Logic and Modelling

— Meta-Theorems of Predicate Logic —

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Meta-Theorems of Predicate Logic

Entailment Syntactically $\vdash$ and Semantically $\models$

$\vdash$ and $\models$ for arbitrary sets Γ of formulas

- $\Gamma \vdash \psi$: there is an natural deduction derivation of ψ that only uses premises in Γ
- $\Gamma \models \psi$: for all $\mathcal{M}$ and ℓ:
  $\forall \phi (\phi \in \Gamma \implies \mathcal{M} \models_{\ell} \phi) \implies \mathcal{M} \models_{\ell} \psi$

(Here Γ can be an infinite set of formulas.)

Soundness and Completeness Theorem

Theorem (Soundness and Completeness)

The syntactic version $\vdash$ and the semantical version $\models$ of logical entailment **coincide**:

$$\Gamma \vdash \phi \iff \Gamma \models \phi$$

holds for all formulas ϕ, and every set Γ of formulas.

Two directions:

$$\begin{aligned} \Longrightarrow &: \text{ correctness (soundness)} \\ \Longleftarrow &: \text{ completeness} \end{aligned}$$

In a symbolic abbreviation:

$$\vdash \; = \; \models$$

Correctness Theorem

Theorem (Correctness of $\vdash$ with respect to $\models$)

For all formulas ϕ, and every set Γ of formulas:

$$\Gamma \vdash \phi \quad \Longrightarrow \quad \Gamma \models \phi$$

Reformulation and explanation:

If there is a natural deduction derivation of ϕ from Γ, then there is no model $\mathcal{M}$ in which all formulas of Γ are true, but ϕ is false.

The derivation rules are **correct** in this sense: It is not possible to derive a false conclusion ϕ from true premises Γ.

Truth in a model is preserved under making deductions.

Proof of the theorem: by **induction on derivation lengths**.

Completeness Theorem

Theorem (Completeness of $\vdash$ with respect to $\models$)

For all formulas ϕ, and every set Γ of formulas:

$$\Gamma \models \phi \quad \Longrightarrow \quad \Gamma \vdash \phi$$

Reformulation and explanation:

The derivations rules are **strong enough** to derive **all valid semantic entailment statements.**

Thus **more** derivation rules are **not necessary**,
and in this sense these rules are **complete.**

Proof of the theorem: non-trivial (first one by Kurt Gödel, 1930)

Kurt Gödel

Kurt Gödel (1906 – 1978)

Consistency and Syntactical Consistency

We already know a **semantical** notion of consistency.

Definition (reminder)

A set Γ of formulas is **consistent** (also: **satisfiable**) if there is a model $\mathcal{M}$ and an environment ℓ such that $\mathcal{M} \models \phi$ for all $\phi \in \Gamma$:

$$\exists \mathcal{M}\ \exists \ell\ \ \forall \phi \in \Gamma\ (\ \mathcal{M} \models_{\ell} \phi\ )$$

or equivalently

$$\exists \mathcal{M} \exists \ell\ \ \mathcal{M} \models_{\ell} \Gamma$$

There is also a **syntactical** variant:

Definition

A set Γ of formulas is **syntactically consistent** if:

$$\Gamma \nvdash \bot$$

(That is, there is no derivation of $\bot$ from Γ.)

Consistency Theorem

Theorem (Consistency)

For every set Γ of formulas it holds:

Γ is **consistent** $\iff$ Γ is **syntactically consistent** $(\star)$

Reformulating this equivalence $(\star)$:

$$\exists \mathcal{M} \exists \ell \;\; \mathcal{M} \vDash_\ell \Gamma \quad \iff \quad \Gamma \nvdash \bot$$

Γ has a model $\iff$ there is no derivation of $\bot$ from Γ

The proof will utilise $\vDash \; = \; \vdash$ (soundness and completeness).

