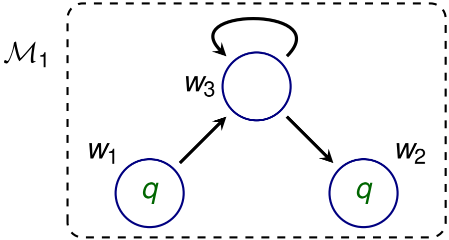
Logic and Modelling

— Modal Logic, Formulas and Frame Properties —

Jörg Endrullis

VU University Amsterdam

Introduction



$w_2 \Vdash \Box\Diamond q$

$w_3 \Vdash q \rightarrow \Box\Diamond q$

$w_2 \Vdash q$

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$w_1 \Vdash \Box\Diamond q$

$\mathcal{M}_1 \models q \rightarrow \Box\Diamond q$

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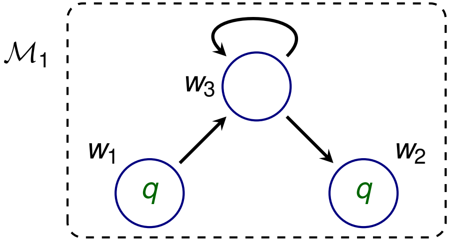

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Question

Can you change the labelling such that such that

$$\mathcal{M} \models q \rightarrow \Box\Diamond q$$

is no longer valid?

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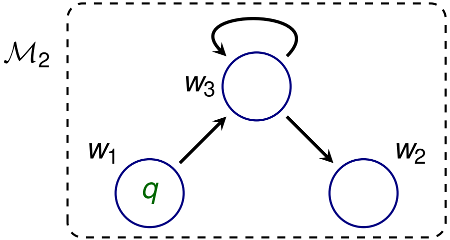

$\mathcal{M}_1 \models q \rightarrow \Box\Diamond q$
$\mathcal{M}_2 \not\models q \rightarrow \Box\Diamond q$

Question

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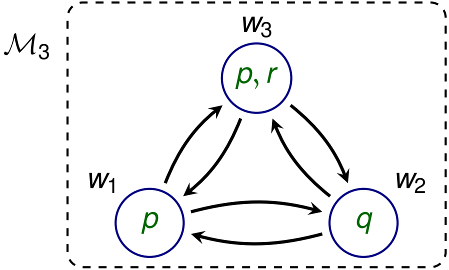
Yes, for example, by setting $L(w_2) = \{\,\}$.

Another Example



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$\mathcal{M}_3 \models q \to \Box\Diamond q$

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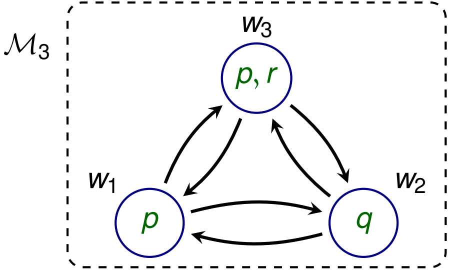

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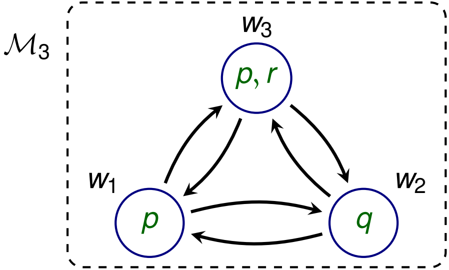

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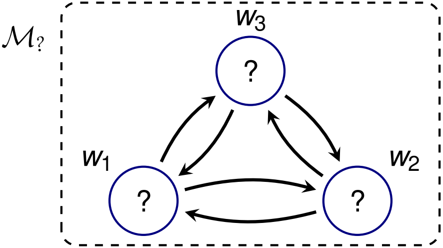
Answer: **No**, that is not possible.

Wherever you put the q's, $q \rightarrow \Box\Diamond q$ always holds!



