

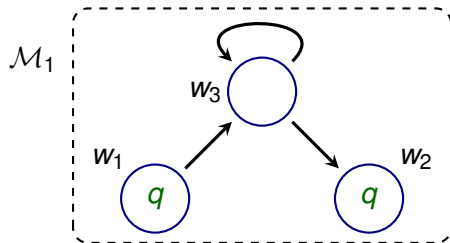
Logic and Modelling

— Modal Logic, Formulas and Frame Properties —

Jörg Endrullis

VU University Amsterdam

Introduction



$$w_2 \Vdash \Box \Diamond q$$

$$w_3 \Vdash q \rightarrow \Box \Diamond q$$

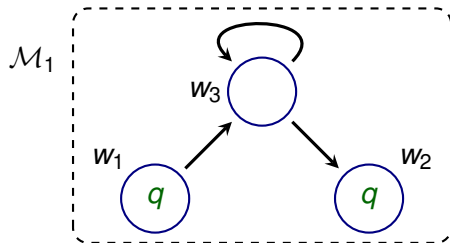
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Introduction



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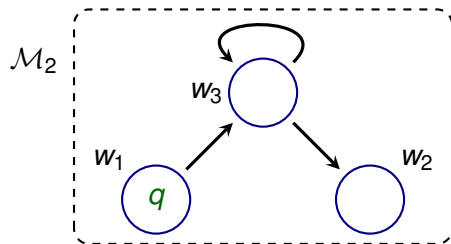
Question

Can you change the labelling such that such that

$$\mathcal{M} \models q \rightarrow \Box \Diamond q$$

is no longer valid?

Introduction



$$\mathcal{M}_1 \models q \rightarrow \square \diamond q$$

$$\mathcal{M}_2 \not\models q \rightarrow \square \diamond q$$

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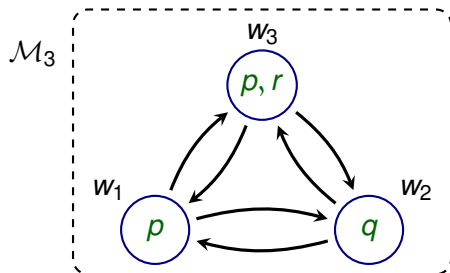
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Yes, for example, by setting $L(w_2) = \{ \}$.

Another Example



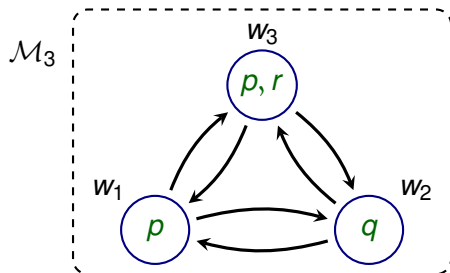
$$w_1 \Vdash \diamond q$$

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$$w_2 \Vdash \Box \diamond q$$

$$\mathcal{M}_3 \models q \rightarrow \Box \diamond q$$

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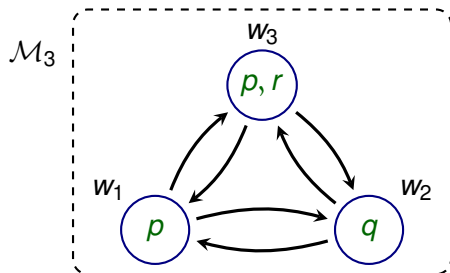
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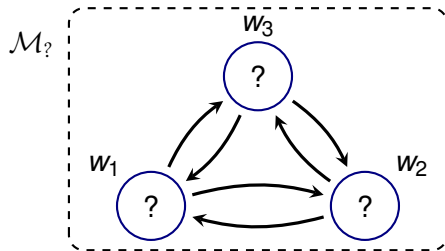
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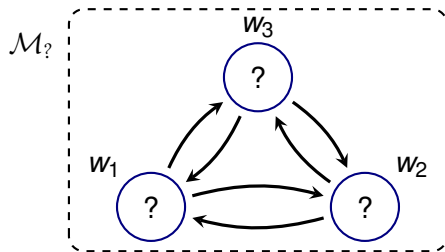
Answer: **No**, that is not possible.

