Logic and Modelling

— Undecidability and Incompleteness of Predicate Logic —

Jörg Endrullis

VU University Amsterdam

Decidability and Undecidability

Decision Problems: Examples

Prime Problem

Determine whether a **number is prime**:

- input: a natural number n
- output: **yes** if n is prime, **no** otherwise

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(Termination means that the program does not run forever.)

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Validity Problem

Determine whether a **formula is valid**:

- input: a formula ϕ of predicate logic
- output: yes if ϕ is valid, no otherwise

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- $I =$ set of all pairs $\langle$program, input$\rangle$
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Can we write a program that on the input of $i \in I$ tells if $i \in Y$?

If such program exists, the predicate Y is called **decidable**.

Decidability

Decidability

A decision problem $Y \subseteq I$ is called **solvable** or **decidable** if there exists a program that tells for every $i \in I$ whether $i \in Y$.

In other words, the program has the following behaviour:

- input: $i \in I$
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Is there such a program for every decision problem?

Can we write a program that decides termination?

Termination Problem (Halting Problem)

Termination

Assume there was a **program** T that decides termination:

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- output: **yes** if P terminates on input w, **no** otherwise

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Note that T_{self} is a special case of T where $w = P$.

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Programs started on itself: `/bin/cat /bin/cat`

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We show that T_{self} does not exist.

Then it follows that T does not exist.

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- if the result is yes, then Z runs forever

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$$\left.\begin{array}{l}\text{program } T_{self}\ \{\ \texttt{p = read input; } \vdots \texttt{ print result;} \\ \texttt{if result = yes then loop forever else terminate}\end{array}\right\}\text{program } Z$$

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Thus T_{self} has made a mistake!

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Termination is undecidable!

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The **termination problem** is unsolvable (undecidable).

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Compare with **Russell's barber paradox**:

$$
\begin{array}{rcl}
\text{barber shaves } x & \Longleftrightarrow & x \text{ does not shave } x \\
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\text{barber shaves barber} & \Longleftrightarrow & \text{barber does not shave barber}
\end{array}
$$

Post’s Correspondence Problem

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Post Correspondence Problem (PCP)

Given n pairs of words:

$$\langle w_1, v_1 \rangle, \ \ldots, \ \langle w_n, v_n \rangle$$

Are there indices $i_1, i_2 \ldots, i_k$ ($k \geq 1$) s.t.

$$w_{i_1} w_{i_2} \cdots w_{i_k} \quad = \quad v_{i_1} v_{i_2} \cdots v_{i_k} \quad ?$$

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- $\langle 1, 101 \rangle, \langle 10, 00 \rangle, \langle 011, 11 \rangle$

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- $\langle 110, 0 \rangle, \langle 00, 1 \rangle$

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- $\langle 1, 111\rangle, \langle 10111, 10\rangle, \langle 10, 0\rangle$ — solution $\langle 2, 1, 1, 3\rangle$

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PCP as decision problem

- PCP $= \{ \ \langle \langle w_1, v_1 \rangle, \ldots, \langle w_k, v_k \rangle \rangle \mid k \geq 1, \ w_i, v_i$ bin. words $\}$
- $Y = \{ \ i \in$ PCP $\mid i$ has a solution $\}$

PCP is Undecidable

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Idea of the Proof.

The termination problem can be **reduced to** PCP.

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Idea of the Proof.

The termination problem can be **reduced to** PCP.

More precisely, there is a computable function r that maps instances of the termination problem to instances of PCP:

$$r: \quad \langle P, w \rangle \quad \longmapsto \quad I_{\langle P, w \rangle}$$

such that it holds:

$$P \text{ terminates on input } w \quad \Longleftrightarrow \quad I_{\langle P, w \rangle} \text{ has a solution}$$

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The termination problem can be **reduced to** PCP.