Towards the Proof of the Consistency Theorem

For all models $\mathcal{M}$ and all environments ℓ :

$$\mathcal{M} \models_\ell \top \qquad\qquad \mathcal{M} \not\models_\ell \bot$$

In other words:

- $\top$ is true in every model
- $\bot$ is not true in any model

Consistency Theorem (Proof)

Theorem (Consistency)

$$\exists \mathcal{M} \exists \ell \;\; \mathcal{M} \models_{\ell} \Gamma \quad \Longleftrightarrow \quad \Gamma \nvdash \bot$$

Proof of $\Rightarrow$

- Let $\mathcal{M}$ and ℓ such that $\mathcal{M} \models_{\ell} \Gamma$.
- Note that $\mathcal{M} \not\models_{\ell} \bot$.
- We conclude $\Gamma \not\models \bot$.
- Then by the soundness theorem: $\Gamma \nvdash \bot$.

Proof of $\Leftarrow$

- Suppose that $\Gamma \nvdash \bot$.
- Then by the completeness theorem: $\Gamma \not\models \bot$.
- Hence there are $\mathcal{M}$ and ℓ such that $\mathcal{M} \models_{\ell} \Gamma$ and $\mathcal{M} \not\models_{\ell} \bot$.
- Hence Γ is consistent.

Towards the Compactness Theorem

$\vdash$ and $\vDash$ for arbitrary sets Γ of formulas

- $\Gamma \vdash \psi$: there is an natural deduction derivation of ψ that only uses premises in Γ
- $\Gamma \vDash \psi$: for all $\mathcal{M}$ and ℓ:
  $\forall\phi(\,\phi \in \Gamma \implies \mathcal{M} \vDash_\ell \phi\,) \implies \mathcal{M} \vDash_\ell \psi$

(Here Γ can be an infinite set of formulas.)

Note: Every natural deduction derivation must be finite.
Hence it can use only finitely many of the premises in Γ.

Compactness Theorem

Theorem (Compactness)

For every set Γ of formulas it holds:

$$\Gamma \text{ is consistent} \iff \text{every finite subset } \Gamma_0 \subseteq \Gamma \text{ is consistent}$$

Proof of $\Rightarrow$

Every model for Γ also is a model for all subsets $\Gamma_0 \subseteq \Gamma$.

Proof of $\Leftarrow$ (by transposition)

- Suppose that Γ is **not** consistent. Then it follows: $\Gamma \models \bot$.
- Then by the completeness theorem: $\Gamma \vdash \bot$.
- Hence there is a derivation $\mathcal{D}$ of $\bot$ from premises in Γ.
- As $\mathcal{D}$ is finite, it can only use finitely many premises $\Gamma_0 \subseteq \Gamma$.
- Then $\mathcal{D}$ also shows $\Gamma_0 \vdash \bot$.
- By the soundness theorem it follows: $\Gamma_0 \models \bot$.
- Hence the finite subset Γ_0 of Γ is **not** consistent.

Definability and Undefinability Results

(Expressive Power of Predicate Logic)

Expressible Frame Properties

Definable frame properties in predicate logic and modal logic:

property	**predicate logic** with $=$	**modal logic**
reflexivity	$\forall x\, R(x, x)$	$\Box p \to p$
symmetry	$\forall x \forall y (R(x, y) \to R(y, x))$	$p \to \Box\Diamond p$
anti-symmetry	$\forall x \forall y ((R(x, y) \land R(y, x) \to x = y)$	×
$\lvert\text{frame}\rvert \geq n$	ϕ_n (see later)	×
$\lvert\text{frame}\rvert \leq n$	ψ_n (see later)	×
every world has ≥ 2 successors	✓	×
McKinsey formula	×	$\Box\Diamond p \to \Diamond\Box p$

Definability and Undefinability Results for Model Cardinality

Reminder: Model Cardinality

Define ϕ_n for $n \in \mathbb{N}$ with $n \geq 2$:

$$\phi_n \;=\; \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

Define ψ_n for $n \in \mathbb{N}$ with $n \geq 1$:

$$\psi_n \;=\; \forall x_1 \ldots \forall x_{n+1} \bigvee_{1 \leq i < j \leq n} x_i = x_j$$

