Logic and Modelling

— Modal Logic, Formulas and Frame Properties —

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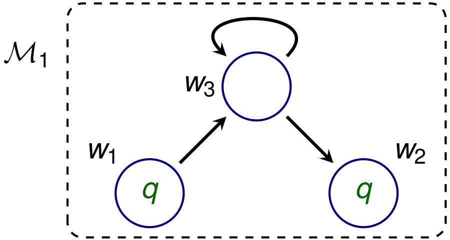
Introduction



$w_2 \Vdash \Box\Diamond q$

$w_3 \Vdash q \rightarrow \Box\Diamond q$

$w_2 \Vdash q$

$w_3 \Vdash \Diamond q$

$w_1 \Vdash \Box\Diamond q$

$\mathcal{M}_1 \models q \rightarrow \Box\Diamond q$

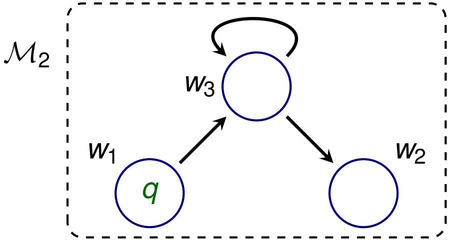
Question

Can you change the labelling such that such that

$$\mathcal{M} \models q \rightarrow \Box\Diamond q$$

is no longer valid?

Introduction



$\mathcal{M}_1 \models q \rightarrow \Box\Diamond q$

$\mathcal{M}_2 \not\models q \rightarrow \Box\Diamond q$

Question

Can you change the labelling such that such that

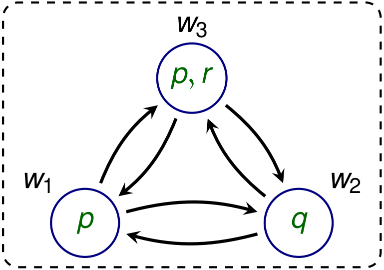
$$\mathcal{M} \models q \rightarrow \Box\Diamond q$$

is no longer valid?

Yes, for example, by setting $L(w_2) = \{\,\}$.

Another Example

$\mathcal{M}_3$



$w_1 \Vdash \Diamond q$

$w_3 \Vdash \Diamond q$

$w_2 \Vdash \Box\Diamond q$

$\mathcal{M}_3 \models q \rightarrow \Box\Diamond q$

Question

Can you change the labelling such that such that

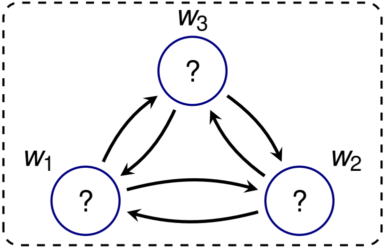
$$\mathcal{M} \models q \rightarrow \Box\Diamond q$$

is no longer valid?

Answer: **No**, that is not possible.

Wherever you put the q's, $q \to \Box\Diamond q$ always holds!

$\mathcal{M}_?$



- $W = \{ w_1, w_2, w_3 \}$
- $R = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle, \langle w_2, w_3\rangle, \langle w_3, w_2\rangle, \langle w_1, w_3\rangle, \langle w_3, w_1\rangle \}$
- $\mathcal{M}_? = (W, R, L_?)$

We check one world (w_1), with and without q

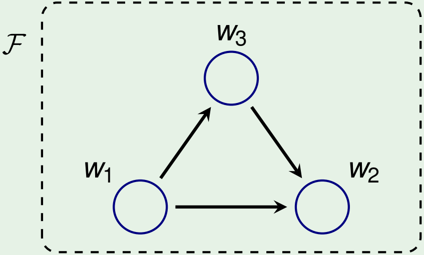
- Assume that $q \notin L_?(w_1)$,
  then $w_1 \Vdash q \to \Box\Diamond q$ since $w_1 \nVdash q$
- Assume that $q \in L_?(w_1)$,
  then $w_1 \Vdash q \to \Box\Diamond q$ since $w_1 \Vdash \Box\Diamond q$
  since $w_2 \Vdash \Diamond q$ and $w_3 \Vdash \Diamond q$
  since $w_1 \Vdash q$

Because of the arrow configuration (always back and forth), $q \to \Box\Diamond q$ is **always valid** wherever you put the q's.