Wherever you put the q 's, $q \rightarrow \square \diamond q$ always holds!



- ▶ $W = \{ w_1, w_2, w_3 \}$
- ▶ $R = \{ \langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle, \langle w_2, w_3 \rangle, \langle w_3, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_3, w_1 \rangle \}$
- ▶ $\mathcal{M}_? = (W, R, L_?)$

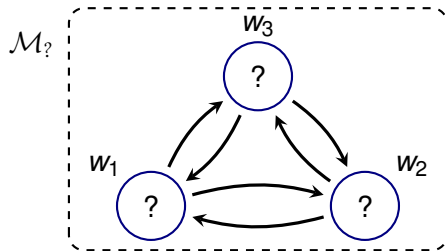
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We check one world (w_1), with and without q

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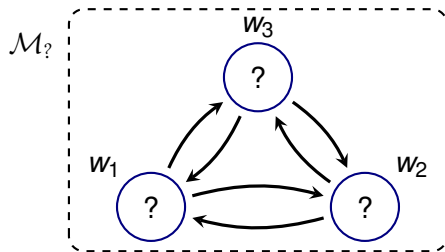


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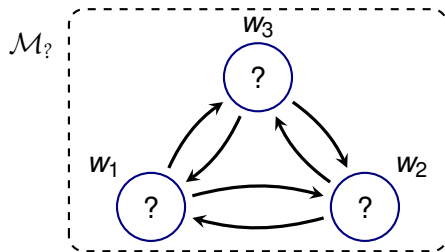


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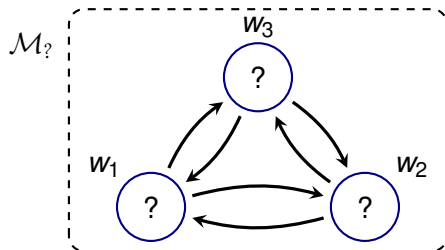


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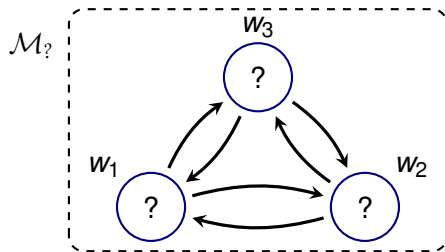
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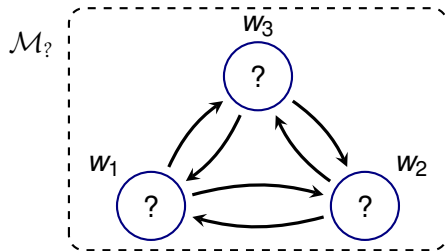
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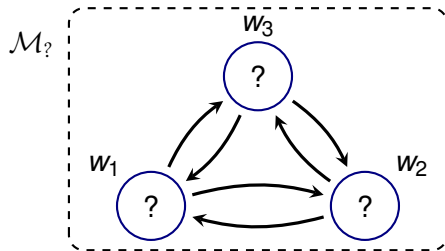
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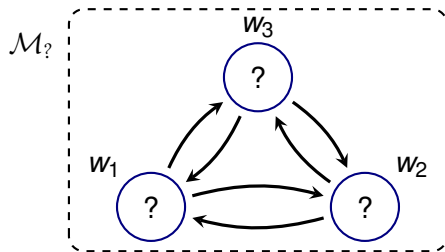


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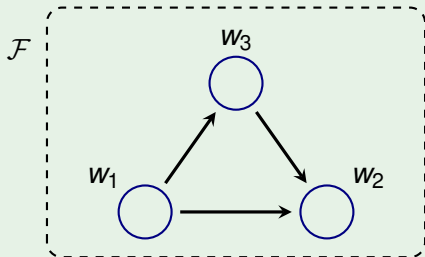
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Because of the arrow configuration (always back and forth),
 $q \rightarrow \Box \Diamond q$ is **always valid** wherever you put the q 's.