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- $\mathcal{M}_? = (W, R, L_?)$

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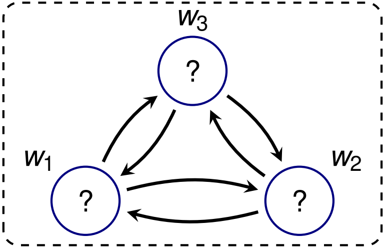

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We check one world (w_1), with and without q

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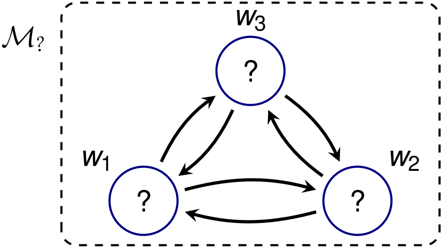

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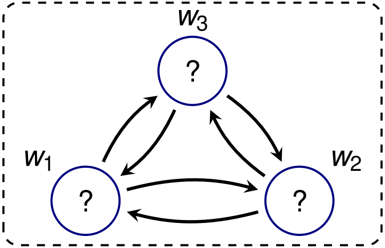

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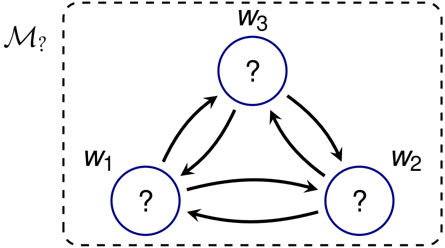

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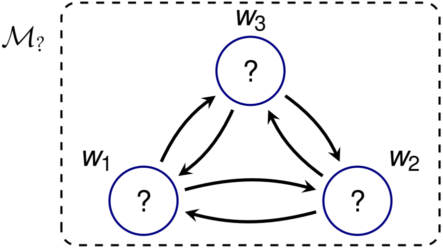

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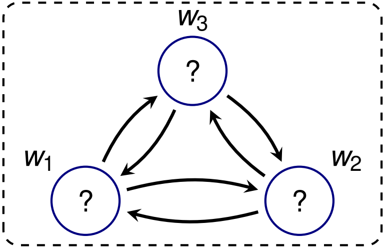

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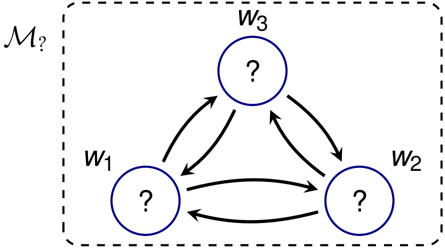

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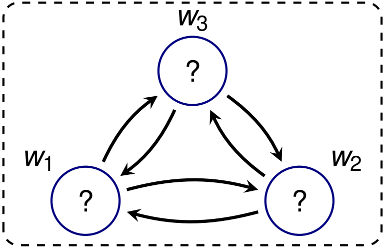

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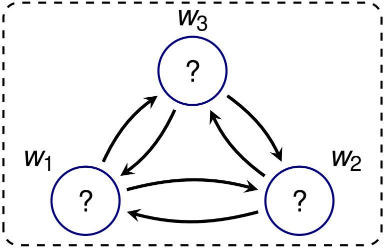

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  since $w_2 \Vdash \Diamond q$ and $w_3 \Vdash \Diamond q$
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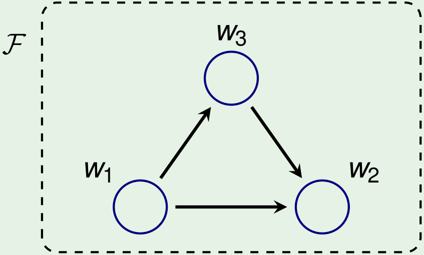
Because of the arrow configuration (always back and forth), $q \rightarrow \Box\Diamond q$ is **always valid** wherever you put the q's.

