Logic and Modelling

— Undecidability and Incompleteness of Predicate Logic —

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Decidability and Undecidability

Decision Problems: Examples

Prime Problem

Determine whether a **number is prime**:

- input: a natural number n
- output: **yes** if n is prime, **no** otherwise

Termination Problem

Decide whether a **program terminates**:

- input: a program P and input w
- output: yes if P started with input w terminates, no else

(Termination means that the program does not run forever.)

Validity Problem

Determine whether a **formula is valid**:

- input: a formula ϕ of predicate logic
- output: yes if ϕ is valid, no otherwise

Decision Problems

A **decision problem** consists of a set I and a predicate $Y \subseteq I$.

Prime Problem

- $I = \mathbb{N}$
- $Y = \{ n \in \mathbb{N} \mid n \text{ is prime} \}$

Termination Problem

- $I =$ set of all pairs $\langle$program, input$\rangle$
- $Y = \{ \langle P, w \rangle \mid P \text{ terminates on input } w \}$

Validity Problem

- $I =$ set of formulas of predicate logic
- $Y =$ set of valid formulas in predicate logic

Can we write a program that on the input of $i \in I$ tells if $i \in Y$?

If such program exists, the predicate Y is called **decidable**.

Decidability

Decidability

A decision problem $Y \subseteq I$ is called **solvable** or **decidable** if there exists a program that tells for every $i \in I$ whether $i \in Y$.

In other words, the program has the following behaviour:

- input: $i \in I$
- output: yes if $i \in Y$
  no if $i \notin Y$

Clearly, the **prime problem** is decidable.

We can write a program deciding whether a number is prime.

Is there such a program for every decision problem?

Can we write a program that decides termination?

Termination Problem (Halting Problem)

Termination

Assume there was a **program** T that decides termination:

- input: a program P and input w
- output: **yes** if P terminates on input w, **no** otherwise

Then using T we can create a program T_{self} :

Then there exists a **program** T_{self} with the behaviour:

- input: a program P
- output: **yes** if P terminates on input P, **no** otherwise

Note that T_{self} is a special case of T where $w = P$.

Programs started on itself: `/bin/cat /bin/cat`

We show that T_{self} does not exist.

Then it follows that T does not exist.

Termination is undecidable!

Assume there would be a **program** T_{self} with the behaviour:

- input: a program P
- output: **yes** if P terminates on input P, **no** otherwise

program T_{self}

```
p = read input;
      ⋮
print result;
```

```
if result = yes
then  loop forever
else  terminate
```

program Z

What happens if we run Z with input Z?

- initial part T_{self} decides whether Z terminates on input Z
- if the result is yes, then Z runs forever ×
- if the result is no, then Z terminates ×

Thus T_{self} has made a mistake! And thus also T!

Termination is Undecidable (Structure of Argument)

Theorem

The **termination problem** is unsolvable (undecidable).

$$
\begin{array}{rcl}
T \text{ outputs yes on input } \langle P, w \rangle & \overset{\times}{\Longleftrightarrow} & P \text{ halts on input } w \\
 & \Downarrow & \\
T_{self} \text{ outputs yes on input } P & \Longleftrightarrow & P \text{ halts on input } P \\
 & \Downarrow & \\
Z \text{ halts on input } P & \Longleftrightarrow & P \text{ does not halt on input } P \\
 & \Downarrow & \\
Z \text{ halts on input } Z & \underset{\times}{\Longleftrightarrow} & Z \text{ does not halt on input } Z
\end{array}
$$

Termination is Undecidable (Structure of Argument)

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\begin{array}{rcl}
Z \text{ halts on input } P & \stackrel{\times}{\Longleftrightarrow} & P \text{ does not halt on input } P \\
 & \Downarrow & \\
Z \text{ halts on input } Z & \underset{\times}{\Longleftrightarrow} & Z \text{ does not halt on input } Z
\end{array}
$$

Compare with **Russell's barber paradox**:

$$
\begin{array}{rcl}
\text{barber shaves } x & \stackrel{\times}{\Longleftrightarrow} & x \text{ does not shave } x \\
 & \Downarrow & \\
\text{barber shaves barber} & \underset{\times}{\Longleftrightarrow} & \text{barber does not shave barber}
\end{array}
$$

Post’s Correspondence Problem

Post's Correspondence Problem

Post Correspondence Problem (PCP)

