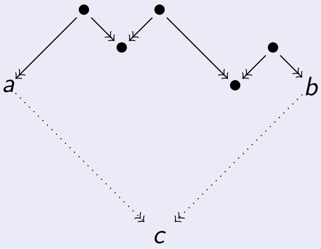

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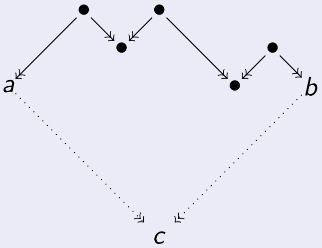

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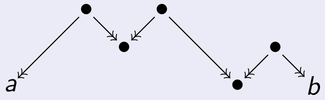

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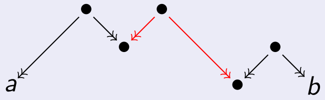

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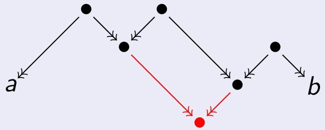

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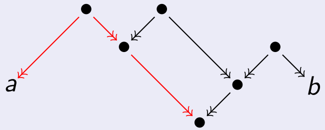

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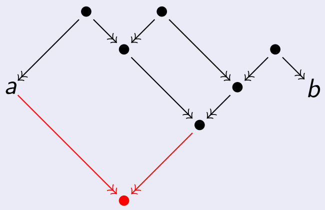

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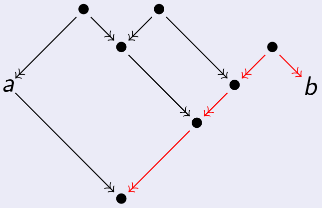

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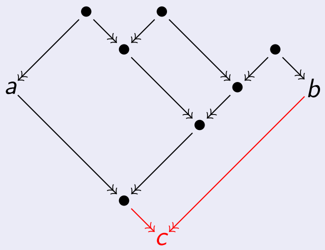

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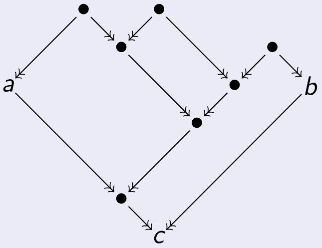

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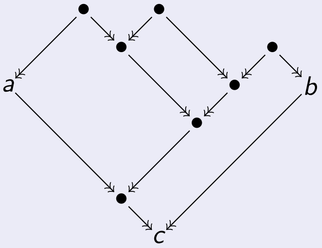

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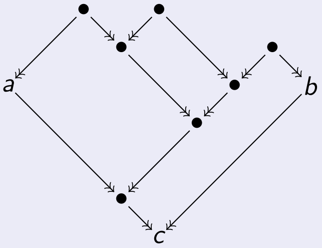

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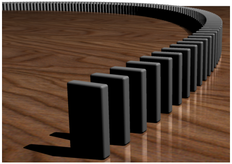
■

Induction

Lemma (Induction)

To prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$ do:

1. *The base case:*
   show that the statement holds for $n = 0$.
2. *The inductive step:*
   show for all n that if the $P(n)$ holds, then also $P(n+1)$ holds.



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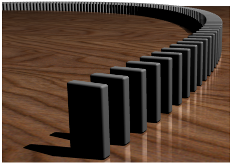

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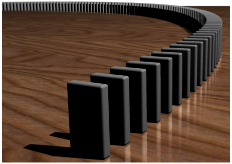

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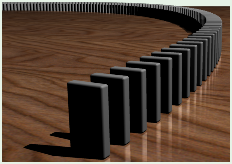

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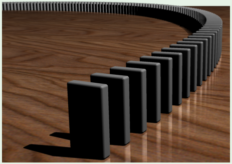

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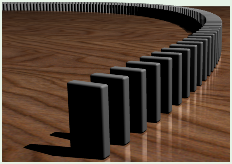

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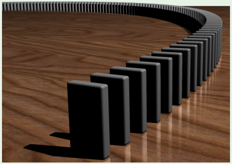

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Then you have proven that all dominoes will fall.

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We use induction to prove that:

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

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Hence the formula holds for all $n \in \mathbb{N}$.

