Logic and Modelling

— Meta-Theorems of Predicate Logic —

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Meta-Theorems of Predicate Logic

Entailment Syntactically $\vdash$ and Semantically $\models$

$\vdash$ and $\models$ for arbitrary sets Γ of formulas

- $\Gamma \vdash \psi$: there is an natural deduction derivation of ψ that only uses premises in Γ
- $\Gamma \models \psi$: for all $\mathcal{M}$ and ℓ:
  $\forall\phi(\,\phi \in \Gamma \implies \mathcal{M} \models_{\ell} \phi\,) \implies \mathcal{M} \models_{\ell} \psi$

(Here Γ can be an infinite set of formulas.)

Soundness and Completeness Theorem

Theorem (Soundness and Completeness)

The syntactic version $\vdash$ and the semantical version $\models$ of logical entailment **coincide**:

$$\Gamma \vdash \phi \iff \Gamma \models \phi$$

holds for all formulas ϕ, and every set Γ of formulas.

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In a symbolic abbreviation:

$$\vdash \; = \; \models$$

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If there is a natural deduction derivation of ϕ from Γ, then there is no model $\mathcal{M}$ in which all formulas of Γ are true, but ϕ is false.

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Proof of the theorem: by **induction on derivation lengths**.

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The derivations rules are **strong enough** to derive **all valid semantic entailment statements**.

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Proof of the theorem: non-trivial (first one by Kurt Gödel, 1930)

Kurt Gödel

Kurt Gödel (1906 – 1978)

Consistency and Syntactical Consistency

We already know a **semantical** notion of consistency.

Definition (reminder)

A set Γ of formulas is **consistent** (also: **satisfiable**) if there is a model $\mathcal{M}$ and an environment ℓ such that $\mathcal{M} \vDash \phi$ for all $\phi \in \Gamma$:

$$\exists \mathcal{M}\ \exists \ell\ \ \forall \phi \in \Gamma\ (\ \mathcal{M} \vDash_{\ell} \phi\ )$$

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There is also a **syntactical** variant:

Definition

A set Γ of formulas is **syntactically consistent** if:

$$\Gamma \nvdash \bot$$

(That is, there is no derivation of $\bot$ from Γ.)

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Theorem (Consistency)

For every set Γ of formulas it holds:

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The proof will utilise $\vDash \; = \; \vdash$ (soundness and completeness).

Towards the Proof of the Consistency Theorem

For all models $\mathcal{M}$ and all environments ℓ:

$$\mathcal{M} \models_\ell \top \qquad\qquad \mathcal{M} \not\models_\ell \bot$$

In other words:

- $\top$ is true in every model
- $\bot$ is not true in any model

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- Let $\mathcal{M}$ and ℓ such that $\mathcal{M} \models_\ell \Gamma$.

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- We conclude $\Gamma \not\models \bot$.

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- Then by the soundness theorem: $\Gamma \nvdash \bot$.

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- Suppose that $\Gamma \nvdash \bot$.

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Proof of $\Leftarrow$

- Suppose that $\Gamma \nvdash \bot$.
- Then by the completeness theorem: $\Gamma \not\models \bot$.
- Hence there are $\mathcal{M}$ and ℓ such that $\mathcal{M} \models_{\ell} \Gamma$ and $\mathcal{M} \not\models_{\ell} \bot$.

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- We conclude $\Gamma \not\models \perp$.
- Then by the soundness theorem: $\Gamma \nvdash \perp$.

Proof of $\Leftarrow$

- Suppose that $\Gamma \nvdash \perp$.
- Then by the completeness theorem: $\Gamma \not\models \perp$.
- Hence there are $\mathcal{M}$ and ℓ such that $\mathcal{M} \models_{\ell} \Gamma$ and $\mathcal{M} \not\models_{\ell} \perp$.
- Hence Γ is consistent.

Towards the Compactness Theorem

$\vdash$ and $\models$ for arbitrary sets Γ of formulas

- $\Gamma \vdash \psi$: there is an natural deduction derivation of ψ that only uses premises in Γ
- $\Gamma \models \psi$: for all $\mathcal{M}$ and ℓ:
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(Here Γ can be an infinite set of formulas.)

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Hence it can use only finitely many of the premises in Γ.

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- As $\mathcal{D}$ is finite, it can only use finitely many premises $\Gamma_0 \subseteq \Gamma$.
- Then $\mathcal{D}$ also shows $\Gamma_0 \vdash \bot$.
- By the soundness theorem it follows: $\Gamma_0 \models \bot$.
- Hence the finite subset Γ_0 of Γ is **not** consistent.