A **Frame** is a Kripke Model without Labelling

A **frame** $\mathcal{F} = (W, R)$ consists of

- ▶ W , the worlds
- ▶ R , the accessibility relation

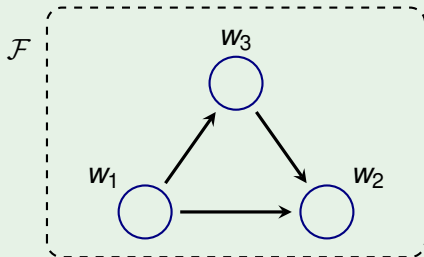


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A Kripke model \mathcal{M} is a frame $\mathcal{F} = (W, R)$ plus a labelling L .

Validity in Frames

Validity in frames

A formula ϕ is **valid in frame** $\mathcal{F} = (W, R)$, denoted

$$\mathcal{F} \models \phi,$$

if **for every labelling** L :

the Kripke model $\mathcal{M} = (W, R, L)$ makes ϕ true ($\mathcal{M} \models \phi$)

We say that \mathcal{M} is a Kripke model on \mathcal{F} .

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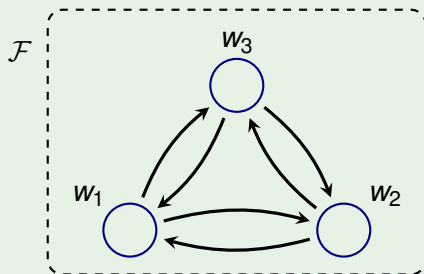
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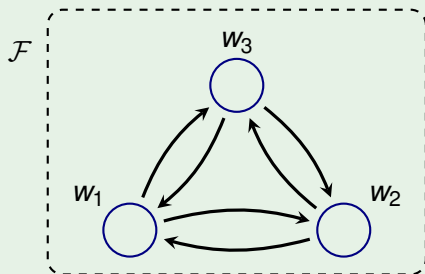
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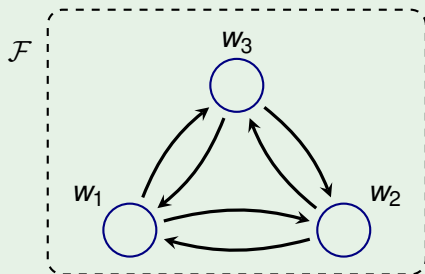
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$$\mathcal{F} \models p \vee \neg p$$

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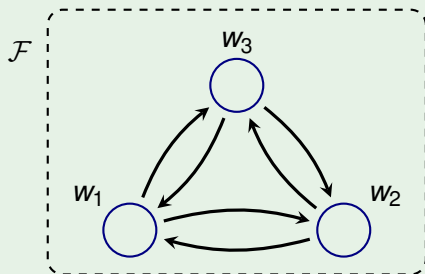
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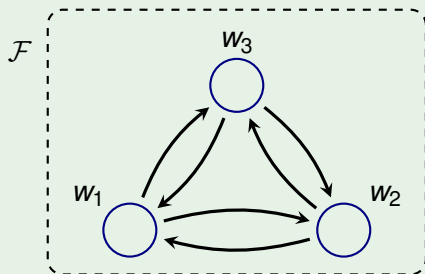
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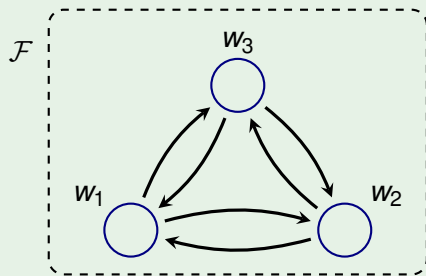
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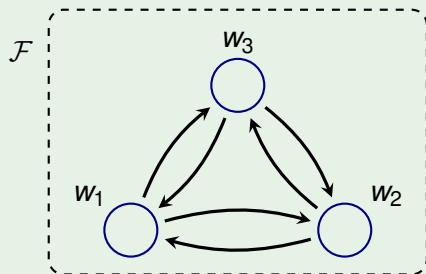
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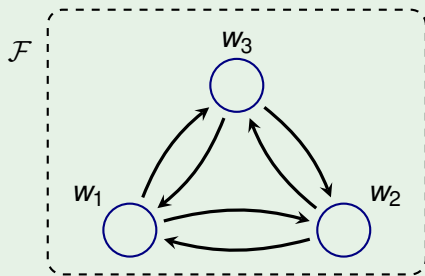
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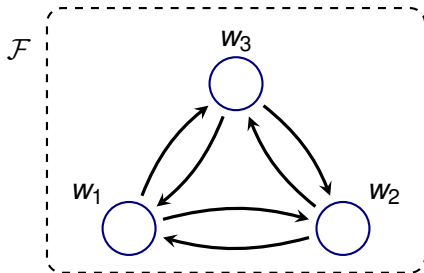
$$\mathcal{F} \models \Box p \rightarrow \Diamond p$$