A **Frame** is a Kripke Model without Labelling

A **frame** $\mathcal{F} = (W, R)$ consists of

- W, the worlds
- R, the accessibility relation



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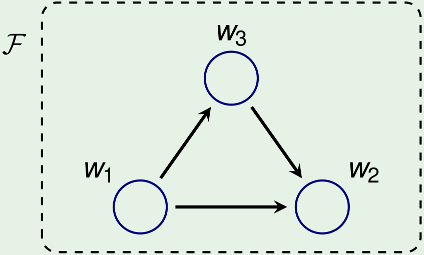

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A Kripke model $\mathcal{M}$ is a frame $\mathcal{F} = (W, R)$ plus a labelling L.

Validity in Frames

Validity in frames

A formula ϕ is **valid in frame** $\mathcal{F} = (W, R)$, denoted

$$\mathcal{F} \models \phi \ ,$$

if **for every labelling** L:

the Kripke model $\mathcal{M} = (W, R, L)$ makes ϕ true $\quad (\mathcal{M} \models \phi)$

We say that $\mathcal{M}$ is a Kripke model on $\mathcal{F}$.

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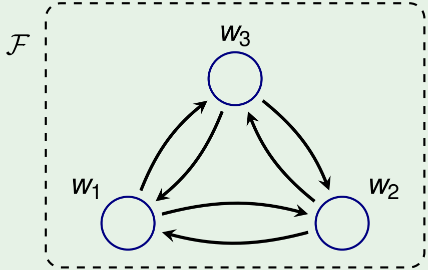

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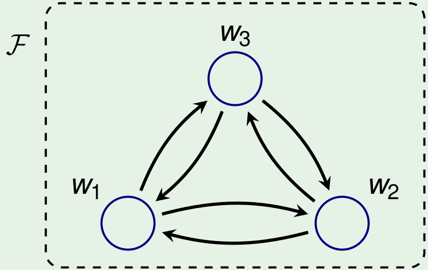

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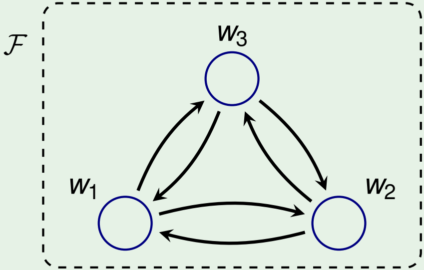

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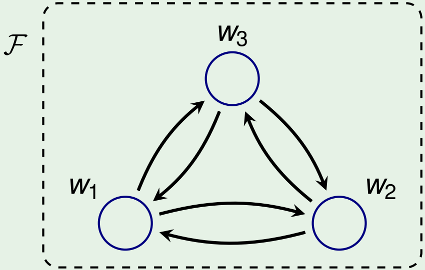

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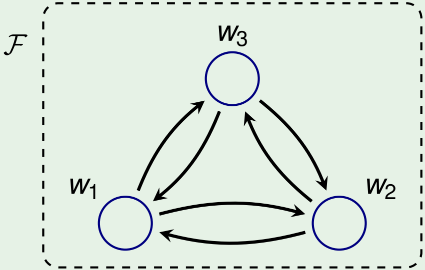

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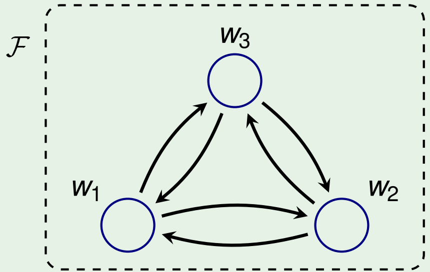

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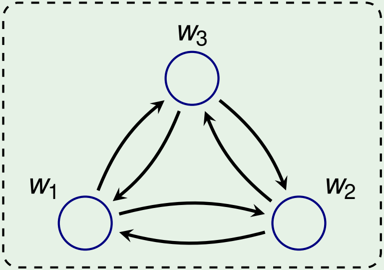
$\mathcal{F}$