More precisely, there is a computable function r that maps instances of the termination problem to instances of PCP:

$$r : \quad \langle P, w \rangle \quad \longmapsto \quad I_{\langle P, w \rangle}$$

such that it holds:

$$P \text{ terminates on input } w \quad \Longleftrightarrow \quad I_{\langle P, w \rangle} \text{ has a solution}$$

Then if we had a **PCP-solver** (decides solvability of PCP-inst.) we would obtain a solver for the termination problem. $\times$ $\square$

Meta-Theorems of Predicate Logic (continued)

Validity is Undecidable

Theorem

The **validity problem** in predicate logic is undecidable.

There cannot be a program that, given any formula ϕ, decides whether or not $\models \phi$ holds.

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We will describe a computable function r that maps instances of PCP to instances of the validity problem:

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Then if we had a program **deciding validity** for predicate logic, we would obtain a PCP-solver. $\times$

Encoding of Binary Words as Terms

The formula ϕ_I will be defined over functions and predicate in:

$$\mathcal{F} = \{ \, e/0, \; f_0/1, f_1/1 \, \} \qquad\qquad \mathcal{P} = \{ \, P/2 \, \}$$

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We consider the encoding of binary strings into $\mathcal{F}$-terms:

binary word	term encoding
ϵ (empty word)	e
0	$f_0(e)$
1	$f_1(e)$
01	$f_0(f_1(e))$
10	$f_1(f_0(e))$
$\vdots$	$\vdots$
$b_1 b_2 \dots b_{l-1} b_l$	$f_{b_1}(f_{b_2}(\dots f_{b_{l-1}}(f_{b_l}(e)) \dots))$

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For binary word $w = b_1 b_2 \ldots b_l$ and $\mathcal{F}$-terms t we abbreviate:

$$f_w(t) \stackrel{\text{def}}{=} f_{b_1}(f_{b_2}(\ldots f_{b_l}(t)\ldots))$$

Encoding a PCP-Instance into a Formula

Now given a PCP instance

$$I = \langle \langle w_1, v_1 \rangle, \langle w_2, v_2 \rangle, \ldots, \langle w_k, v_k \rangle \rangle$$

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The idea behind P is:

$P(x, y) \iff \langle x, y \rangle$ can be constructed using dominos in I

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Interpreting $\phi_2 \stackrel{\text{def}}{=} \forall x \forall y \big(P(x, y) \to \bigwedge_{i=1,\ldots,k} P(f_{w_i}(x), f_{v_i}(y)) \big)$:

$$\mathcal{M} \models_\ell \phi_2 \iff \ldots$$

$$\iff \Big(\forall w, v \in B : \langle w, v \rangle \in P^{\mathcal{M}} \implies \bigwedge_{i=1,\ldots,k} \langle w_i w, v_i v \rangle \in P^{\mathcal{M}} \Big)$$

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Interpreting $\phi_3 \stackrel{\text{def}}{=} \exists z\, P(z, z)$ in $\mathcal{M}$ we find:

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- domain B = set of binary words
- $e^{\mathcal{M}} = \epsilon$ (empty word)
- ${f_0}^{\mathcal{M}}(w) = 0w$
- ${f_1}^{\mathcal{M}}(w) = 1w$
- $P^{\mathcal{M}} = \{ \langle x, y \rangle \mid$ there are indices $\langle i_1, i_2, \ldots, i_n \rangle$ such that $x = w_{i_1} w_{i_2} \cdots w_{i_n}$ and $y = v_{i_1} v_{i_2} \cdots v_{i_n} \}$

For all $w \in B$ it holds: $(f_w(e))^{\mathcal{M}} = w$.

Interpreting $\phi_3 \overset{\text{def}}{=} \exists z\, P(z, z)$ in $\mathcal{M}$ we find:

$\mathcal{M} \models_{\ell} \phi_3$

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$$\iff \text{PCP instance } I \text{ is solvable}$$

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Validity is Undecidable

Theorem

The **validity problem** in predicate logic is undecidable.