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- Base case $n = 0$: $a\,(\leftarrow^* \cdot \rightarrow^*)^0\, b$. Then $a = b$ and hence $a \rightarrow^* \cdot {}^*\!\leftarrow b$.
- Induction step $n + 1$: (assume it holds for n, show it for $n + 1$)

  Let $a\,(\leftarrow^* \cdot \rightarrow^*)^{n+1}\, b$. Then $a\,(\leftarrow^* \cdot \rightarrow^*)^n\, d \leftarrow^* e \rightarrow^* b$ for some d, e.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

$\Leftarrow$ Assume $\leftrightarrow^* \subseteq \downarrow$. We have $\uparrow \subseteq \leftrightarrow^* \subseteq \downarrow$. Hence $\uparrow \subseteq \downarrow$.

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  - Hence $a \rightarrow^* f \, {}^*\!\leftarrow d$ for some f by induction hypothesis.

Lemma

$$\uparrow \subseteq \downarrow \iff \leftrightarrow^* \subseteq \downarrow$$

Second proof: induction.

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  We conclude $a \rightarrow^* \cdot {}^*\!\leftarrow b$, that is, $a \downarrow b$.

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$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftrightarrow^* \subseteq \downarrow$$

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Hence we have shown $\uparrow^* \subseteq \downarrow$.

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  We conclude $a \rightarrow^* \cdot {}^*\!\leftarrow b$, that is, $a \downarrow b$.

Hence we have shown $\uparrow^* \subseteq \downarrow$. It follows $\leftrightarrow^* \subseteq \downarrow$ since $\leftrightarrow^* \subseteq \uparrow^*$. ∎

Lemma

Confluence $\uparrow \subseteq \downarrow$ *is equivalent to:*

- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$

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- $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$
- $\forall a, b, c$

  $\exists d$

$$\begin{tikzcd} a \arrow[r, two heads] \arrow[d] & b \arrow[d, dotted, two heads] \\ c \arrow[r, dotted, two heads] & d \end{tikzcd}$$

Lemma

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- $\leftarrow \cdot \rightarrow^* \ \subseteq \ \downarrow$
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  $\exists d$

$$\begin{tikzcd} a \arrow[rr, two heads] \arrow[d] & & b \arrow[d, two heads, dotted] \\ c \arrow[rr, two heads, dotted] & & d \end{tikzcd}$$

- $\forall a, b, c. \quad a \rightarrow^* b \ \wedge \ a \rightarrow c \quad \Rightarrow \quad \exists d. \quad b \rightarrow^* d \ \wedge \ c \rightarrow^* d$

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Lemma

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Proof.

Lemma

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Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

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- $\Leftarrow$ Assume $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$. We show $\uparrow \subseteq \downarrow$.

$$\begin{array}{lcl} a & \xrightarrow{\qquad\quad} \!\!\!\!\rightarrow & b \\ \downarrow & & \\ \downarrow & & \\ \downarrow & & \\ \downarrow & & \\ \downarrow & & \\ c & & \end{array}$$

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

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$$\begin{tikzcd}
a \arrow[rr, two heads] \arrow[d] & & b \arrow[d, dashed, two heads] \\
{} \arrow[rr, dashed, two heads] \arrow[d] & & {} \\
{} \arrow[d] & & \\
{} \arrow[d] & & \\
{} \arrow[d] & & \\
c & &
\end{tikzcd}$$

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a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \\
{} \arrow[d] & \\
{} \arrow[d] & \\
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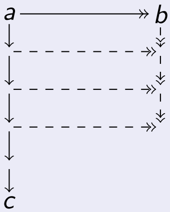

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$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
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a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
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{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
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\end{tikzcd}$$

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By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
c \arrow[r, dashed, two heads] & {}
\end{tikzcd}$$

Lemma

$$\uparrow \subseteq \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

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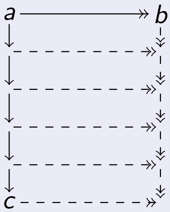

By induction on n we show: ${}^n\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^0\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.

Lemma

$$\uparrow \subseteq \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \subseteq \downarrow$$

Proof.