A **Frame** is a Kripke Model without Labelling

A **frame** $\mathcal{F} = (W, R)$ consists of

- W, the worlds
- R, the accessibility relation



- $W = \{ w_1, w_2, w_3 \}$
- $R = \{ \langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_3, w_2 \rangle \}$

A Kripke model $\mathcal{M}$ is a frame $\mathcal{F} = (W, R)$ plus a labelling L.

Validity in Frames

Validity in frames

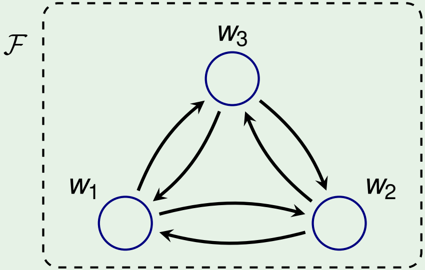
A formula ϕ is **valid in frame** $\mathcal{F} = (W, R)$, denoted

$$\mathcal{F} \models \phi \ ,$$

if **for every labelling** L:

the Kripke model $\mathcal{M} = (W, R, L)$ makes ϕ true $\quad (\mathcal{M} \models \phi)$

We say that $\mathcal{M}$ is a Kripke model on $\mathcal{F}$.



$\mathcal{F} \models q \to \Box\Diamond q$

$\mathcal{F} \models p \vee \neg p$

$\mathcal{F} \not\models \Diamond p \to \Box p$

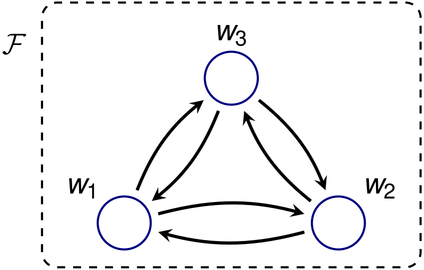
$\mathcal{F} \models \Box p \to \Diamond p$

Symmetric Frames: $q \to \Box\Diamond q$

We can now make precise why it is impossible to make

$$q \to \Box\Diamond q$$

false on the frame $\mathcal{F}$.



The reason is that R is **symmetric**: $\forall x\ \forall y\ \big(R(x,y) \to R(y,x)\big)$

Symmetric Frames: $q \rightarrow \Box\Diamond q$

Theorem

$$\mathcal{F} \models q \rightarrow \Box\Diamond q \qquad \Longleftrightarrow \qquad \text{the frame } \mathcal{F} \text{ is symmetric}$$

Note that the formula characterises a property of the frame!

Proof ($\Leftarrow$)

Let $\mathcal{F} = \langle W, R \rangle$ be symmetric. We show that $\mathcal{F} \models q \rightarrow \Box\Diamond q$.

That is: $\mathcal{M} \models q \rightarrow \Box\Diamond q$ for $\mathcal{M} = \langle W, R, L \rangle$ with arbitrary L.

Let $x \in W$ be a world. Then (we reason as before)

- Assume that $q \notin L(x)$,
  then $x \Vdash q \rightarrow \Box\Diamond q$ since $x \nVdash q$
- Assume that $q \in L(x)$,
  then $x \Vdash q \rightarrow \Box\Diamond q$ since $x \Vdash \Box\Diamond q$
  since $x' \Vdash \Diamond q$ for all x' with $R(x, x')$
  since $x \Vdash q$

Hence $x \Vdash q \rightarrow \Box\Diamond q$.

Symmetric Frames: $q \to \Box\Diamond q$

Theorem

$$\mathcal{F} \models q \to \Box\Diamond q \quad \iff \quad \text{the frame } \mathcal{F} \text{ is symmetric}$$

Note that the formula characterises a property of the frame!

Proof ($\Rightarrow$)

Let $\mathcal{F} \models q \to \Box\Diamond q$. We show that $\mathcal{F}$ is symmetric.

For a contradiction, assume that $\mathcal{F}$ is not symmetric.

- Let $a, b \in W$ with $R(a, b)$ and $\neg R(b, a)$.
- Define $\mathcal{M} = \langle W, R, L\rangle$ with labelling L given by
  $$L(a) = \{\, q \,\} \qquad\qquad L(x) = \emptyset \quad \text{for all } x \neq a$$
- Then
  - $a \Vdash q$ since $q \in L(a)$
  - $b \nVdash \Diamond q$ since q is valid only in a and $\neg R(b, a)$
  - $a \nVdash \Box\Diamond q$ since $b \nVdash \Diamond q$ and $R(a, b)$
- Hence $\mathcal{F} \not\models q \to \Box\Diamond q$; a contradiction.

