

Logic and Modelling

— Meta-Theorems of Predicate Logic —

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Meta-Theorems of Predicate Logic

Entailment Syntactically \vdash and Semantically \models

\vdash and \models for arbitrary sets Γ of formulas

- ▶ $\Gamma \vdash \psi$: there is a natural deduction derivation of ψ that only uses premises in Γ
- ▶ $\Gamma \models \psi$: for all \mathcal{M} and ℓ :
$$\forall \phi (\phi \in \Gamma \implies \mathcal{M} \models_{\ell} \phi) \implies \mathcal{M} \models_{\ell} \psi$$

(Here Γ can be an infinite set of formulas.)

Soundness and Completeness Theorem

Theorem (Soundness and Completeness)

The syntactic version \vdash and the semantical version \models of logical entailment **coincide**:

$$\Gamma \vdash \phi \iff \Gamma \models \phi$$

holds for all formulas ϕ , and every set Γ of formulas.

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\implies : correctness (soundness)

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In a symbolic abbreviation:

$$\vdash = \models$$

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Reformulation and explanation:

If there is a natural deduction derivation of ϕ from Γ , then there is no model \mathcal{M} in which all formulas of Γ are true, but ϕ is false.

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Proof of the theorem: by **induction on derivation lengths**.

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The derivations rules are **strong enough** to derive **all valid semantic entailment statements**.

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Thus **more** derivation rules are **not necessary**, and in this sense these rules are **complete**.

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Proof of the theorem: non-trivial (first one by Kurt Gödel, 1930)

Kurt Gödel



Kurt Gödel (1906–1978)

Consistency and Syntactical Consistency

We already know a **semantical** notion of consistency.

Definition (reminder)

A set Γ of formulas is **consistent** (also: **satisfiable**) if there is a model \mathcal{M} and an environment ℓ such that $\mathcal{M} \models \phi$ for all $\phi \in \Gamma$:

$$\exists \mathcal{M} \exists \ell \forall \phi \in \Gamma (\mathcal{M} \models_{\ell} \phi)$$

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There is also a **syntactical** variant:

Definition

A set Γ of formulas is **syntactically consistent** if:

$$\Gamma \not\vdash \perp$$

(That is, there is no derivation of \perp from Γ .)

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Theorem (Consistency)

For every set Γ of formulas it holds:

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Reformulating this equivalence (\star) :

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The proof will utilise $\models = \vdash$ (soundness and completeness).

Towards the Proof of the Consistency Theorem

For all models \mathcal{M} and all environments ℓ :

$$\mathcal{M} \models_{\ell} \top$$

$$\mathcal{M} \not\models_{\ell} \perp$$

In other words:

- ▶ \top is true in every model
- ▶ \perp is not true in any model

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- ▶ Then by the soundness theorem: $\Gamma \not\models \perp$.

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- ▶ Suppose that $\Gamma \not\vdash \perp$.

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- ▶ Suppose that $\Gamma \not\vdash \perp$.
- ▶ Then by the completeness theorem: $\Gamma \not\models \perp$.
- ▶ Hence there are \mathcal{M} and ℓ such that $\mathcal{M} \models_{\ell} \Gamma$ and $\mathcal{M} \not\models_{\ell} \perp$.

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- ▶ Suppose that $\Gamma \not\vdash \perp$.
- ▶ Then by the completeness theorem: $\Gamma \not\models \perp$.
- ▶ Hence there are \mathcal{M} and ℓ such that $\mathcal{M} \models_{\ell} \Gamma$ and $\mathcal{M} \not\models_{\ell} \perp$.
- ▶ Hence Γ is consistent.

Towards the Compactness Theorem

\vdash and \models for arbitrary sets Γ of formulas

- ▶ $\Gamma \vdash \psi$: there is a natural deduction derivation of ψ that only uses premises in Γ
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Note: Every natural deduction derivation must be finite.

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Note: Every natural deduction derivation must be finite. Hence it can use only finitely many of the premises in Γ .

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- ▶ As \mathcal{D} is finite, it can only use finitely many premises $\Gamma_0 \subseteq \Gamma$.
- ▶ Then \mathcal{D} also shows $\Gamma_0 \vdash \perp$.
- ▶ By the soundness theorem it follows: $\Gamma_0 \vDash \perp$.
- ▶ Hence the finite subset Γ_0 of Γ is **not** consistent.