$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

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$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d] & b \arrow[d, dashed, two heads] \\
{} \arrow[r, dashed, two heads] \arrow[d] & {} \arrow[d, dashed, two heads] \\
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By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
- Induction step $n + 1$:

Lemma

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$\Rightarrow$ Assume $\uparrow \subseteq \downarrow$. We have $\leftarrow \cdot \rightarrow^* \subseteq \uparrow \subseteq \downarrow$. Hence $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$.

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$$\begin{tikzcd} a \arrow[r, ->>] \arrow[d, "n+1"] & b \\ c & \end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow \cdot \rightarrow^* \subseteq \downarrow$ for all n.

- Base case $n = 0$: ${}^{0}\!\leftarrow \cdot \rightarrow^* = \rightarrow^* \subseteq \downarrow$.
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$$\uparrow\ \subseteq\ \downarrow \quad\Longleftrightarrow\quad \leftarrow\cdot\rightarrow^*\ \subseteq\ \downarrow$$

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$\Rightarrow$ Assume $\uparrow\ \subseteq\ \downarrow$. We have $\leftarrow\cdot\rightarrow^*\ \subseteq\ \uparrow\ \subseteq\ \downarrow$. Hence $\leftarrow\cdot\rightarrow^*\ \subseteq\ \downarrow$.

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$$\begin{tikzcd} a \arrow[r, two heads] \arrow[d, "n"] & b \\ d \arrow[d] & \\ c & \end{tikzcd}$$

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  Then $c \leftarrow d\ {}^{n}\!\leftarrow a \rightarrow^* b$ for some d, and:

Lemma

$$\uparrow\ \subseteq\ \downarrow \quad\Longleftrightarrow\quad \leftarrow\cdot\rightarrow^{*}\ \subseteq\ \downarrow$$

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$\Rightarrow$ Assume $\uparrow\ \subseteq\ \downarrow$. We have $\leftarrow\cdot\rightarrow^{*}\ \subseteq\ \uparrow\ \subseteq\ \downarrow$. Hence $\leftarrow\cdot\rightarrow^{*}\ \subseteq\ \downarrow$.

$\Leftarrow$ Assume $\leftarrow\cdot\rightarrow^{*}\ \subseteq\ \downarrow$. We show $\uparrow\ \subseteq\ \downarrow$.

$$\begin{tikzcd}
a \arrow[r, two heads] \arrow[d, "n"] \arrow[dr, phantom, "\mathrm{IH}"] & b \arrow[d, dashed, two heads] \\
d \arrow[r, dashed, two heads] \arrow[d] & e \\
c &
\end{tikzcd}$$

By induction on n we show: ${}^{n}\!\leftarrow\cdot\rightarrow^{*}\ \subseteq\ \downarrow$ for all n.

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  - By induction hypothesis $d \rightarrow^{*} e\ {}^{*}\!\leftarrow b$.

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$$\uparrow \; \subseteq \; \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \; \subseteq \; \downarrow$$

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$$\uparrow \; \subseteq \; \downarrow \quad \Longleftrightarrow \quad \leftarrow \cdot \rightarrow^* \; \subseteq \; \downarrow$$

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  - By induction hypothesis $d \rightarrow^* e \; {}^{*}\!\leftarrow b$.
  - Then $c \rightarrow^* f \; {}^{*}\!\leftarrow e$ since $\leftarrow \cdot \rightarrow^* \; \subseteq \; \downarrow$.

  Hence $c \rightarrow^* \cdot \; {}^{*}\!\leftarrow b$.

$\blacksquare$

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN
$$\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$$
3. CR $\Longrightarrow$ NF $\Longrightarrow$ UN $\Longrightarrow$ $\mathrm{UN}^{\rightarrow}$
4. UN $\not\Longleftarrow$ $\mathrm{UN}^{\rightarrow}$
$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$$
5. NF $\not\Longleftarrow$ UN
$$\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$$

Lemmata

1. $\mathsf{SN} \Longrightarrow \mathsf{WN}$
2. $\mathsf{SN} \not\Longleftarrow \mathsf{WN}$ $\quad \circlearrowright a \longrightarrow b$
3. $\mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \not\Longleftarrow \mathsf{UN}^{\rightarrow}$ $\quad a \leftarrow b \rightarrow \overset{\circlearrowleft}{c} \leftarrow d \rightarrow e$
5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\quad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$