Proposition

For all models $\mathcal{M}$ and all $n \geq 2$ it holds:

(i) $\mathcal{M} \models \phi_n \iff A$ has at least n elements (i.e. $|A| \geq n$)

(ii) $\mathcal{M} \models \psi_n \iff A$ has at most n elements (i.e. $|A| \leq n$)

(iii) $\mathcal{M} \models \phi_n \wedge \psi_n \iff A$ has precisely n elements ($|A| = n$)

Model Infiniteness is Definable by Set of Formulas

Proposition

There is a set Δ of sentences such that for all $\mathcal{M}$:

$$\mathcal{M} \models \Delta \iff \mathcal{M} \text{ has an infinite domain}$$

Proof.

Let $\Delta = \{ \phi_2, \phi_3, \phi_4, \dots \}$ where ϕ_n express 'at least n values'.

Then it holds for all models $\mathcal{M}$:

$$\begin{aligned} \mathcal{M} \models \Delta &\iff \mathcal{M} \models \phi_n \text{ for all } n \geq 2 \\ &\iff \mathcal{M} \text{ has at least } n \text{ values, for all } n \in \mathbb{N} \\ &\iff \mathcal{M} \text{ has infinitely many values} \\ &\iff \mathcal{M} \text{ has an infinite domain} \end{aligned}$$

$\square$

Model Finiteness is Undefinable (Single Formula)

Theorem (Finiteness is Undefinable)

There is no sentence ψ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \psi \iff \mathcal{M} \text{ has a finite domain} \qquad (\star)$$

Proof

(A) Suppose sentence ψ expresses finiteness in the sense $(\star)$.

Consider the set $\Delta = \{ \psi \} \cup \{ \phi_2, \phi_3, \phi_4, \ldots \}$.

(B) The set Δ is **inconsistent**:

- $\mathcal{M} \models \psi \iff \mathcal{M}$ is finite
- $\mathcal{M} \models \{ \phi_2, \phi_3, \phi_4, \ldots \} \iff \mathcal{M}$ is infinite

Model Finiteness is Undefinable (Single Formula)

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$$\mathcal{M} \models \psi \iff \mathcal{M} \text{ has a finite domain} \qquad (\star)$$

Proof

(A) Suppose sentence ψ expresses finiteness in the sense $(\star)$.

Consider the set $\Delta = \{\psi\} \cup \{\phi_2, \phi_3, \phi_4, \dots\}$.

(B) The set Δ is **inconsistent**.

(C) Yet every finite $\Delta_0 \subseteq \Delta$ is **consistent**:

- Let n_0 be the largest number such that $\phi_{n_0} \in \Delta_0$.
- Take a model $\mathcal{M}$ with n_0 elements.
- Then $\mathcal{M} \models \phi_n$ for all $\phi_n \in \Delta_0$, and $\mathcal{M} \models \psi$.

Model Finiteness is Undefinable (Single Formula)

Theorem (Finiteness is Undefinable)

There is no sentence ψ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \psi \iff \mathcal{M} \text{ has a finite domain} \qquad (\star)$$

Proof

(A) Suppose sentence ψ expresses finiteness in the sense $(\star)$.

Consider the set $\Delta = \{\psi\} \cup \{\phi_2, \phi_3, \phi_4, \dots\}$.

(B) The set Δ is **inconsistent.**

(C) Yet every finite $\Delta_0 \subseteq \Delta$ is **consistent.**

A set Δ with (B) and (C) contradicts the compactness theorem.

The problem must be assumption (A).

Hence **there cannot be such a formula** ψ.

Model Finiteness is Undefinable (Set of Formulas)

Theorem (Finiteness is Undefinable)

There is no set of formulas Γ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \Gamma \iff \mathcal{M} \text{ has a finite domain} \qquad (\star)$$

Proof

(A) Suppose a set Γ expresses finiteness in the sense $(\star)$.