Properties of Relations

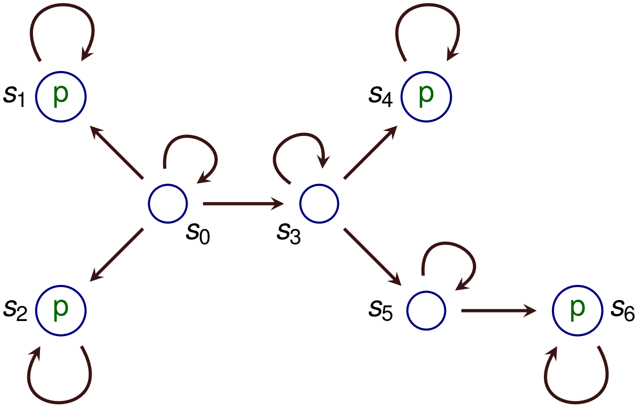
Properties of a relation R

Reflexive	$\forall x \; R(x, x)$
Symmetric	$\forall x \, \forall y \; \big(\; R(x, y) \to R(y, x) \; \big)$
Transitive	$\forall x \, \forall y \, \forall z \; \big(\; (R(x, y) \land R(y, z)) \to R(x, z) \; \big)$
Serial	$\forall x \, \exists y \; R(x, y)$
Functional	$\forall x \, \exists y \; \big(\; R(x, y) \land \forall z \; (R(x, z) \to z = y) \; \big)$

An **equivalence relation** is a relation that is

- reflexive,
- symmetric and
- transitive.

A Kripke Model on a Reflexive Frame



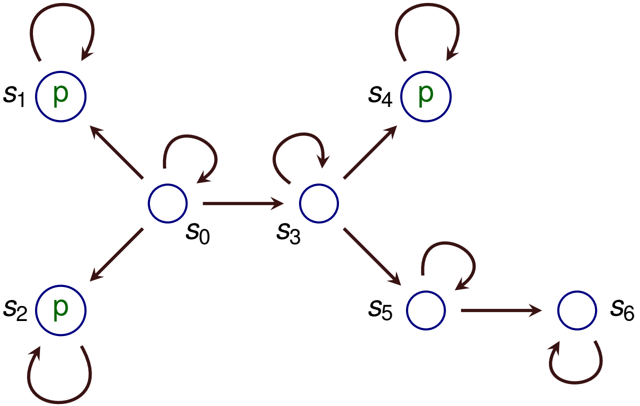
$\mathcal{M} = \langle W, R, L \rangle$

- $\mathcal{M} \models \Box p \rightarrow p$
- $\mathcal{M} \models \Box \Diamond p$

$\mathcal{F} = \langle W, R \rangle$

- $\mathcal{F} \models \Box p \rightarrow p$
- $\mathcal{F} \not\models \Box \Diamond p$

The Same Frame, a Different Labelling



$\mathcal{M}' = \langle W, R, L \rangle$

- $\mathcal{M}' \models \Box p \to p$
- $\mathcal{M}' \not\models \Box \Diamond p$

$\mathcal{F} = \langle W, R \rangle$

- $\mathcal{F} \models \Box p \to p$
- $\mathcal{F} \not\models \Box \Diamond p$

Reflexive Frames: $\Box p \to p$

Recall that **reflexivity** means: $\forall x \; R(x, x)$.

Theorem

$$\mathcal{F} \models \Box p \to p \quad \iff \quad \text{the frame } \mathcal{F} \text{ is reflexive}$$

Proof

$\Leftarrow$ Assume that $\mathcal{F} = \langle W, R \rangle$ is reflexive.

We show that $\mathcal{F} \models \Box p \to p$.

Let L be an arbitrary labelling and $\mathcal{M} = \langle W, R, L \rangle$.

We show for every world x: $\mathcal{M}, x \Vdash \Box p \to p$.

If $x \Vdash \Box p$, then $x \Vdash p$ since $R(x, x)$.

Hence $\mathcal{M}, x \Vdash \Box p \to p$.

Reflexive Frames: $\Box p \to p$

Recall that **reflexivity** means: $\forall x\ R(x,x)$.