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1. SN $\Longrightarrow$ WN
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6. CR $\not\Longleftarrow$ NF $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$

Lemmata

1. $\mathrm{SN} \implies \mathrm{WN}$
2. $\mathrm{SN} \not\Longleftarrow \mathrm{WN}$ $\qquad$ $\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$
3. $\mathrm{CR} \implies \mathrm{NF} \implies \mathrm{UN} \implies \mathrm{UN}^{\rightarrow}$
4. $\mathrm{UN} \not\Longleftarrow \mathrm{UN}^{\rightarrow}$ $\qquad$ $\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$
5. $\mathrm{NF} \not\Longleftarrow \mathrm{UN}$ $\qquad$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$
6. $\mathrm{CR} \not\Longleftarrow \mathrm{NF}$ $\qquad$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \arrow[loop right] \end{tikzcd}$
7. $\mathrm{CR} \iff {\leftrightarrow^*} \subseteq {\downarrow}$

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4. $\mathrm{UN} \;\cancel{\Longleftarrow}\; \mathrm{UN}^{\rightarrow}$ $\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$
5. $\mathrm{NF} \;\cancel{\Longleftarrow}\; \mathrm{UN}$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$
6. $\mathrm{CR} \;\cancel{\Longleftarrow}\; \mathrm{NF}$ $\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \arrow[loop right] \end{tikzcd}$
7. $\mathrm{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$

Lemmata

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5. $\mathsf{NF} \not\Longleftarrow \mathsf{UN}$ $\quad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \not\Longleftarrow \mathsf{NF}$ $\quad \circlearrowright a \longleftarrow b \longrightarrow c \circlearrowleft$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathsf{WN}\ \&\ \mathsf{UN}^{\rightarrow} \implies \mathsf{CR}$

Definitions

- WCR local confluence or weak Church-Rosser property
    - $\leftarrow \cdot \rightarrow \ \subseteq \ \downarrow$

Definitions

- WCR local confluence or weak Church-Rosser property
  - $\leftarrow \cdot \rightarrow \; \subseteq \; \downarrow$
  - $\forall a, b, c$ $\exists d$

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dotted, two heads] & & c \arrow[dl, dotted, two heads] \\
 & d &
\end{tikzcd}$$

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2. $\mathsf{SN} \;\not\Longleftarrow\; \mathsf{WN}$ $\qquad \circlearrowleft a \longrightarrow b$
3. $\mathsf{CR} \implies \mathsf{NF} \implies \mathsf{UN} \implies \mathsf{UN}^{\rightarrow}$
4. $\mathsf{UN} \;\not\Longleftarrow\; \mathsf{UN}^{\rightarrow}$ $\qquad a \longleftarrow b \longrightarrow \overset{\circlearrowleft}{c} \longleftarrow d \longrightarrow e$
5. $\mathsf{NF} \;\not\Longleftarrow\; \mathsf{UN}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c$
6. $\mathsf{CR} \;\not\Longleftarrow\; \mathsf{NF}$ $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$
7. $\mathsf{CR} \iff \leftrightarrow^* \subseteq \downarrow \iff \leftrightarrow^* = \downarrow \iff \leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. $\mathsf{WN} \;\&\; \mathsf{UN}^{\rightarrow} \implies \mathsf{CR}$
9. $\mathsf{CR} \implies \mathsf{WCR}$

Lemmata

1. SN $\implies$ WN
2. SN $\not\Longleftarrow$ WN $\qquad \circlearrowleft a \longrightarrow b$
3. CR $\implies$ NF $\implies$ UN $\implies$ UN$^{\rightarrow}$
4. UN $\not\Longleftarrow$ UN$^{\rightarrow}$ $\qquad a \leftarrow b \rightarrow \circlearrowleft c \leftarrow d \rightarrow e$
5. NF $\not\Longleftarrow$ UN $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c$
6. CR $\not\Longleftarrow$ NF $\qquad \circlearrowleft a \longleftarrow b \longrightarrow c \circlearrowright$
7. CR $\iff$ $\leftrightarrow^* \subseteq \downarrow$ $\iff$ $\leftrightarrow^* = \downarrow$ $\iff$ $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. WN & UN$^{\rightarrow}$ $\implies$ CR
9. CR $\implies$ WCR
10. CR $\not\Longleftarrow$ WCR