Definability and Undefinability Results

(Expressive Power of Predicate Logic)

Expressible Frame Properties

Definable frame properties in predicate logic and modal logic:

property	predicate logic with =	modal logic
reflexivity	$\forall x R(x, x)$	$\Box p \rightarrow p$
symmetry	$\forall x \forall y (R(x, y) \rightarrow R(y, x))$	

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every world has ≥ 2 successors		

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anti-symmetry	$\forall x \forall y ((R(x, y) \wedge R(y, x) \rightarrow x = y)$	×
$ \text{frame} \geq n$	ϕ_n (see later)	×
$ \text{frame} \leq n$	ψ_n (see later)	×
every world has ≥ 2 successors	✓	

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Definable frame properties in predicate logic and modal logic:

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Definability and Undefinability Results for Model Cardinality

Reminder: Model Cardinality

Define ϕ_n for $n \in \mathbb{N}$ with $n \geq 2$:

$$\phi_n = \exists x_1 \dots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

Define ψ_n for $n \in \mathbb{N}$ with $n \geq 1$:

$$\psi_n = \forall x_1 \dots \forall x_{n+1} \bigvee_{1 \leq i < j \leq n} x_i = x_j$$

Proposition

For all models \mathcal{M} and all $n \geq 2$ it holds:

- (i) $\mathcal{M} \models \phi_n \iff A$ has **at least** n elements (i.e. $|A| \geq n$)
- (ii) $\mathcal{M} \models \psi_n \iff A$ has **at most** n elements (i.e. $|A| \leq n$)
- (iii) $\mathcal{M} \models \phi_n \wedge \psi_n \iff A$ has **precisely** n elements ($|A| = n$)

Model Infiniteness is Definable by Set of Formulas

Proposition

There is a set Δ of sentences such that for all \mathcal{M} :

$$\mathcal{M} \models \Delta \iff \mathcal{M} \text{ has an infinite domain}$$

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There is **no** sentence ψ such that for all \mathcal{M} :

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We conclude that the **assumption** was **wrong**. □

(Un)Definability Results: Current Overview

property	definable by a sentence	definable by a set of sentences
at least	✓	✓
at most	✓	✓
finiteness	✗	✗
infiniteness	✗	✓
reachability		
unreachability		

Definability and Undefinability Results for Reachability

Reachability via R

For a binary predicate symbol R we want to express:

v is reachable via R from u

Reachability via R

For a binary predicate symbol R we want to express:

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Thinking of R as arrows, this means: there is a path from v to u .

Reachability via R

For a binary predicate symbol R we want to express:

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Thinking of R as arrows, this means: there is a path from v to u .

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If we consider the relation R :

$R(x, y) : x$ is child of y

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If we consider the relation R :

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Then reachability via R is the relation:

' x is descendant of, or is the same person as y '

Reachable in n Steps

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We work with constants c, d : we write $\chi(c, d)$ for $\chi[c/u][d/v]$.

$\chi_2(c, d)$ denotes the formula $\exists x_1 (R(c, x_1) \wedge R(x_1, d))$.

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Theorem

It holds for all models \mathcal{M} :

$$\mathcal{M} \models \chi_n(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } n R^{\mathcal{M}}\text{-steps}$$

Reachability is Undefinable (Proof)

There is **no** sentence $\chi(c, d)$ such that for all \mathcal{M} :

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$$\Delta = \{ \chi(c, d) \} \cup \{ \neg\chi_0(c, d), \neg\chi_1(c, d), \neg\chi_2(c, d), \dots \}$$

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- ▶ Consequently $\mathcal{M} \not\models \Delta$.

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 - ▶ Then $\neg\chi_{n_0}(c, d) \notin \Delta_0$ for some n_0 .

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There is **no** sentence $\chi(c, d)$ such that for all \mathcal{M} :

$$\mathcal{M} \models \chi(c, d) \iff d^{\mathcal{M}} \text{ reachable from } c^{\mathcal{M}} \text{ by } R^{\mathcal{M}}\text{-steps} \quad (\star)$$

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 - ▶ Then $\mathcal{M} \models \chi(c, d)$

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 - ▶ Then $\mathcal{M} \models \chi(c, d)$ and $\mathcal{M} \models \neg\chi_n(c, d)$ for all $n \neq n_0$.
 - ▶ Hence $\mathcal{M} \models \Delta_0$, and we conclude that Δ_0 is consistent.

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(5) A set Δ of formulas with (3) and (4) **contradicts** the **compactness theorem**.

(6) The problem must be **assumption (1)**.

So we conclude that **there cannot be such a formula χ** .

Definability Results for Reachability/Unreachability

Proposition

Let R be a binary relation symbol.

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Proof.

Similar to definability and undefinability results (see earlier) for finiteness and infiniteness. □

(Un)Definability Results: Overview

property	definable by a sentence	definable by a set of sentences
at least	✓	✓
at most	✓	✓
finiteness	✗	✗
infiniteness	✗	✓
reachability	✗	✗
unreachability	✗	✓