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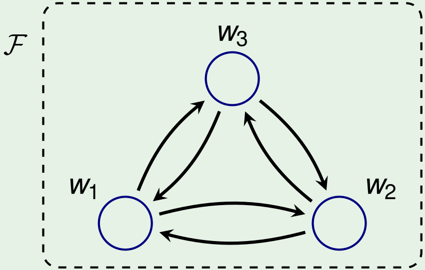

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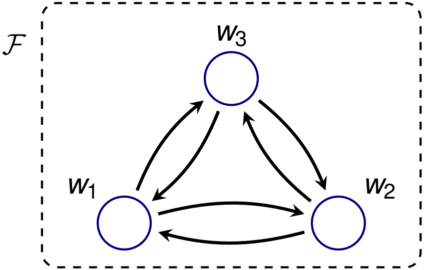
Symmetric Frames: $q \rightarrow \Box\Diamond q$

We can now make precise why it is impossible to make

$$q \rightarrow \Box\Diamond q$$

false on the frame $\mathcal{F}$.



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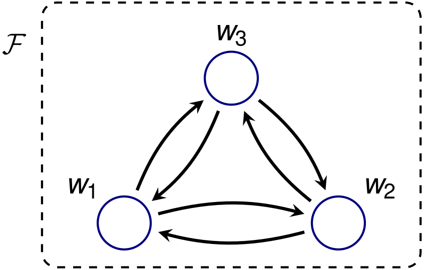

The reason is that R is **symmetric**: $\forall x\, \forall y\, \big(R(x,y) \to R(y,x)\big)$

Symmetric Frames: $q \rightarrow \Box\Diamond q$

Theorem

$$\mathcal{F} \models q \rightarrow \Box\Diamond q \quad \Longleftrightarrow \quad \text{the frame } \mathcal{F} \text{ is symmetric}$$

Note that the formula characterises a property of the frame!

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Proof ($\Leftarrow$)

Let $\mathcal{F} = \langle W, R \rangle$ be symmetric.

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Proof ($\Leftarrow$)

Let $\mathcal{F} = \langle W, R \rangle$ be symmetric. We show that $\mathcal{F} \models q \to \Box\Diamond q$.

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Let $\mathcal{F} = \langle W, R \rangle$ be symmetric. We show that $\mathcal{F} \models q \rightarrow \Box\Diamond q$.

That is: $\mathcal{M} \models q \rightarrow \Box\Diamond q$ for $\mathcal{M} = \langle W, R, L \rangle$ with arbitrary L.

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Let $x \in W$ be a world. Then (we reason as before)

- Assume that $q \notin L(x)$,
  then $x \Vdash q \to \Box\Diamond q$ since $x \nVdash q$
- Assume that $q \in L(x)$,
  then $x \Vdash q \to \Box\Diamond q$ since $x \Vdash \Box\Diamond q$
  since $x' \Vdash \Diamond q$ for all x' with $R(x, x')$
  since $x \Vdash q$

Hence $x \Vdash q \to \Box\Diamond q$.

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  - $a \nVdash \Box\Diamond q$ since $b \nVdash \Diamond q$ and $R(a, b)$
- Hence $\mathcal{F} \not\models q \to \Box\Diamond q$; a contradiction.

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Properties of a relation R

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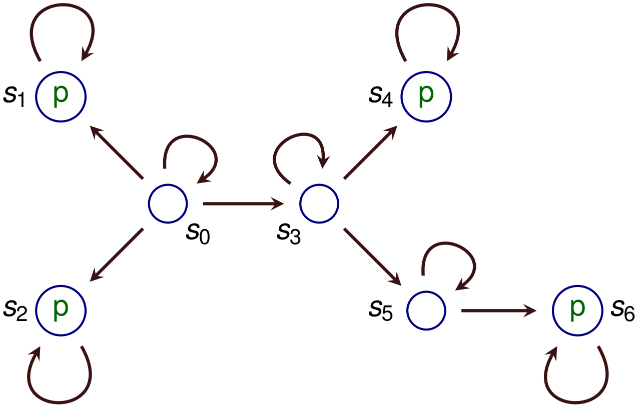
An **equivalence relation** is a relation that is

- reflexive,
- symmetric and
- transitive.