Lemmata

1. SN $\Longrightarrow$ WN
2. SN $\not\Longleftarrow$ WN
$$\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$$
3. CR $\Longrightarrow$ NF $\Longrightarrow$ UN $\Longrightarrow$ $\text{UN}^{\rightarrow}$
4. UN $\not\Longleftarrow$ $\text{UN}^{\rightarrow}$
$$\begin{tikzcd} a & b \arrow[l] \arrow[r] & c \arrow[loop above] & d \arrow[l] \arrow[r] & e \end{tikzcd}$$
5. NF $\not\Longleftarrow$ UN
$$\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \end{tikzcd}$$
6. CR $\not\Longleftarrow$ NF
$$\begin{tikzcd} a \arrow[loop left] & b \arrow[l] \arrow[r] & c \arrow[loop right] \end{tikzcd}$$
7. CR $\Longleftrightarrow$ $\leftrightarrow^* \subseteq \downarrow$ $\Longleftrightarrow$ $\leftrightarrow^* = \downarrow$ $\Longleftrightarrow$ $\leftarrow \cdot \rightarrow^* \subseteq \downarrow$
8. WN & $\text{UN}^{\rightarrow}$ $\Longrightarrow$ CR
9. CR $\Longrightarrow$ WCR
10. CR $\not\Longleftarrow$ WCR
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11. $\mathsf{SN} \;\&\; \mathsf{WCR} \Longrightarrow \mathsf{CR}$

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11. $\mathsf{SN}\ \&\ \mathsf{WCR} \implies \mathsf{CR}$ $\qquad$ Newman's Lemma

Summary

$$
\begin{tikzcd}
\mathrm{SN} \arrow[rrrr] \arrow[ddrr, bend right] & & & & \mathrm{WN} \arrow[ddll, bend left] & \\
& \& & & \& & \\
& & \mathrm{CR} \arrow[ddll, bend left] \arrow[ddr, bend right] & & & \\
& & & & & \\
\mathrm{WCR} \arrow[uurr, bend left] & & & \mathrm{NF} \arrow[r] & \mathrm{UN} \arrow[r] & \mathrm{UN}^{\rightarrow} \arrow[uullll, bend right]
\end{tikzcd}
$$

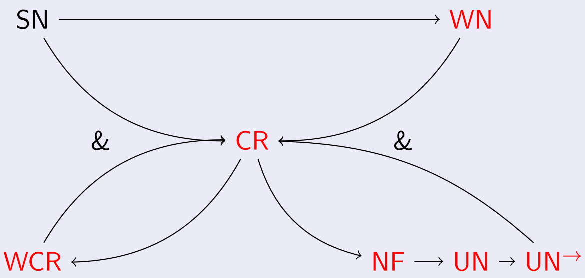
Summary

$$
\begin{tikzcd}
\mathsf{SN} \arrow[rrr] \arrow[dr, bend right] & & & \mathsf{WN} \arrow[dll, bend left] & \\
\& & \mathsf{CR} \arrow[dl, bend left] \arrow[dr, bend right] & & \& \\
\mathsf{WCR} \arrow[ur, bend left] & & \mathsf{NF} \arrow[r] & \mathsf{UN} \arrow[r] & \mathsf{UN}^{\rightarrow} \arrow[ulll, bend right]
\end{tikzcd}
$$

Definitions

- semi-completeness
  - CR & WN

Summary



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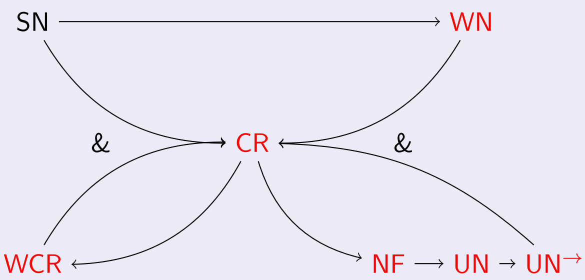