Symmetric Frames: $q \rightarrow \Box \Diamond q$

We can now make precise why it is impossible to make

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false on the frame \mathcal{F} .

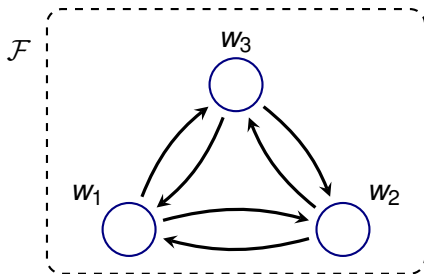


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The reason is that R is **symmetric**: $\forall x \forall y (R(x, y) \rightarrow R(y, x))$

Symmetric Frames: $q \rightarrow \Box \Diamond q$

Theorem

$\mathcal{F} \models q \rightarrow \Box \Diamond q \iff$ the frame \mathcal{F} is symmetric

Note that the formula characterises a property of the frame!

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Proof (\Leftarrow)

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Let $\mathcal{F} = \langle W, R \rangle$ be symmetric. We show that $\mathcal{F} \models q \rightarrow \Box \Diamond q$.

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Let $x \in W$ be a world. Then (we reason as before)

- ▶ Assume that $q \notin L(x)$,
then $x \Vdash q \rightarrow \Box \Diamond q$ since $x \nVdash q$
- ▶ Assume that $q \in L(x)$,
then $x \Vdash q \rightarrow \Box \Diamond q$ since $x \Vdash \Box \Diamond q$
since $x' \Vdash \Diamond q$ for all x' with $R(x, x')$
since $x \Vdash q$

Hence $x \Vdash q \rightarrow \Box \Diamond q$.

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$\mathcal{F} \models q \rightarrow \Box \Diamond q \iff$ the frame \mathcal{F} is symmetric

Note that the formula characterises a property of the frame!

Proof (\Rightarrow)

Let $\mathcal{F} \models q \rightarrow \Box \Diamond q$. We show that \mathcal{F} is symmetric.

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- ▶ Hence $\mathcal{F} \not\models q \rightarrow \Box \Diamond q$; a contradiction.

Properties of Relations

Properties of a relation R

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Transitive $\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$

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Symmetric	$\forall x \forall y (R(x, y) \rightarrow R(y, x))$
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Serial	$\forall x \exists y R(x, y)$