A Kripke Model on a Reflexive Frame



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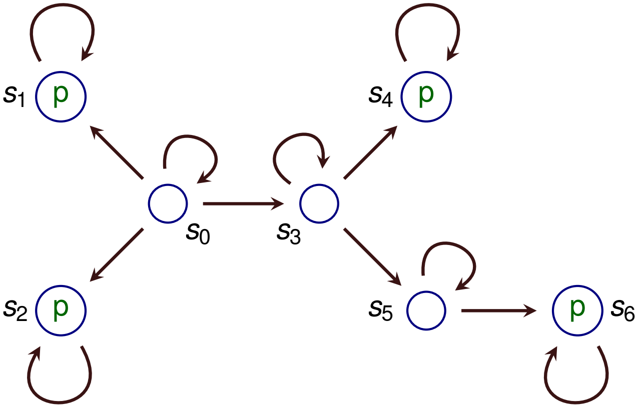

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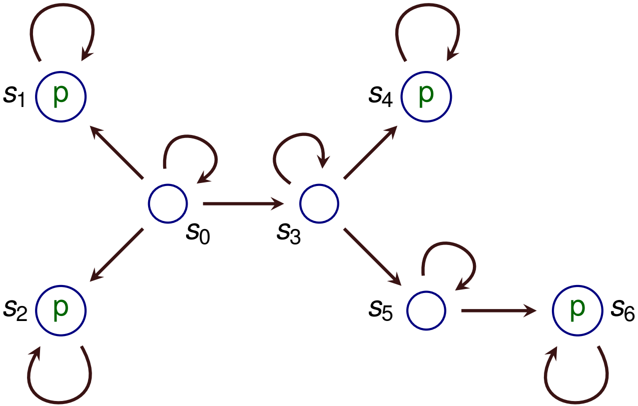

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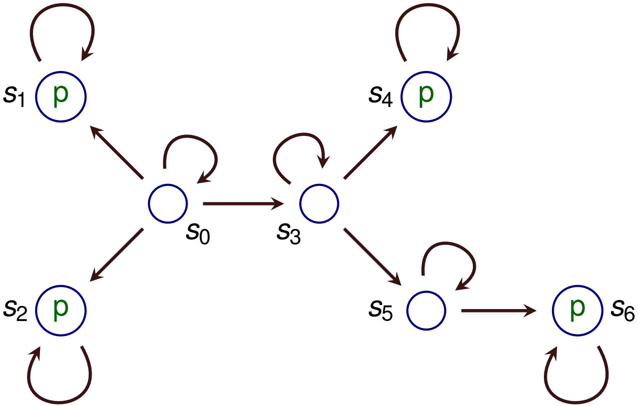

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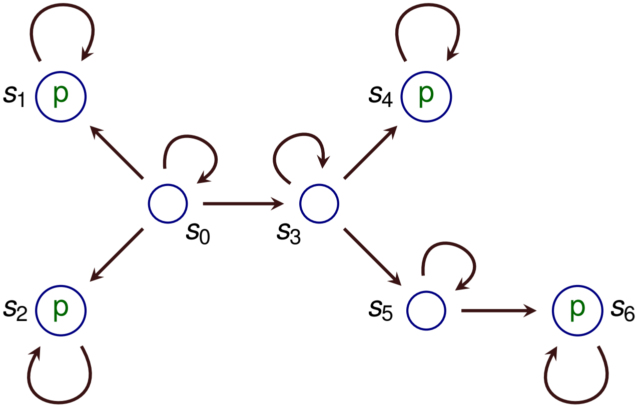