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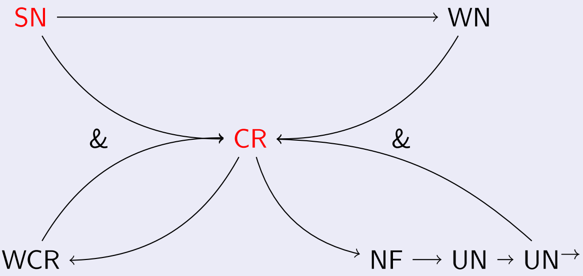

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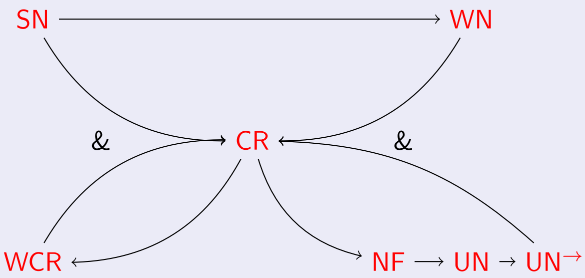

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- diamond property
  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$

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$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\
 & d &
\end{tikzcd}$$

$\exists d$

Definition

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  - $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$
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\end{tikzcd}$$

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b \arrow[dr, dotted] & & c \arrow[dl, dotted] \\
 & d &
\end{tikzcd}$$

Lemma

An ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if $\rightarrow$ has the diamond property.

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Proof.

Exercise. ∎

Lemma

ARS $\mathcal{A} = \langle A, \to_1 \rangle$ *is confluent if*

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$\Longrightarrow \quad {}_1^*\!\leftarrow \cdot \to_1^* \subseteq \to_1^* \cdot {}_1^*\!\leftarrow$ ∎

Outline

Well-Founded Induction

given

- strongly normalizing ARS $\mathcal{A} = \langle A, \to \rangle$

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$$\Big(\forall a\colon \big(\forall b\colon a \to b \implies \mathsf{P}(b)\big) \implies \mathsf{P}(a)\Big) \quad\implies\quad \forall a\colon \mathsf{P}(a)$$

Newman's Lemma

$\mathrm{SN}(\mathcal{A})$ & $\mathrm{WCR}(\mathcal{A}) \quad \Longrightarrow \quad \mathrm{CR}(\mathcal{A})$

Proof.

$$
\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
 & b_1 \arrow[dl] & & c_1 \arrow[dr] & \\
b & & & & c
\end{tikzcd}
$$

$\blacksquare$

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$$

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$$\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
 & b_1 \arrow[dl] \arrow[dr, red] & \color{red}{\mathrm{WCR}} & c_1 \arrow[dl, red] \arrow[dr] & \\
b & & \color{red}{d_1} & & c
\end{tikzcd}$$

∎

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$$\begin{tikzcd}
& & a \arrow[dl, ->>] \arrow[dr, ->>] \arrow[dd, phantom, "\mathrm{WCR}"] & & \\
& b_1 \arrow[dl, ->>] \arrow[dr, ->>] \arrow[dd, phantom, "\color{red}\mathrm{IH}"] & & c_1 \arrow[dl, ->>] \arrow[dr, ->>] & \\
b \arrow[dr, ->>, red] & & d_1 \arrow[dl, ->>, red] & & c \\
& \color{red}d_2 & & &
\end{tikzcd}$$

induction hypothesis $\mathrm{CR}(b_1)$

$\forall a' :\ \text{if } a \to a' \text{ then } \mathrm{CR}(a')$

∎

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$$\begin{tikzcd}
& & a \arrow[dl] \arrow[dr] & & \\
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b \arrow[dr, ->>] & \mathrm{IH} & \color{red}{d_1} \arrow[dl, ->>, red] & & c \\
& \color{red}{d_2} & & &
\end{tikzcd}$$

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b \arrow[dr] & \mathrm{IH} & d_1 \arrow[dl] & & c \\
 & \color{red}{d_2} & & &
\end{tikzcd}
$$

induction hypothesis

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$\blacksquare$

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Proof.

