

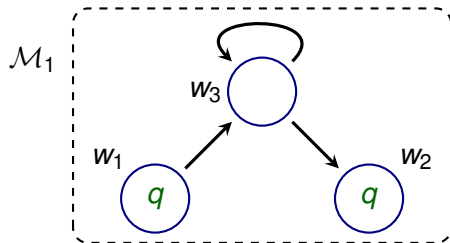
Logic and Modelling

— Modal Logic, Formulas and Frame Properties —

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Introduction



$$w_2 \Vdash \Box \Diamond q$$

$$w_3 \Vdash q \rightarrow \Box \Diamond q$$

$$w_2 \Vdash q$$

$$w_3 \Vdash \Diamond q$$

$$w_1 \Vdash \Box \Diamond q$$

$$\mathcal{M}_1 \models q \rightarrow \Box \Diamond q$$

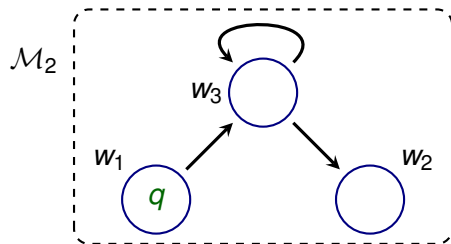
Question

Can you change the labelling such that such that

$$\mathcal{M} \models q \rightarrow \Box \Diamond q$$

is no longer valid?

Introduction



$$\mathcal{M}_1 \models q \rightarrow \square \diamond q$$

$$\mathcal{M}_2 \not\models q \rightarrow \square \diamond q$$

Question

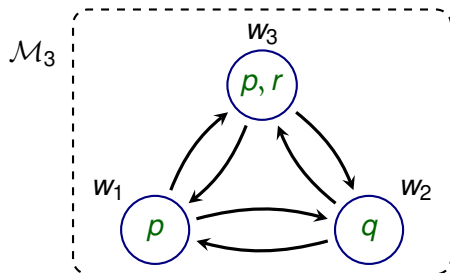
Can you change the labelling such that such that

$$\mathcal{M} \models q \rightarrow \square \diamond q$$

is no longer valid?

Yes, for example, by setting $L(w_2) = \{ \}$.

Another Example



$$w_1 \Vdash \diamond q$$

$$w_3 \Vdash \diamond q$$

$$w_2 \Vdash \Box \diamond q$$

$$\mathcal{M}_3 \models q \rightarrow \Box \diamond q$$

Question

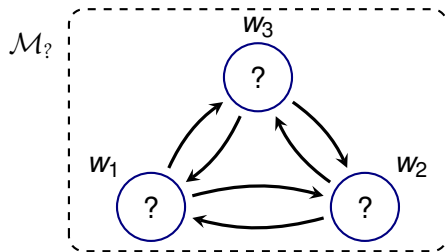
Can you change the labelling such that such that

$$\mathcal{M} \models q \rightarrow \Box \diamond q$$

is no longer valid?

Answer: **No**, that is not possible.

Wherever you put the q 's, $q \rightarrow \Box \Diamond q$ always holds!



- ▶ $W = \{ w_1, w_2, w_3 \}$
- ▶ $R = \{ \langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle, \langle w_2, w_3 \rangle, \langle w_3, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_3, w_1 \rangle \}$
- ▶ $\mathcal{M}_? = (W, R, L_?)$

We check one world (w_1), with and without q

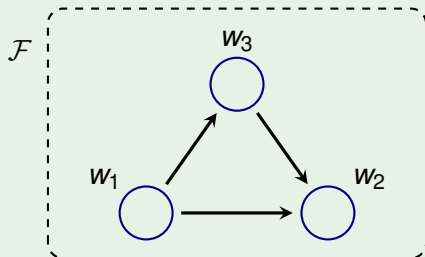
- ▶ Assume that $q \notin L_?(w_1)$,
then $w_1 \Vdash q \rightarrow \Box \Diamond q$ since $w_1 \nVdash q$
- ▶ Assume that $q \in L_?(w_1)$,
then $w_1 \Vdash q \rightarrow \Box \Diamond q$ since $w_1 \Vdash \Box \Diamond q$
since $w_2 \Vdash \Diamond q$ and $w_3 \Vdash \Diamond q$
since $w_1 \Vdash q$

Because of the arrow configuration (always back and forth),
 $q \rightarrow \Box \Diamond q$ is **always valid** wherever you put the q 's.