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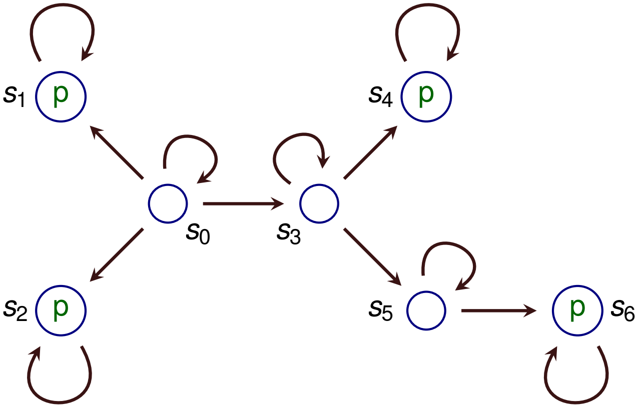

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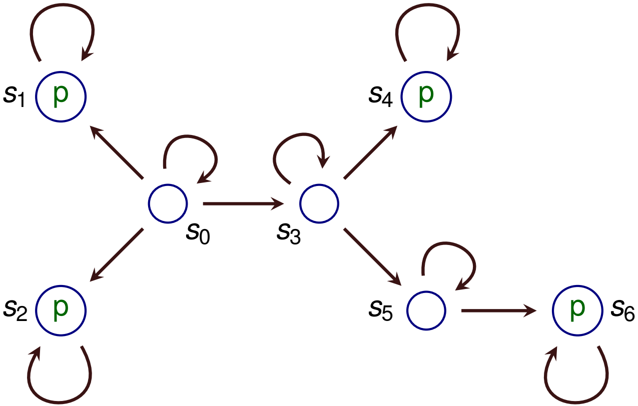

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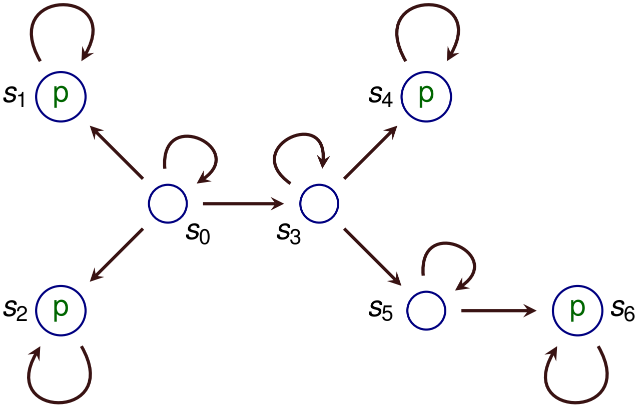

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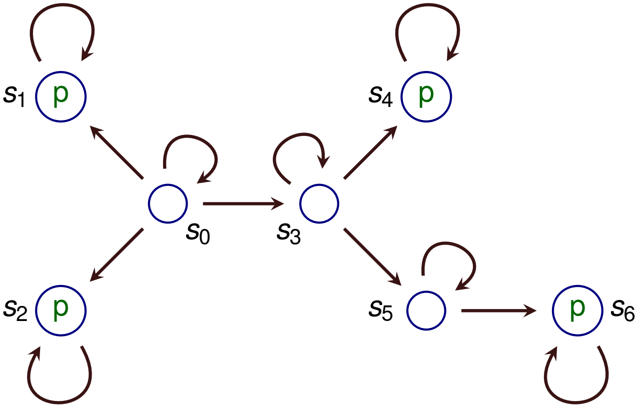

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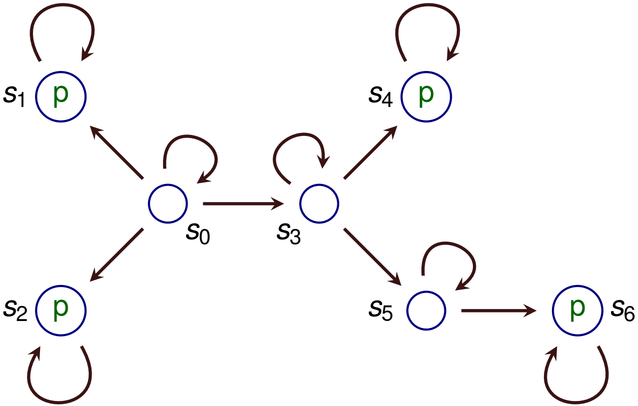