Proof structure.

There is a computable function r (see previous slides) that maps instances of PCP to instances of the validity problem:

$$r : \; I \;\longmapsto\; \phi_I$$

such that it holds:

$$I \text{ has a solution} \iff \models \phi_I \quad (\text{i.e. } \phi_I \text{ is valid}) \qquad (\star)$$

Then if we had a program **deciding validity** for predicate logic, we would obtain a PCP-solver. $\times$

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'$\Longrightarrow$' in $(\star)$: see Huth & Ryan p. 134, 135

Alan Turing

Alan Mathison Turing (1912–1954)

Undecidability of Validity and Provability

Theorem (Church, Turing, 1936/37)

The **validity problem** in predicate logic is undecidable.

There cannot be a program that, given any formula ϕ, decides whether or not $\models \phi$ holds.

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Using the soundness and completeness theorem we obtain:

Corollary (Undecidability of Provability)

The **provability problem** in predicate logic is undecidable.

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- limits the power of theorem provers
- building better theorem provers is an open-ended endeavour (creativity will always be needed)

Also Satisfiability is Undecidable

Proposition

For sentences ϕ it holds:

$$\phi \text{ is unsatisfiable} \iff \neg\phi \text{ is valid}$$

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Also Satisfiability is Undecidable

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For sentences ϕ it holds:

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Since this defines an easy reduction of the validity problem to the satisfiability problem. It follows immediately:

Theorem

The **satisfiability problem** in predicate logic is undecidable.

Undecidability of $\models$ and $\vdash$

Deduction Theorem

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Easy connection between $\models$ and validity:

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Easy connection between $\models$ and validity:

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$$\phi_1, \phi_2, \ldots, \phi_n \models \psi \iff \models \phi_1 \land \phi_2 \land \ldots \land \phi_n \to \psi$$

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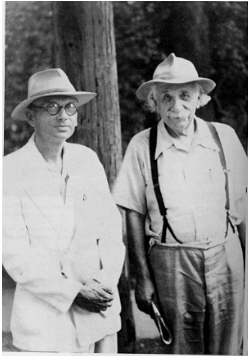
Corollary (Undecidability of entailment relations $\models$ and $\vdash$)

The relations $\models$ and $\vdash$ in predicate logic are undecidable.

There cannot be a program that, given formulas $\phi_1, \ldots, \phi_n, \psi$ decides whether or not $\phi_1, \ldots, \phi_n \models \psi$ (or $\phi_1, \ldots, \phi_n \vdash \psi$).

Kurt Gödel



Kurt Gödel with Albert Einstein (Princeton, around 1950)

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We consider the sets of function and predicate symbols:

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One would like to have a

complete theory (**deduction system**) $\vdash$ for $\mathcal{N}$

that allows to derive all formulas that are true in $\mathcal{N}$.

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Yet it turned out that this is impossible.

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First incompleteness theorem (Gödel, 1931)

Every axiomatizable and sound theory $\vdash$ of first-order logic for number theory with language $\langle \mathcal{F}, \mathcal{P} \rangle$ is incomplete. That is, it contains sentences ϕ that are true in $\mathcal{N}$, but unprovable in $\vdash$:

$$\mathcal{N} \vDash \phi, \quad \text{yet} \quad \nvdash \phi$$

Second Incompleteness Theorem

Theorem (Gödel, von Neumann, 1930-31)

For every axiomatizable theory $\vdash$ of first-order logic for number theory with language $\langle \mathcal{F}, \mathcal{P} \rangle$

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it holds that either:

- $\vdash \bot$ ($\vdash$ is inconsistent) , or
- $\nvdash \phi_{\vdash}$ (hence $\vdash$ is incomplete)

$\Longrightarrow$ First-order theories (based on predicate logic) of number theory are not able to prove their own consistency.

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1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'