Given n pairs of words:

$$\langle w_1, v_1 \rangle, \ \ldots, \ \langle w_n, v_n \rangle$$

Are there indices $i_1, i_2 \ldots, i_k$ ($k \geq 1$) s.t.

$$w_{i_1} w_{i_2} \cdots w_{i_k} \quad = \quad v_{i_1} v_{i_2} \cdots v_{i_k} \ \ ?$$

- $\langle 1, 101 \rangle, \langle 10, 00 \rangle, \langle 011, 11 \rangle$ — solution $\langle 1, 3, 2, 3 \rangle$
- $\langle 110, 0 \rangle, \langle 00, 1 \rangle$ — no solution
- $\langle 1, 111 \rangle, \langle 10111, 10 \rangle, \langle 10, 0 \rangle$ — solution $\langle 2, 1, 1, 3 \rangle$
- $\langle 001, 0 \rangle, \langle 01, 011 \rangle, \langle 01, 101 \rangle, \langle 10, 001 \rangle$ — solution length 66

PCP as decision problem

- PCP $= \{ \ \langle \langle w_1, v_1 \rangle, \ldots, \langle w_k, v_k \rangle \rangle \mid k \geq 1, \ w_i, v_i$ bin. words $\}$
- $Y = \{ \ i \in$ PCP $\mid i$ has a solution $\}$

PCP is Undecidable

Theorem

Post's Correspondence Problem is undecidable.

Idea of the Proof.

The termination problem can be **reduced to** PCP.

More precisely, there is a computable function r that maps instances of the termination problem to instances of PCP:

$$r : \quad \langle P, w \rangle \quad \longmapsto \quad I_{\langle P, w \rangle}$$

such that it holds:

$$P \text{ terminates on input } w \quad \Longleftrightarrow \quad I_{\langle P, w \rangle} \text{ has a solution}$$

Then if we had a **PCP-solver** (decides solvability of PCP-inst.) we would obtain a solver for the termination problem. $\times$ $\square$

Meta-Theorems of Predicate Logic (continued)

Validity is Undecidable

Theorem

The **validity problem** in predicate logic is undecidable.

There cannot be a program that, given any formula ϕ, decides whether or not $\models \phi$ holds.

Proof structure.

PCP can be **encoded into** (**reduced to**) the validity problem.

We will describe a computable function r that maps instances of PCP to instances of the validity problem:

$$r : \; I \;\longmapsto\; \phi_I$$

such that it holds:

$$I \text{ has a solution} \iff \models \phi_I \quad (\text{i.e. } \phi_I \text{ is valid})$$

Then if we had a program **deciding validity** for predicate logic, we would obtain a PCP-solver. $\times$

Encoding of Binary Words as Terms

The formula ϕ_I will be defined over functions and predicate in:

$$\mathcal{F} = \{\ e/0,\ f_0/1, f_1/1\ \} \qquad\qquad \mathcal{P} = \{\ P/2\ \}$$

We consider the encoding of binary strings into $\mathcal{F}$-terms:

binary word	term encoding
ϵ (empty word)	e
0	$f_0(e)$
1	$f_1(e)$
01	$f_0(f_1(e))$
10	$f_1(f_0(e))$
$\vdots$	$\vdots$
$b_1 b_2 \ldots b_{l-1} b_l$	$f_{b_1}(f_{b_2}(\ldots f_{b_{l-1}}(f_{b_l}(e)) \ldots))$

For binary word $w = b_1 b_2 \ldots b_l$ and $\mathcal{F}$-terms t we abbreviate:

$$f_w(t) \stackrel{\text{def}}{=} f_{b_1}(f_{b_2}(\ldots f_{b_l}(t) \ldots))$$

Encoding a PCP-Instance into a Formula

Now given a PCP instance

$$I = \langle \langle w_1, v_1 \rangle, \langle w_2, v_2 \rangle, \ldots, \langle w_k, v_k \rangle \rangle$$

we encode I into a formula ϕ_I of predicate logic as follows:

$$\phi_I \stackrel{\text{def}}{=} \phi_1 \wedge \phi_2 \to \phi_3$$

$$\phi_1 \stackrel{\text{def}}{=} \bigwedge_{i=1,\ldots,k} P(f_{w_i}(e), f_{v_i}(e))$$

$$\phi_2 \stackrel{\text{def}}{=} \forall x \forall y \big(P(x, y) \to \bigwedge_{i=1,\ldots,k} P(f_{w_i}(x), f_{v_i}(y)) \big)$$

$$\phi_3 \stackrel{\text{def}}{=} \exists z \, P(z, z)$$

The idea behind P is:

$P(x, y) \iff \langle x, y \rangle$ can be constructed using dominos in I

Encoding a PCP-Instance into a Formula

To understand the formulas we consider the model $\mathcal{M}$:

- domain $B =$ set of binary words
- $e^{\mathcal{M}} = \epsilon$ (empty word)
- ${f_0}^{\mathcal{M}}(w) = 0w$
- ${f_1}^{\mathcal{M}}(w) = 1w$
- $P^{\mathcal{M}} = \{ \langle x, y \rangle \mid$ there are indices $\langle i_1, i_2, \dots, i_n \rangle$ such that $x = w_{i_1} w_{i_2} \cdots w_{i_n}$ and $y = v_{i_1} v_{i_2} \cdots v_{i_n} \}$

For all $w \in B$ it holds: $(f_w(e))^{\mathcal{M}} = w$.

Encoding a PCP-Instance into a Formula

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For all $w \in B$ it holds: $(f_w(e))^{\mathcal{M}} = w$.

Interpreting $\phi_1 \stackrel{\text{def}}{=} \bigwedge_{i=1,\ldots,k} P(f_{w_i}(e), f_{v_i}(e))$ in $\mathcal{M}$ we find:

$$\begin{aligned}
\mathcal{M} \models_\ell \phi_1 &\iff \forall i \text{ with } 1 \leq i \leq k\text{: } \mathcal{M} \models_\ell P(f_{w_i}(e), f_{v_i}(e)) \\
&\iff \forall i \text{ with } 1 \leq i \leq k\text{: } \langle (f_{w_i}(e))^{\mathcal{M}}, (f_{v_i}(e))^{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\
&\iff \forall i \text{ with } 1 \leq i \leq k\text{: } \langle w_i, v_i \rangle \in P^{\mathcal{M}} \quad \checkmark
\end{aligned}$$

Encoding a PCP-Instance into a Formula

To understand the formulas we consider the model $\mathcal{M}$:

- domain B = set of binary words
- $e^{\mathcal{M}} = \epsilon$ (empty word)
- $f_0{}^{\mathcal{M}}(w) = 0w$
- $f_1{}^{\mathcal{M}}(w) = 1w$
- $P^{\mathcal{M}} = \{ \langle x, y \rangle \mid$ there are indices $\langle i_1, i_2, \ldots, i_n \rangle$ such that $x = w_{i_1} w_{i_2} \cdots w_{i_n}$ and $y = v_{i_1} v_{i_2} \cdots v_{i_n} \}$

For all $w \in B$ it holds: $(f_w(e))^{\mathcal{M}} = w$.

Interpreting $\phi_2 \stackrel{\text{def}}{=} \forall x \forall y \big(P(x, y) \to \bigwedge_{i=1,\ldots,k} P(f_{w_i}(x), f_{v_i}(y)) \big)$:

$$\mathcal{M} \vDash_\ell \phi_2 \iff \ldots$$

$$\iff \Big(\forall w, v \in B \colon \langle w, v \rangle \in P^{\mathcal{M}} \Longrightarrow \bigwedge_{i=1,\ldots,k} \langle w_i w, v_i v \rangle \in P^{\mathcal{M}} \Big)$$

$$\iff \Big(\langle w_{i_1} \cdots w_{i_n}, v_{i_1} \cdots v_{i_n} \rangle \in P^{\mathcal{M}} \Longrightarrow \langle w_i w_{i_1} \cdots w_{i_n}, v_i v_{i_1} \cdots v_{i_n} \rangle \in P^{\mathcal{M}_l} \Big) \checkmark$$

Encoding a PCP-Instance into a Formula

To understand the formulas we consider the model $\mathcal{M}$:

- domain B = set of binary words
- $e^{\mathcal{M}} = \epsilon$ (empty word)
- $f_0{}^{\mathcal{M}}(w) = 0w$
- $f_1{}^{\mathcal{M}}(w) = 1w$
- $P^{\mathcal{M}} = \{ \langle x, y \rangle \mid$ there are indices $\langle i_1, i_2, \ldots, i_n \rangle$ such that $x = w_{i_1} w_{i_2} \cdots w_{i_n}$ and $y = v_{i_1} v_{i_2} \cdots v_{i_n} \}$

For all $w \in B$ it holds: $(f_w(e))^{\mathcal{M}} = w$.