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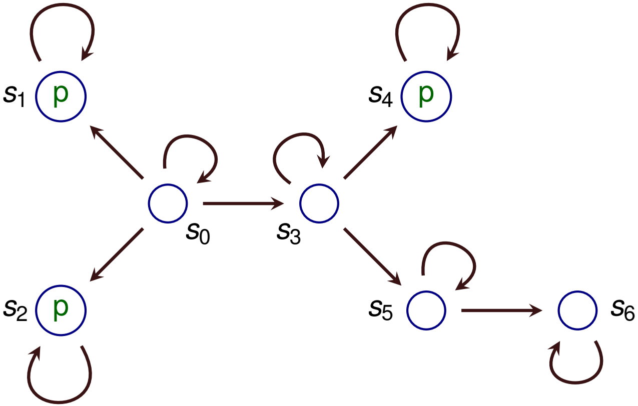

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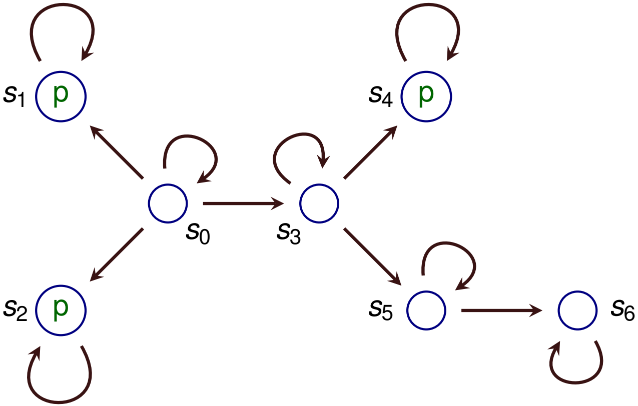

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Recall that **reflexivity** means: $\forall x\ R(x,x)$.

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Then $a \Vdash \Box p$ since p holds in all worlds $\neq a$ and $\neg R(a,a)$.

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Assume, for a contradiction, $\mathcal{F}$ was not reflexive.

Then there is a world a with $\neg R(a,a)$.

Let $\mathcal{M} = \langle W, R, L \rangle$ where L is given by:

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Reflexive Frames: $\Box p \to p$

Recall that **reflexivity** means: $\forall x\ R(x, x)$.

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Correspondence of Formulas and Frame Properties

We now know that the formula

$$\Box p \to p$$

is valid **precisely** on the reflexive frames:

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We say that the formula $\Box p \to p$ **corresponds** with the frame property **reflexivity.**

In general:

A **modal formula** ϕ **corresponds** with a **frame property** E if:

$$\mathcal{F} \models \phi \quad \Longleftrightarrow \quad \mathcal{F} \text{ has property } E$$

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Correspondences are not unique

A few alternatives:

Reflexive	$p \rightarrow \Diamond p$
Symmetric	$\Diamond\Box p \rightarrow p$
Transitive	$\Diamond\Diamond p \rightarrow \Diamond p$
Serial	$\Box p \rightarrow \Diamond p$

Transitive Frames: $\Diamond\Diamond p \rightarrow \Diamond p$

Theorem

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- Then
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  - $a \Vdash \Diamond\Diamond p$ since $b \Vdash \Diamond p$ en $R(a, b)$
  - $a \nVdash \Diamond p$ since p is valid only in c, and $\neg R(a, c)$
- Thus $\mathcal{M} \not\models \Diamond\Diamond p \to \Diamond p$; contradicting $\mathcal{F} \models \Diamond\Diamond p \to \Diamond p$.