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Properties of Relations

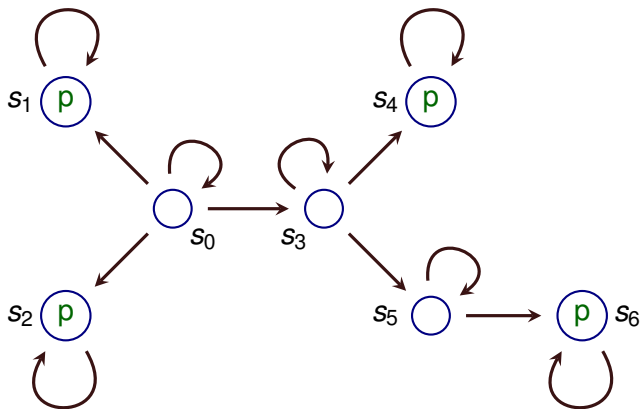
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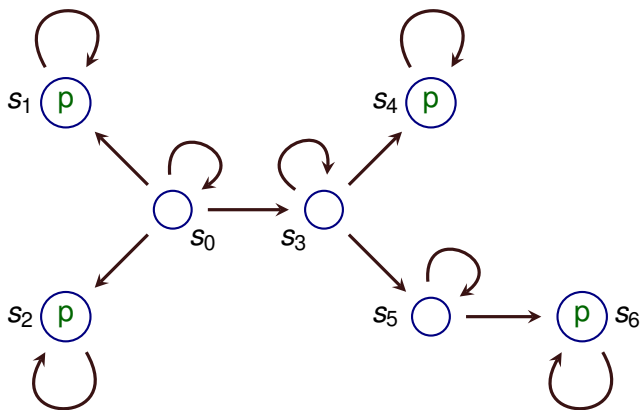
An **equivalence relation** is a relation that is

- ▶ reflexive,
- ▶ symmetric and
- ▶ transitive.

A Kripke Model on a Reflexive Frame



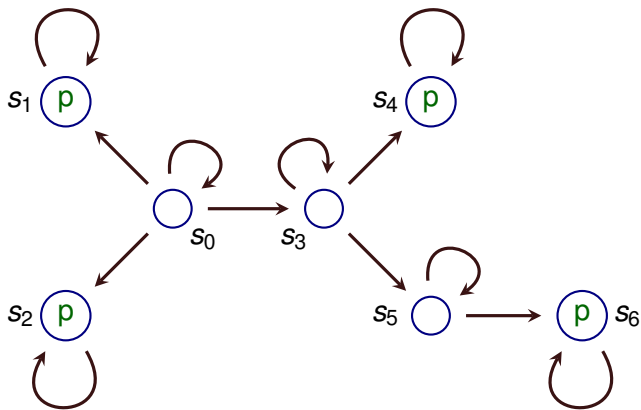
A Kripke Model on a Reflexive Frame



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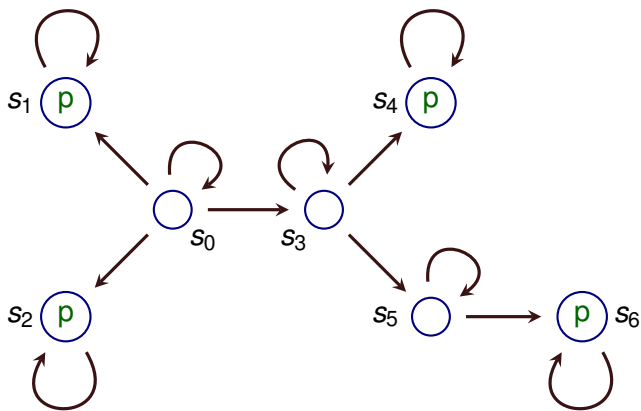


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► \mathcal{M} ◻ $p \rightarrow p$

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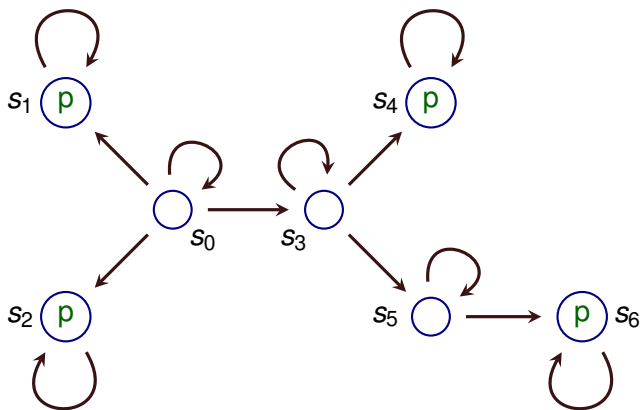