Theorem

$$\mathcal{F} \models \Box p \to p \quad \iff \quad \text{the frame } \mathcal{F} \text{ is reflexive}$$

Proof

$\Rightarrow$ Assume that $\mathcal{F} \models \Box p \to p$. We show that $\mathcal{F}$ is reflexive.

Assume, for a contradiction, $\mathcal{F}$ was not reflexive.

Then there is a world a with $\neg R(a,a)$.

Let $\mathcal{M} = \langle W, R, L \rangle$ where L is given by:

$$L(a) = \emptyset \qquad L(w) = \{\, p \,\} \quad \text{for every world } w \neq a$$

Then $a \Vdash \Box p$ since p holds in all worlds $\neq a$ and $\neg R(a,a)$.

But $a \nVdash p$. Thus $a \nVdash \Box p \to p$.

Thus $\mathcal{F} \nVdash \Box p \to p$; a contradiction! Hence $\mathcal{F}$ is reflexive.

Correspondence of Formulas and Frame Properties

We now know that the formula

$$\Box p \to p$$

is valid **precisely** on the reflexive frames:

$$\mathcal{F} \models \Box p \to p \quad \Longleftrightarrow \quad \mathcal{F} \text{ is reflexive}$$

We say that the formula $\Box p \to p$ **corresponds** with the frame property **reflexivity.**

In general:

A **modal formula** ϕ **corresponds** with a **frame property** E if:

$$\mathcal{F} \models \phi \quad \Longleftrightarrow \quad \mathcal{F} \text{ has property } E$$

Correspondence Table

Correspondence Table

Reflexive	$\Box p \rightarrow p$
Symmetric	$p \rightarrow \Box\Diamond p$
Transitive	$\Box p \rightarrow \Box\Box p$
Serial	$\Diamond\top$
Functional	$\Box p \leftrightarrow \Diamond p$

Correspondences are not unique

A few alternatives:

Reflexive	$p \rightarrow \Diamond p$
Symmetric	$\Diamond\Box p \rightarrow p$
Transitive	$\Diamond\Diamond p \rightarrow \Diamond p$
Serial	$\Box p \rightarrow \Diamond p$

Transitive Frames: $\Diamond\Diamond p \to \Diamond p$

Theorem

$$\mathcal{F} \models \Diamond\Diamond p \to \Diamond p \quad \Longleftrightarrow \quad \text{the frame } \mathcal{F} \text{ is transitive}$$

Proof ($\Leftarrow$)

- Let $\mathcal{F} = (W, R)$ be a frame where R is transitive.
- Let L be an arbitrary labelling, and x a world in W.
- We show $x \Vdash \Diamond\Diamond p \to \Diamond p$.
  That is, if $x \Vdash \Diamond\Diamond p$, then also $x \Vdash \Diamond p$.
- Thus assume that $x \Vdash \Diamond\Diamond p$.
- Then there exists $y \in W$ with $R(x, y)$ and $y \Vdash \Diamond p$.
- Then there exists $z \in W$ with $R(y, z)$ and $z \Vdash p$.
- Because of transitivity of R we have $R(x, z)$.
- Hence $x \Vdash \Diamond p$.

Transitive Frames: $\Diamond\Diamond p \to \Diamond p$

Theorem

$$\mathcal{F} \models \Diamond\Diamond p \to \Diamond p \quad \iff \quad \text{the frame } \mathcal{F} \text{ is transitive}$$

Proof ($\Rightarrow$)

- Let $\mathcal{F} = \langle W, R \rangle$ be a frame with $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$.
- For a contradiction, assume that R is not transitive.
- Since R is not transitive, there are $a, b, c \in W$ with
  $$R(a, b) \qquad R(b, c) \qquad \neg R(a, c)$$
- Define $\mathcal{M} = \langle W, R, L \rangle$ with labelling L given by:
  $$L(c) = \{ p \} \qquad L(w) = \emptyset \quad \text{for every } w \neq c$$
- Then
  - $b \Vdash \Diamond p$ since $c \Vdash p$ and $R(b, c)$
  - $a \Vdash \Diamond\Diamond p$ since $b \Vdash \Diamond p$ en $R(a, b)$
  - $a \nVdash \Diamond p$ since p is valid only in c, and $\neg R(a, c)$
- Thus $\mathcal{M} \not\models \Diamond\Diamond p \to \Diamond p$; contradicting $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$.