A **Frame** is a Kripke Model without Labelling

A **frame** $\mathcal{F} = (W, R)$ consists of

- ▶ W , the worlds
- ▶ R , the accessibility relation



- ▶ $W = \{ w_1, w_2, w_3 \}$
- ▶ $R = \{ \langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_3, w_2 \rangle \}$

A Kripke model \mathcal{M} is a frame $\mathcal{F} = (W, R)$ plus a labelling L .

Validity in Frames

Validity in frames

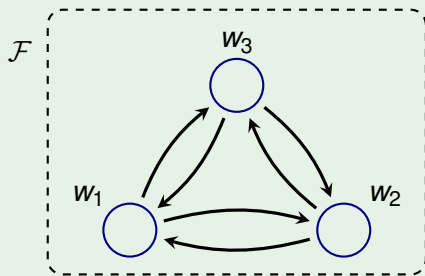
A formula ϕ is **valid in frame** $\mathcal{F} = (W, R)$, denoted

$$\mathcal{F} \models \phi,$$

if **for every labelling** L :

the Kripke model $\mathcal{M} = (W, R, L)$ makes ϕ true ($\mathcal{M} \models \phi$)

We say that \mathcal{M} is a Kripke model on \mathcal{F} .



$$\mathcal{F} \models q \rightarrow \Box \Diamond q$$

$$\mathcal{F} \models p \vee \neg p$$

$$\mathcal{F} \not\models \Diamond p \rightarrow \Box p$$

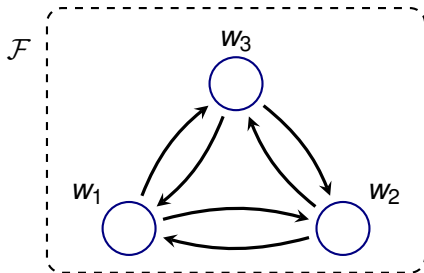
$$\mathcal{F} \models \Box p \rightarrow \Diamond p$$

Symmetric Frames: $q \rightarrow \square \diamond q$

We can now make precise why it is impossible to make

$$q \rightarrow \square \diamond q$$

false on the frame \mathcal{F} .



The reason is that R is **symmetric**: $\forall x \forall y (R(x, y) \rightarrow R(y, x))$

Symmetric Frames: $q \rightarrow \Box \Diamond q$

Theorem

$$\mathcal{F} \models q \rightarrow \Box \Diamond q \iff \text{the frame } \mathcal{F} \text{ is symmetric}$$

Note that the formula characterises a property of the frame!

Proof (\Leftarrow)

Let $\mathcal{F} = \langle W, R \rangle$ be symmetric. We show that $\mathcal{F} \models q \rightarrow \Box \Diamond q$.

That is: $\mathcal{M} \models q \rightarrow \Box \Diamond q$ for $\mathcal{M} = \langle W, R, L \rangle$ with arbitrary L .

Let $x \in W$ be a world. Then (we reason as before)

- ▶ Assume that $q \notin L(x)$,
then $x \Vdash q \rightarrow \Box \Diamond q$ since $x \nVdash q$
- ▶ Assume that $q \in L(x)$,
then $x \Vdash q \rightarrow \Box \Diamond q$ since $x \Vdash \Box \Diamond q$
since $x' \Vdash \Diamond q$ for all x' with $R(x, x')$
since $x \Vdash q$

Hence $x \Vdash q \rightarrow \Box \Diamond q$.

Symmetric Frames: $q \rightarrow \Box \Diamond q$

Theorem

$\mathcal{F} \models q \rightarrow \Box \Diamond q \iff$ the frame \mathcal{F} is symmetric

Note that the formula characterises a property of the frame!

Proof (\Rightarrow)

Let $\mathcal{F} \models q \rightarrow \Box \Diamond q$. We show that \mathcal{F} is symmetric.

For a contradiction, assume that \mathcal{F} is not symmetric.

- ▶ Let $a, b \in W$ with $R(a, b)$ and $\neg R(b, a)$.
- ▶ Define $\mathcal{M} = \langle W, R, L \rangle$ with labelling L given by
$$L(a) = \{ q \} \qquad L(x) = \emptyset \quad \text{for all } x \neq a$$
- ▶ Then
 - ▶ $a \Vdash q$ since $q \in L(a)$
 - ▶ $b \not\Vdash \Diamond q$ since q is valid only in a and $\neg R(b, a)$
 - ▶ $a \not\Vdash \Box \Diamond q$ since $b \not\Vdash \Diamond q$ and $R(a, b)$
- ▶ Hence $\mathcal{F} \not\models q \rightarrow \Box \Diamond q$; a contradiction.