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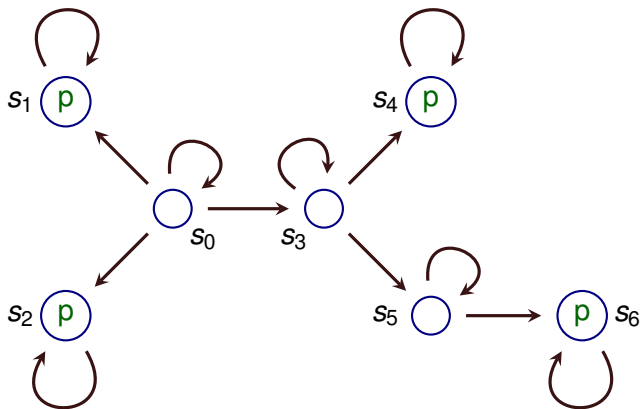
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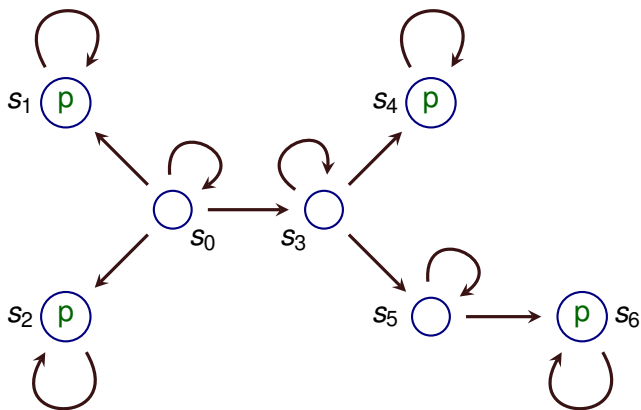
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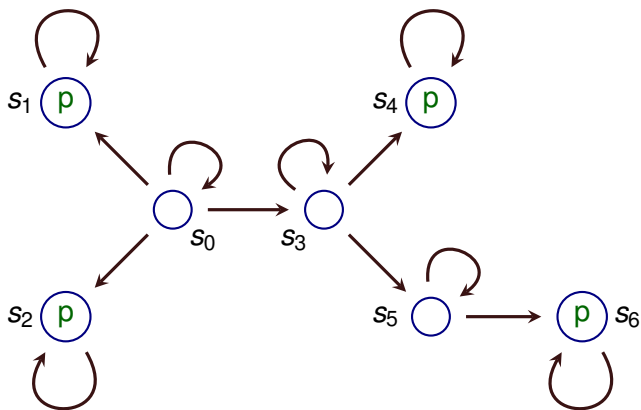
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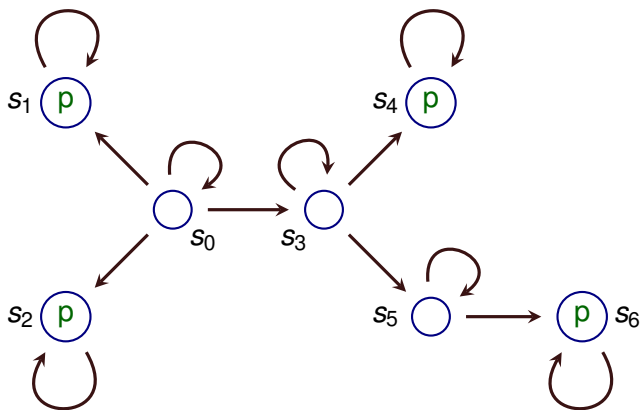
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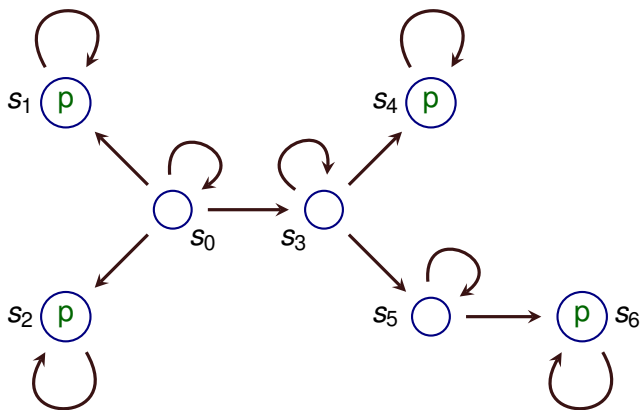
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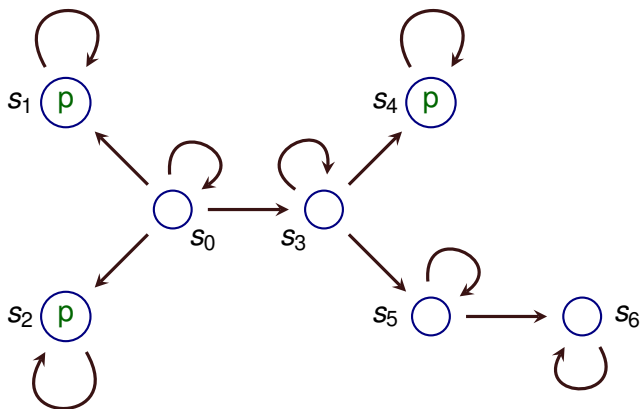
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The Same Frame, a Different Labelling