Definability and Undefinability Results

(Expressive Power of Predicate Logic)

Expressible Frame Properties

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$\lvert\text{frame}\rvert \geq n$	ϕ_n (see later)	$\times$
$\lvert\text{frame}\rvert \leq n$	ψ_n (see later)	

Expressible Frame Properties

Definable frame properties in predicate logic and modal logic:

property	**predicate logic** with $=$	**modal logic**
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Definability and Undefinability Results for Model Cardinality

Reminder: Model Cardinality

Define ϕ_n for $n \in \mathbb{N}$ with $n \geq 2$:

$$\phi_n \;=\; \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

Define ψ_n for $n \in \mathbb{N}$ with $n \geq 1$:

$$\psi_n \;=\; \forall x_1 \ldots \forall x_{n+1} \bigvee_{1 \leq i < j \leq n} x_i = x_j$$

Proposition

For all models $\mathcal{M}$ and all $n \geq 2$ it holds:

(i) $\mathcal{M} \models \phi_n \iff$ A has at least n elements (i.e. $|A| \geq n$)

(ii) $\mathcal{M} \models \psi_n \iff$ A has at most n elements (i.e. $|A| \leq n$)

(iii) $\mathcal{M} \models \phi_n \wedge \psi_n \iff$ A has precisely n elements ($|A| = n$)

Model Infiniteness is Definable by Set of Formulas

Proposition

There is a set Δ of sentences such that for all $\mathcal{M}$:

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Let $\Delta = \{ \phi_2, \phi_3, \phi_4, \ldots \}$ where ϕ_n express 'at least n values'.

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Then it holds for all models $\mathcal{M}$:

$$\mathcal{M} \models \Delta$$

□

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Hence **there cannot be such a formula** ψ.

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Theorem (Finiteness is Undefinable)

There is no set of formulas Γ such that for all $\mathcal{M}$:

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- Then $\mathcal{M} \models \phi_n$ for all $\phi_n \in \Delta_0$, and $\mathcal{M} \models \Gamma$. Thus $\mathcal{M} \models \Delta_0$.

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Corollary

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Corollary

There is no sentence ψ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \psi \quad \Longleftrightarrow \quad \mathcal{M} \text{ has an infinite domain} \qquad (\star)$$

Proof.

Suppose that there exists a sentence ψ such that $(\star)$ holds.

Model Infiniteness is Undefinable by a Single Formula

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There is no sentence ψ such that for all $\mathcal{M}$:

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Proof.

Suppose that there exists a sentence ψ such that $(\star)$ holds.

Then we find that $\neg\psi$ actually defines model finiteness:

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Proof.

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Then we find that $\neg\psi$ actually defines model finiteness:

$$\mathcal{M} \models \neg\psi$$

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There is no sentence ψ such that for all $\mathcal{M}$:

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Suppose that there exists a sentence ψ such that $(\star)$ holds.

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$$\mathcal{M} \models \neg\psi \iff \text{not: } \mathcal{M} \models \psi$$

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We know that such sentence does not exist.

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Suppose that there exists a sentence ψ such that $(\star)$ holds.

Then we find that $\neg\psi$ actually defines model finiteness:

$$\begin{aligned} \mathcal{M} \models \neg\psi &\iff \text{not: } \mathcal{M} \models \psi \\ &\iff \text{not: } \mathcal{M} \text{ has an infinite domain} \\ &\iff \mathcal{M} \text{ has a finite domain} \end{aligned}$$

We know that such sentence does not exist.

We conclude that the assumption was wrong. □

(Un)Definability Results: Current Overview

property	definable by a sentence	definable by a set of sentences
at least	✓	✓
at most	✓	✓
finiteness	×	×
infiniteness	×	✓
reachability		
unreachability		

Definability and Undefinability Results for Reachability

Reachability via R

For a binary predicate symbol R we want to express:

v is reachable via R from u

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If we consider the relation R :

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Reachability via R

For a binary predicate symbol R we want to express:

v is reachable via R from u

Thinking of R as arrows, this means: there is a path from v to u.

Example

If we consider the relation R :

$$R(x, y) : \quad x \text{ is child of } y$$

Then reachability via R is the relation:

'x is descendant of, or is the same person as y'

Reachable in n Steps

We search for formulas χ_n that express reachability in n steps:

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\chi_1 &: \quad R(u, v) \\
\chi_2 &: \quad \exists x_1 (R(u, x_1) \wedge R(x_1, v)) \\
\chi_3 &: \quad \exists x_1 \exists x_2 (R(u, x_1) \wedge R(x_1, x_2) \wedge R(x_2, v))
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\vdots & & \vdots \\
\chi_n & : & \exists x_1 \exists x_2 \ldots \exists x_{n-1} (R(u, x_1) \land R(x_1, x_2) \land \ldots \land R(x_{n-1}, v))
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\end{array}$$

We work with constants c, d: we write $\chi(c, d)$ for $\chi[c/u][d/v]$.