Interpreting $\phi_3 \overset{\text{def}}{=} \exists z\, P(z, z)$ in $\mathcal{M}$ we find:

$$
\begin{aligned}
\mathcal{M} \vDash_{\ell} \phi_3 \quad &\Longleftrightarrow \quad \text{there is a } w \in B \text{ such that } \langle w, w \rangle \in P^{\mathcal{M}} \\
&\Longleftrightarrow \quad \text{there is a sequence } \langle i_1, i_2, \ldots, i_n \rangle \text{ of indices} \\
&\qquad\quad \text{such that } w_{i_1} w_{i_2} \cdots w_{i_n} = v_{i_1} v_{i_2} \cdots v_{i_n} \\
&\Longleftrightarrow \quad \text{PCP instance } I \text{ is solvable}
\end{aligned}
$$

Encoding a PCP-Instance into a Formula

So we have shown:

$$\begin{aligned} \mathcal{M} &\models \phi_1 \\ \mathcal{M} &\models \phi_2 \\ \mathcal{M} &\models \phi_3 \iff I \text{ is solvable} \end{aligned}$$

Hence we conclude:

$$\begin{aligned} \mathcal{M} \models \phi_I &\iff \mathcal{M} \models \phi_1 \wedge \phi_2 \rightarrow \phi_3 \\ &\iff I \text{ is solvable} \end{aligned}$$

and furthermore:

$$\begin{aligned} \phi_I \text{ is valid} &\implies \mathcal{M} \models \phi_I \\ &\implies I \text{ is solvable} \end{aligned}$$

Validity is Undecidable

Theorem

The **validity problem** in predicate logic is undecidable.

Proof structure.

There is a computable function r (see previous slides) that maps instances of PCP to instances of the validity problem:

$$r : \ I \ \longmapsto \ \phi_I$$

such that it holds:

$$I \text{ has a solution} \iff \ \models \phi_I \quad (\text{i.e. } \phi_I \text{ is valid}) \qquad (\star)$$

Then if we had a program **deciding validity** for predicate logic, we would obtain a PCP-solver. $\times$

'$\Longleftarrow$' in $(\star)$: just shown ✓

'$\Longrightarrow$' in $(\star)$: see Huth & Ryan p. 134, 135

Alan Turing

Alan Mathison Turing (1912–1954)

Undecidability of Validity and Provability

Theorem (Church, Turing, 1936/37)

The **validity problem** in predicate logic is undecidable.

There cannot be a program that, given any formula ϕ, decides whether or not $\models \phi$ holds.

Using the soundness and completeness theorem we obtain:

Corollary (Undecidability of Provability)

The **provability problem** in predicate logic is undecidable.

There cannot be a program that, given any formula ϕ, decides whether or not $\vdash \phi$ holds.

- limits the power of theorem provers
- building better theorem provers is an open-ended endeavour (creativity will always be needed)

Also Satisfiability is Undecidable

Proposition

For sentences ϕ it holds:

$$\begin{aligned} \phi \text{ is unsatisfiable} &\iff \neg\phi \text{ is valid} \\ \phi \text{ is satisfiable} &\iff \neg\phi \text{ is not valid} \end{aligned}$$

Since this defines an easy reduction of the validity problem to the satisfiability problem. It follows immediately:

Theorem

The **satisfiability problem** in predicate logic is undecidable.

Undecidability of $\models$ and $\vdash$

Deduction Theorem

Easy connection between $\models$ and validity:

$$\begin{aligned} \phi_1,\phi_2,\ldots,\phi_n \models \psi \quad &\Longleftrightarrow \quad \models \phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \to \psi \\ &\Longleftrightarrow \quad \phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \to \psi \quad \text{is valid} \end{aligned}$$

and between $\vdash$ and provability:

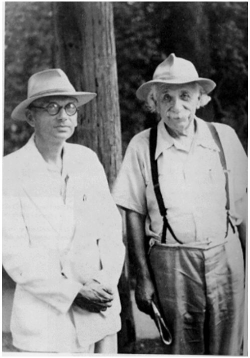
$$\begin{aligned} \phi_1,\phi_2,\ldots,\phi_n \vdash \psi \quad &\Longleftrightarrow \quad \vdash \phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \to \psi \\ &\Longleftrightarrow \quad \phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \to \psi \quad \text{is provable} \end{aligned}$$

Corollary (Undecidability of entailment relations $\models$ and $\vdash$)

The relations $\models$ and $\vdash$ in predicate logic are undecidable.