Properties of Relations

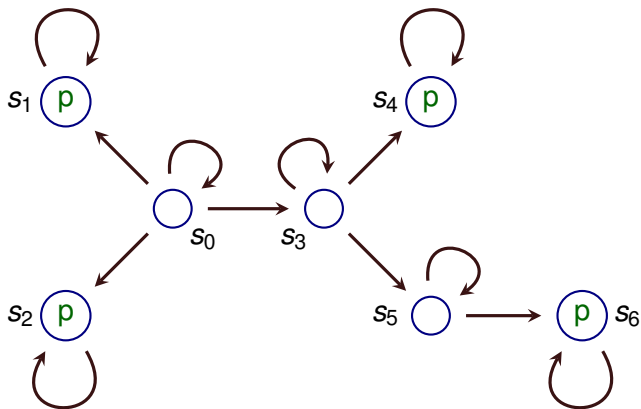
Properties of a relation R

Reflexive	$\forall x R(x, x)$
Symmetric	$\forall x \forall y (R(x, y) \rightarrow R(y, x))$
Transitive	$\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$
Serial	$\forall x \exists y R(x, y)$
Functional	$\forall x \exists y (R(x, y) \wedge \forall z (R(x, z) \rightarrow z = y))$

An **equivalence relation** is a relation that is

- ▶ reflexive,
- ▶ symmetric and
- ▶ transitive.

A Kripke Model on a Reflexive Frame



$\mathcal{M} = \langle W, R, L \rangle$

▶ $\mathcal{M} \models \Box p \rightarrow p$

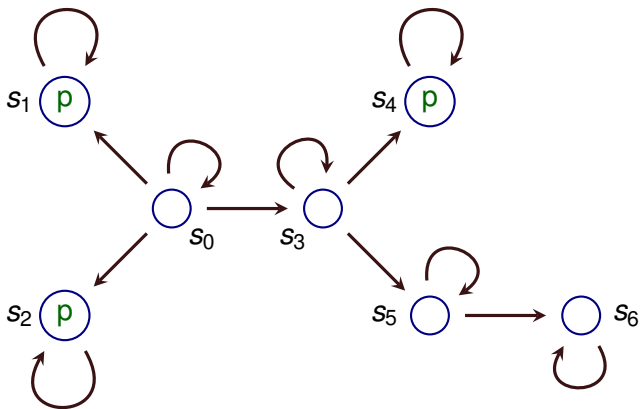
▶ $\mathcal{M} \models \Box \Diamond p$

$\mathcal{F} = \langle W, R \rangle$

▶ $\mathcal{F} \models \Box p \rightarrow p$

▶ $\mathcal{F} \not\models \Box \Diamond p$

The Same Frame, a Different Labelling



$$\mathcal{M}' = \langle W, R, L \rangle$$

$$\triangleright \mathcal{M}' \models \Box p \rightarrow p$$

$$\triangleright \mathcal{M}' \not\models \Box \Diamond p$$

$$\mathcal{F} = \langle W, R \rangle$$

$$\triangleright \mathcal{F} \models \Box p \rightarrow p$$

$$\triangleright \mathcal{F} \not\models \Box \Diamond p$$

Reflexive Frames: $\Box p \rightarrow p$

Recall that **reflexivity** means: $\forall x R(x, x)$.

Theorem

$\mathcal{F} \models \Box p \rightarrow p \iff$ the frame \mathcal{F} is reflexive

Proof

\Leftarrow Assume that $\mathcal{F} = \langle W, R \rangle$ is reflexive.

We show that $\mathcal{F} \models \Box p \rightarrow p$.

Let L be an arbitrary labelling and $\mathcal{M} = \langle W, R, L \rangle$.

We show for every world x : $\mathcal{M}, x \Vdash \Box p \rightarrow p$.

If $x \Vdash \Box p$, then $x \Vdash p$ since $R(x, x)$.

Hence $\mathcal{M}, x \Vdash \Box p \rightarrow p$.

Reflexive Frames: $\Box p \rightarrow p$

Recall that **reflexivity** means: $\forall x R(x, x)$.