$\chi_2(c, d)$ denotes the formula $\exists x_1 (R(c, x_1) \wedge R(x_1, d))$.

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Theorem

It holds for all models $\mathcal{M}$:

$$\mathcal{M} \models \chi_n(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } n\ R^{\mathcal{M}}\text{-steps}$$

Reachability is Undefinable (Proof)

There is no sentence $\chi(c, d)$ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \chi(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \quad (\star)$$

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(1) Suppose that there exists $\chi(c, d)$ with $(\star)$.

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(3) The set Δ is **inconsistent**:

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$$\mathcal{M} \vDash \chi(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \qquad (\star)$$

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- Hence $\mathcal{M} \models \chi_m(c, d)$,

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- If $\mathcal{M} \models \chi(c, d)$, there are R-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$, say m.
- Hence $\mathcal{M} \models \chi_m(c, d)$, and so $\mathcal{M} \not\models \neg\chi_m(c, d)$.

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- Consequently $\mathcal{M} \not\models \Delta$.

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   - Then $\neg\chi_{n_0}(c, d) \notin \Delta_0$ for some n_0.

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  - Let $\Delta_0 \subseteq \Delta$ be finite.
  - Then $\neg\chi_{n_0}(c, d) \notin \Delta_0$ for some n_0.
  - Let $\mathcal{M}$ a model with exactly n_0 $R^{\mathcal{M}}$-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$.
  - Then $\mathcal{M} \models \chi(c, d)$

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   - Let $\mathcal{M}$ a model with exactly n_0 $R^{\mathcal{M}}$-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$.
   - Then $\mathcal{M} \models \chi(c, d)$ and $\mathcal{M} \models \neg\chi_n(c, d)$ for all $n \neq n_0$.

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There is no sentence $\chi(c,d)$ such that for all $\mathcal{M}$:

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- Let $\mathcal{M}$ a model with exactly n_0 $R^{\mathcal{M}}$-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$.
- Then $\mathcal{M} \models \chi(c,d)$ and $\mathcal{M} \models \neg\chi_n(c,d)$ for all $n \neq n_0$.
- Hence $\mathcal{M} \models \Delta_0$,

Reachability is Undefinable (Proof)

There is no sentence $\chi(c, d)$ such that for all $\mathcal{M}$:

$$\mathcal{M} \vDash \chi(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \quad (\star)$$

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- Let $\mathcal{M}$ a model with exactly n_0 $R^{\mathcal{M}}$-steps from $c^{\mathcal{M}}$ to $d^{\mathcal{M}}$.
- Then $\mathcal{M} \vDash \chi(c, d)$ and $\mathcal{M} \vDash \neg\chi_n(c, d)$ for all $n \neq n_0$.
- Hence $\mathcal{M} \vDash \Delta_0$, and we conclude that Δ_0 is consistent.

Reachability is Undefinable (Proof)

There is no sentence $\chi(c, d)$ such that for all $\mathcal{M}$:

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(5) A set Δ of formulas with (3) and (4) contradicts the compactness theorem.

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(6) The problem must be assumption (1).

So we conclude that there cannot be such a formula χ.

Definability Results for Reachability/Unreachability

Proposition

Let R be a binary relation symbol.

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1. In predicate logic, **reachability** by R-steps is:

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   - not definable by a sentence.
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Definability Results for Reachability/Unreachability

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Let R be a binary relation symbol.

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   - not definable by a sentence.
   - not definable by a set of sentences.
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Definability Results for Reachability/Unreachability

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Let R be a binary relation symbol.

1. In predicate logic, **reachability** by R-steps is:
   - not definable by a sentence.
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2. In predicate logic, **unreachability** by R-steps is:
   - not definable by a single sentence.
   - **definable** by a set of sentences.

Definability Results for Reachability/Unreachability

Proposition

Let R be a binary relation symbol.

1. In predicate logic, **reachability** by R-steps is:
   - not definable by a sentence.
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2. In predicate logic, **unreachability** by R-steps is:
   - not definable by a single sentence.
   - **definable** by a set of sentences.

Proof.

Similar to definability and undefinability results (see earlier) for finiteness and infiniteness. $\square$

(Un)Definability Results: Overview

property	definable by a sentence	definable by a set of sentences
at least	✓	✓
at most	✓	✓
finiteness	×	×
infiniteness	×	✓
reachability	×	×
unreachability	×	✓