There cannot be a program that, given formulas $\phi_1, \ldots, \phi_n, \psi$ decides whether or not $\phi_1,\ldots,\phi_n \models \psi$ (or $\phi_1,\ldots,\phi_n \vdash \psi$).

Kurt Gödel



Kurt Gödel with Albert Einstein (Princeton, around 1950)

Incompleteness Theorem

We consider the sets of function and predicate symbols:

$$\mathcal{F} = \{\, 0,\ S, +, \cdot \,\} \qquad \mathcal{P} = \{\, < \,\}$$

with as intended model **number theory $\mathcal{N}$**:

- domain of $\mathcal{N}$ is $\mathbb{N}$, the natural numbers (with 0)
- $0^{\mathcal{N}} = 0$
- $S^{\mathcal{N}}(n) = n + 1$
- $+^{\mathcal{N}}(n, m) = n + m$
- $\cdot^{\mathcal{N}}(n, m) = n \cdot m$
- $<^{\mathcal{N}} = \{\, \langle n, m \rangle \mid n, m \in \mathbb{N} \text{ such that } n < m \,\}$

Example formula: $\forall n.\ \forall m.\ \exists o.\ (n < m + o)$

One would like to have a

complete theory (**deduction system**) $\vdash$ for $\mathcal{N}$

that allows to derive all formulas that are true in $\mathcal{N}$.

First Incompleteness Theorem

Desirable: an **axiomatizable first-order theory** $\vdash$ for $\mathcal{N}$:

- an effective list $\mathcal{A}$ of axioms (pred. logic $\langle \mathcal{F}, \mathcal{P} \rangle$-formulas)
- finitely many rules $\mathcal{R}$
- $\vdash \phi$ means: there is a derivation of ϕ from axioms $\mathcal{A}$ that only used rules from $\mathcal{R}$

that is sound and complete for $\mathcal{N}$:

$$\mathcal{N} \models \phi \iff \vdash \phi \qquad \text{(for all } \langle \mathcal{F}, \mathcal{P} \rangle\text{-formulas } \phi)$$

Yet it turned out that this is impossible.

First incompleteness theorem (Gödel, 1931)

Every axiomatizable and sound theory $\vdash$ of first-order logic for number theory with language $\langle \mathcal{F}, \mathcal{P} \rangle$ is incomplete. That is, it contains sentences ϕ that are true in $\mathcal{N}$, but unprovable in $\vdash$:

$$\mathcal{N} \models \phi, \quad \text{yet} \quad \nvdash \phi$$

Second Incompleteness Theorem

Theorem (Gödel, von Neumann, 1930-31)

For every axiomatizable theory $\vdash$ of first-order logic for number theory with language $\langle \mathcal{F}, \mathcal{P} \rangle$

- that is rich enough to express its own consistency by a sentence $\phi_{\vdash}$

it holds that either:

- $\vdash \bot$ ($\vdash$ is inconsistent) , or
- $\nvdash \phi_{\vdash}$ (hence $\vdash$ is incomplete)

$\Longrightarrow$ First-order theories (based on predicate logic) of number theory are not able to prove their own consistency.

Timeline: From Logic to Computability

1900	Hilbert's 23 Problems in mathematics
1921	Schönfinkel: Combinatory logic
1928	Hilbert/Ackermann: formulate completeness/decision problems for the predicate calculus (the latter called 'Entscheidungsproblem')
1929	Presburger: completeness/decidability of theory of addition on $\mathbb{Z}$
1930	Gödel: completeness theorem of predicate calculus
1931	Gödel: incompleteness theorems for first-order arithmetic
1932	Church: λ-calculus
1933/34	Herbrand/Gödel: general recursive functions
1936	Church/Kleene: λ-definable $\sim$ general recursive Church Thesis: 'effectively calculable' be defined as either Church shows: the 'Entscheidungsproblem' is unsolvable
1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'