Consider the set $\Delta = \Gamma \cup \{ \phi_2, \phi_3, \phi_4, \ldots \}$.

(B) The set Δ is **inconsistent**:

- $\mathcal{M} \models \Gamma \iff \mathcal{M}$ is finite
- $\mathcal{M} \models \{ \phi_2, \phi_3, \phi_4, \ldots \} \iff \mathcal{M}$ is infinite

Model Finiteness is Undefinable (Set of Formulas)

Theorem (Finiteness is Undefinable)

There is no set of formulas Γ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \Gamma \iff \mathcal{M} \text{ has a finite domain} \qquad (\star)$$

Proof

(A) Suppose a set Γ expresses finiteness in the sense $(\star)$.

Consider the set $\Delta = \Gamma \cup \{\, \phi_2, \phi_3, \phi_4, \ldots \,\}$.

(B) The set Δ is **inconsistent**.

(C) Yet every finite $\Delta_0 \subseteq \Delta$ is **consistent**:

- Let n_0 be the largest number such that $\phi_{n_0} \in \Delta_0$.
- Take a model $\mathcal{M}$ with n_0 elements.
- Then $\mathcal{M} \models \phi_n$ for all $\phi_n \in \Delta_0$, and $\mathcal{M} \models \Gamma$. Thus $\mathcal{M} \models \Delta_0$.

Model Finiteness is Undefinable (Set of Formulas)

Theorem (Finiteness is Undefinable)

There is no set of formulas Γ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \Gamma \iff \mathcal{M} \text{ has a finite domain} \qquad (\star)$$

Proof

(A) Suppose a set Γ expresses finiteness in the sense $(\star)$.

Consider the set $\Delta = \Gamma \cup \{ \phi_2, \phi_3, \phi_4, \dots \}$.

(B) The set Δ is **inconsistent.**

(C) Yet every finite $\Delta_0 \subseteq \Delta$ is **consistent.**

A set Δ with (B) and (C) contradicts the compactness theorem.

The problem must be assumption (A).

Hence **there cannot be such a set of formulas** Γ.

Model Infiniteness is Undefinable by a Single Formula

Corollary

There is no sentence ψ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \psi \iff \mathcal{M} \text{ has an infinite domain} \qquad (\star)$$

Proof.

Suppose that there exists a sentence ψ such that $(\star)$ holds.

Then we find that $\neg\psi$ actually defines model finiteness:

$$\begin{aligned} \mathcal{M} \models \neg\psi &\iff \text{not: } \mathcal{M} \models \psi \\ &\iff \text{not: } \mathcal{M} \text{ has an infinite domain} \\ &\iff \mathcal{M} \text{ has a finite domain} \end{aligned}$$

We know that such sentence does not exist.

We conclude that the assumption was wrong. □

(Un)Definability Results: Current Overview

property	**definable by a sentence**	**definable by a set of sentences**
at least	✓	✓
at most	✓	✓
finiteness	×	×
infiniteness	×	✓
reachability		
unreachability		

Definability and Undefinability Results for Reachability

Reachability via R

For a binary predicate symbol R we want to express:

v is reachable via R from u

Thinking of R as arrows, this means: there is a path from v to u.

Example

If we consider the relation R :

$$R(x, y) : \quad x \text{ is child of } y$$

Then reachability via R is the relation:

'x is descendant of, or is the same person as y'

Reachable in n Steps

We search for formulas χ_n that express reachability in n steps:

$$\begin{aligned}
\chi_0 &: \quad u = v \\
\chi_1 &: \quad R(u, v) \\
\chi_2 &: \quad \exists x_1 (R(u, x_1) \wedge R(x_1, v)) \\
\chi_3 &: \quad \exists x_1 \exists x_2 (R(u, x_1) \wedge R(x_1, x_2) \wedge R(x_2, v)) \\
\vdots & \qquad \vdots \\
\chi_n &: \quad \exists x_1 \exists x_2 \ldots \exists x_{n-1} (R(u, x_1) \wedge R(x_1, x_2) \wedge \ldots \wedge R(x_{n-1}, v))
\end{aligned}$$

We work with constants c, d: we write $\chi(c, d)$ for $\chi[c/u][d/v]$.