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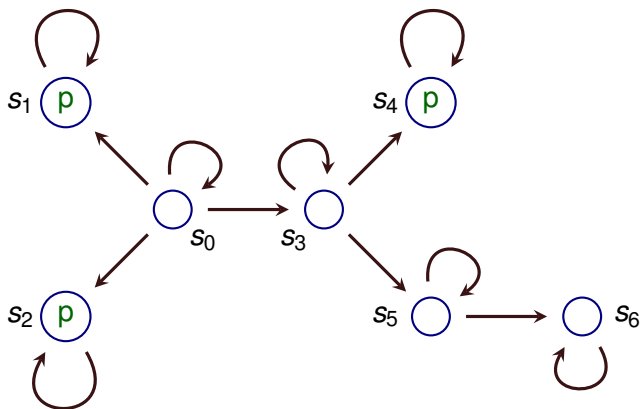
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Recall that **reflexivity** means: $\forall x R(x, x)$.

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But $a \not\models p$.

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Thus $\mathcal{F} \not\models \Box p \rightarrow p$; a contradiction!

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Then $a \Vdash \Box p$ since p holds in all worlds $\neq a$ and $\neg R(a, a)$.

But $a \not\Vdash p$. Thus $a \not\Vdash \Box p \rightarrow p$.

Thus $\mathcal{F} \not\models \Box p \rightarrow p$; a contradiction! Hence \mathcal{F} is reflexive.

Correspondence of Formulas and Frame Properties

We now know that the formula

$$\Box p \rightarrow p$$

is valid **precisely** on the reflexive frames:

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In general:

A modal formula ϕ **corresponds** with a frame property E if:

$$\mathcal{F} \models \phi \iff \mathcal{F} \text{ has property } E$$

Correspondence Table

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Reflexive $\square p \rightarrow p$

Correspondence Table

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Reflexive	$\Box p \rightarrow p$
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Correspondences are not unique

A few alternatives:

Reflexive	$p \rightarrow \Diamond p$
Symmetric	$\Diamond \Box p \rightarrow p$
Transitive	$\Diamond \Diamond p \rightarrow \Diamond p$
Serial	$\Box p \rightarrow \Diamond p$

Transitive Frames: $\diamond\diamond p \rightarrow \diamond p$

Theorem

$\mathcal{F} \models \diamond\diamond p \rightarrow \diamond p \iff$ the frame \mathcal{F} is transitive

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 - ▶ $b \Vdash \diamond p$ since $c \Vdash p$ and $R(b, c)$

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- ▶ $a \Vdash \diamond\diamond p$ since $b \Vdash \diamond p$ en $R(a, b)$
- ▶ $a \not\Vdash \diamond p$ since p is valid only in c , and $\neg R(a, c)$

- ▶ Thus $\mathcal{M} \not\models \diamond\diamond p \rightarrow \diamond p$; contradicting $\mathcal{F} \models \diamond\diamond p \rightarrow \diamond p$.