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 & d_2 & & &
\end{tikzcd}$$

induction hypothesis

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$$
\begin{tikzcd}
 & & a \arrow[dl] \arrow[dr] & & \\
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b \arrow[dr, ->>] & \mathrm{IH} & d_1 \arrow[dl, ->>] & & c \arrow[ddll, ->>, red] \\
 & d_2 \arrow[dr, ->>, red] & & \color{red}{\mathrm{IH}} & \\
 & & \color{red}{d_3} & &
\end{tikzcd}
$$

induction hypothesis $\color{red}{\mathrm{CR}(c_1)}$

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Definitions (Properties of Elements)

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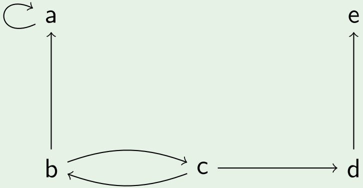
An ARS has the property if all its elements have the respective property.

Example



Example

$$\begin{tikzcd}
\color{red}{a} \arrow[loop left] & & e \\
b \arrow[u] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & d \arrow[u]
\end{tikzcd}$$

Example

CR,WCR

$$\begin{tikzcd}
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\end{tikzcd}$$

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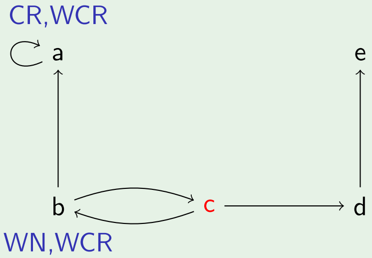

Example

$$\begin{tikzcd}
\text{CR,WCR} & & \\
a \arrow[loop left] & & e \\
b \arrow[u] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & \color{red}{d} \arrow[u] \\
\text{WN,WCR} & \text{WN,WCR} &
\end{tikzcd}$$

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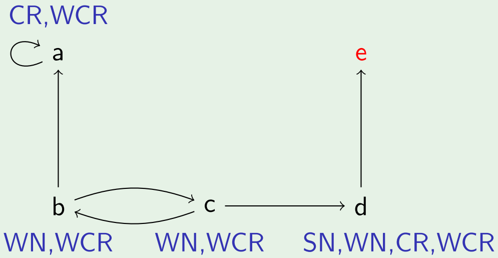

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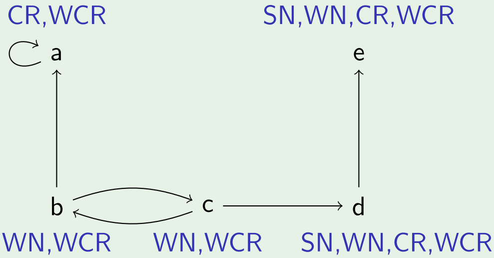

ARSs with Multiple Relations

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$$
\begin{tikzcd}
a \arrow[dr, "1"] & b \arrow[l, "2"'] \arrow[r, "1"] & c \arrow[r, "1"] \arrow[d, "2"] & d \\
 & e \arrow[u, "1"] \arrow[dr, "1"] & f \arrow[l, "3"'] \arrow[d, "1"] & \\
 & & g &
\end{tikzcd}
$$

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& e \arrow[u, "1"] \arrow[dr, "1"] & f \arrow[l, "3"'] \arrow[d, "1"] & \\
& & g &
\end{tikzcd}
$$

ARS $\mathcal{A} = \langle A, \to_1, \to_2, \to_3 \rangle$

- $A = \{a, b, c, d, e, f, g\}$
- $\to_1 = \left\{ \begin{matrix} (a, e), (b, c), (c, d), \\ (e, b), (e, g), (f, g) \end{matrix} \right\}$
- $\to_2 = \{(b, a), (c, f)\}$
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- $\to \; = \; \to_1 \cup \to_2 \cup \to_3$

ARSs with Multiple Relations

Definitions

- abstract rewrite system (ARS) is set A with binary relations $\to_i$ for $i \in \mathcal{I}$

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Definition

Let $\mathcal{A} = \langle A, \to_1, \to_2 \rangle$ be an ARS.

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