$\chi_2(c, d)$ denotes the formula $\exists x_1 (R(c, x_1) \wedge R(x_1, d))$.

Theorem

It holds for all models $\mathcal{M}$:

$$\mathcal{M} \models \chi_n(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } n\ R^{\mathcal{M}}\text{-steps}$$

Reachability is Undefinable (Proof)

There is **no** sentence $\chi(c, d)$ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \chi(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \quad (\star)$$

(1) Suppose that there exists $\chi(c, d)$ with $(\star)$.

(2) Then we consider the set Δ of formulas:

$$\Delta = \{\, \chi(c, d) \,\} \cup \{\, \neg\chi_0(c, d), \neg\chi_1(c, d), \neg\chi_2(c, d), \ldots \,\}$$

(3) The set Δ is **inconsistent**:

- If $\mathcal{M} \models \chi(c, d)$, there are R-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$, say m.
- Hence $\mathcal{M} \models \chi_m(c, d)$, and so $\mathcal{M} \not\models \neg\chi_m(c, d)$.
- Consequently $\mathcal{M} \not\models \Delta$.

Reachability is Undefinable (Proof)

There is **no** sentence $\chi(c, d)$ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \chi(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \quad (\star)$$

(1) Suppose that there exists $\chi(c, d)$ with $(\star)$.

(2) Then we consider the set Δ of formulas:

$$\Delta = \{\, \chi(c, d) \,\} \cup \{\, \neg\chi_0(c, d), \neg\chi_1(c, d), \neg\chi_2(c, d), \dots \,\}$$

(3) The set Δ is **inconsistent.**

(4) Yet every finite $\Delta_0 \subseteq \Delta$ is **consistent**:
- Let $\Delta_0 \subseteq \Delta$ be finite.
- Then $\neg\chi_{n_0}(c, d) \notin \Delta_0$ for some n_0.
- Let $\mathcal{M}$ a model with exactly n_0 $R^{\mathcal{M}}$-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$.
- Then $\mathcal{M} \models \chi(c, d)$ and $\mathcal{M} \models \neg\chi_n(c, d)$ for all $n \neq n_0$.
- Hence $\mathcal{M} \models \Delta_0$, and we conclude that Δ_0 is consistent.

Reachability is Undefinable (Proof)

There is no sentence $\chi(c,d)$ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \chi(c,d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \quad (\star)$$

(1) Suppose that there exists $\chi(c,d)$ with $(\star)$.

(2) Then we consider the set Δ of formulas:

$$\Delta = \{\, \chi(c,d) \,\} \cup \{\, \neg\chi_0(c,d), \neg\chi_1(c,d), \neg\chi_2(c,d), \ldots \,\}$$

(3) The set Δ is **inconsistent.**

(4) Yet every finite $\Delta_0 \subseteq \Delta$ is **consistent.**

(5) A set Δ of formulas with (3) and (4) contradicts the compactness theorem.

(6) The problem must be assumption (1).

So we conclude that there cannot be such a formula χ.

Definability Results for Reachability/Unreachability

Proposition

Let R be a binary relation symbol.

1. In predicate logic, **reachability** by R-steps is:
   - not definable by a sentence.
   - not definable by a set of sentences.
2. In predicate logic, **unreachability** by R-steps is:
   - not definable by a single sentence.
   - **definable** by a set of sentences.

Proof.

Similar to definability and undefinability results (see earlier) for finiteness and infiniteness. □

(Un)Definability Results: Overview

property	definable by a sentence	definable by a set of sentences
at least	✓	✓
at most	✓	✓
finiteness	×	×
infiniteness	×	✓
reachability	×	×
unreachability	×	✓