Theorem

$\mathcal{F} \models \Box p \rightarrow p \iff$ the frame \mathcal{F} is reflexive

Proof

\Rightarrow Assume that $\mathcal{F} \models \Box p \rightarrow p$. We show that \mathcal{F} is reflexive.

Assume, for a contradiction, \mathcal{F} was not reflexive.

Then there is a world a with $\neg R(a, a)$.

Let $\mathcal{M} = \langle W, R, L \rangle$ where L is given by:

$$L(a) = \emptyset \quad L(w) = \{ p \} \quad \text{for every world } w \neq a$$

Then $a \Vdash \Box p$ since p holds in all worlds $\neq a$ and $\neg R(a, a)$.

But $a \not\Vdash p$. Thus $a \not\Vdash \Box p \rightarrow p$.

Thus $\mathcal{F} \not\models \Box p \rightarrow p$; a contradiction! Hence \mathcal{F} is reflexive.

Correspondence of Formulas and Frame Properties

We now know that the formula

$$\Box p \rightarrow p$$

is valid **precisely** on the reflexive frames:

$$\mathcal{F} \models \Box p \rightarrow p \iff \mathcal{F} \text{ is reflexive}$$

We say that the formula $\Box p \rightarrow p$ **corresponds** with the frame property **reflexivity**.

In general:

A modal formula ϕ **corresponds** with a frame property E if:

$$\mathcal{F} \models \phi \iff \mathcal{F} \text{ has property } E$$

Correspondence Table

Correspondence Table

Reflexive	$\Box p \rightarrow p$
Symmetric	$p \rightarrow \Box \Diamond p$
Transitive	$\Box p \rightarrow \Box \Box p$
Serial	$\Diamond \top$
Functional	$\Box p \leftrightarrow \Diamond p$

Correspondences are not unique

A few alternatives:

Reflexive	$p \rightarrow \Diamond p$
Symmetric	$\Diamond \Box p \rightarrow p$
Transitive	$\Diamond \Diamond p \rightarrow \Diamond p$
Serial	$\Box p \rightarrow \Diamond p$

Transitive Frames: $\diamond\diamond p \rightarrow \diamond p$

Theorem

$\mathcal{F} \models \diamond\diamond p \rightarrow \diamond p \iff$ the frame \mathcal{F} is transitive

Proof (\Leftarrow)

- ▶ Let $\mathcal{F} = (W, R)$ be a frame where R is transitive.
- ▶ Let L be an arbitrary labelling, and x a world in W .
- ▶ We show $x \Vdash \diamond\diamond p \rightarrow \diamond p$.
That is, if $x \Vdash \diamond\diamond p$, then also $x \Vdash \diamond p$.
- ▶ Thus assume that $x \Vdash \diamond\diamond p$.
- ▶ Then there exists $y \in W$ with $R(x, y)$ and $y \Vdash \diamond p$.
- ▶ Then there exists $z \in W$ with $R(y, z)$ and $z \Vdash p$.
- ▶ Because of transitivity of R we have $R(x, z)$.
- ▶ Hence $x \Vdash \diamond p$.

Transitive Frames: $\diamond\diamond p \rightarrow \diamond p$

Theorem

$\mathcal{F} \models \diamond\diamond p \rightarrow \diamond p \iff$ the frame \mathcal{F} is transitive

Proof (\Rightarrow)

▶ Let $\mathcal{F} = \langle W, R \rangle$ be a frame with $\mathcal{F} \models \diamond\diamond p \rightarrow \diamond p$.

▶ For a contradiction, assume that R is not transitive.

▶ Since R is not transitive, there are $a, b, c \in W$ with

$$R(a, b) \quad R(b, c) \quad \neg R(a, c)$$

▶ Define $\mathcal{M} = \langle W, R, L \rangle$ with labelling L given by:

$$L(c) = \{ p \} \quad L(w) = \emptyset \quad \text{for every } w \neq c$$

▶ Then

- ▶ $b \Vdash \diamond p$ since $c \Vdash p$ and $R(b, c)$
- ▶ $a \Vdash \diamond\diamond p$ since $b \Vdash \diamond p$ en $R(a, b)$
- ▶ $a \not\Vdash \diamond p$ since p is valid only in c , and $\neg R(a, c)$

▶ Thus $\mathcal{M} \not\models \diamond\diamond p \rightarrow \diamond p$; contradicting $\mathcal{F} \models \diamond\diamond p \rightarrow \diamond p$.