Logic and Modelling

— Modal Logic —

Jörg Endrullis

VU University Amsterdam

Modal Logic

In propositional logic and predicate logic, the **world is static.**

Modal Logic allows to reason about **dynamics**:

- possible futures,
- knowledge and beliefs,
- different locations/worlds (with different properties),
- ...

Modalities

Modal logic introduces **modalities**

- box $\Box$
- diamond $\Diamond$

$\Box$ "Box"	$\Diamond$ "Diamond"
sure	*possibly*
always	*sometimes*
has to be	*maybe*
knows	*believes is possible*
guaranteed result	*possible result*

Modal Formulas

Modal Formulas

Modal logic extends propositional logic with

$$\Box \qquad \text{and} \qquad \Diamond$$

as unary (having one argument) connectives.

Both $\Box$ and $\Diamond$ have the same binding strength as $\neg$.

Example formulas

$$\Diamond p$$

$$\Box p \to p$$

$$\neg \Box \neg p \to \Diamond p$$

$$\Diamond p \wedge \Box \neg q$$

$$\Box (p \to q) \wedge \Diamond p$$

Modal Logic

Which of the following formulas are valid?

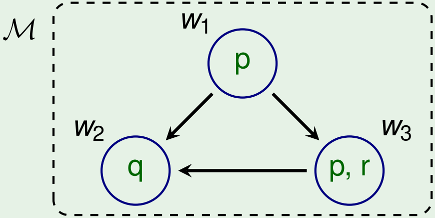
- $\Box p \leftrightarrow \neg\Diamond\neg p$
- $\Diamond p \wedge \Diamond q \rightarrow \Diamond(p \wedge q)$
- $\Box p \rightarrow \Diamond p$
- $\Box p \rightarrow p$
- $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
- $\Diamond\Diamond p \rightarrow \Diamond p$
- $\Box \neg \perp$

That depends on the interpretation of the modal operators!

Kripke Models

A **Kripke model** $\mathcal{M} = (W, R, L)$ consists of

- W, the **worlds**
- R, the **accessibility relation**
- L, the **labelling function**



Formally:

- $W = \{ w_1, \ w_2, \ w_3 \}$
- $R = \{ \langle w_1, w_2 \rangle, \ \langle w_1, w_3 \rangle, \ \langle w_3, w_2 \rangle \}$
- $L(w_1) = \{ p \} \quad L(w_2) = \{ q \} \quad L(w_3) = \{ p, \ r \}$

Kripke Models: Truth in Worlds

The notation

$$\mathcal{M}, w \Vdash \phi$$

means: formula ϕ is true in the world w of Kripke model $\mathcal{M}$.

We often abbreviate

$$\mathcal{M}, w \Vdash \phi$$

as

$$w \Vdash \phi$$

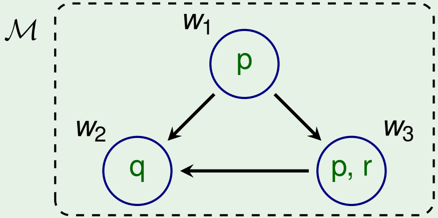
if the Kripke model $\mathcal{M}$ is clear from the context.

Kripke Models: Labelling Function

The **labelling function** L tells which propositional letters are true in which world:

$$w \Vdash p \iff p \in L(w)$$

$L(w)$ are the propositional letters that are true in world w!



$$L(w_1) = \{\, p \,\}$$
$$L(w_2) = \{\, q \,\}$$
$$L(w_3) = \{\, p,\; r \,\}$$

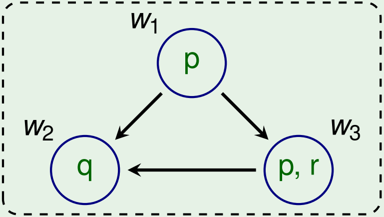
Hence

$$w_1 \Vdash p \qquad w_2 \Vdash q \qquad w_3 \Vdash p \qquad w_3 \Vdash r \qquad w_3 \nVdash q$$

Truth in Worlds

Connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ behave as in propositional logic.

$\mathcal{M}$



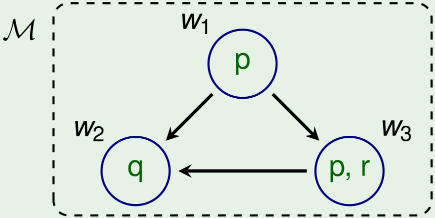
- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
- $w_2 \Vdash p \vee q$ since $w_2 \Vdash q$
- $w_2 \nVdash q \rightarrow r$ since $w_2 \Vdash q$ and $w_2 \nVdash r$
- $w_1 \Vdash q \rightarrow r$ since $w_1 \nVdash q$
- $w_3 \Vdash p \wedge r$ since $w_3 \Vdash p$ and $w_3 \Vdash r$

Truth of Diamonds: $\Diamond\phi$

$w \Vdash \Diamond\phi$

The formula $\Diamond\phi$ is true in world w if there exists a world w' such that $R(w, w')$ and ϕ is true in w'.

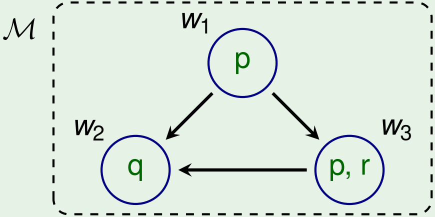
As a formula: $w \Vdash \Diamond\phi \iff \exists w' \left(R(w,w') \land w' \Vdash \phi\right)$



For example,

$w_1 \Vdash \Diamond p \qquad w_3 \nVdash \Diamond p \qquad w_1 \Vdash \Diamond q \qquad w_3 \Vdash \Diamond q$

Truth of Diamonds: $\Diamond\phi$



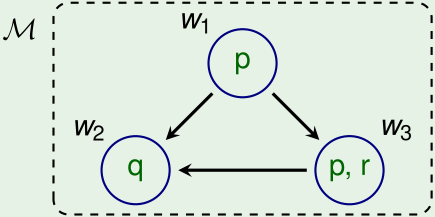
- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$
- $w_1 \Vdash \Diamond p$ since $R(w_1, w_3)$ and $w_3 \Vdash p$
- $w_1 \Vdash \Diamond p \wedge \Diamond q$ since $w_1 \Vdash \Diamond p$ en $w_1 \Vdash \Diamond q$
- $w_1 \nVdash \Diamond(p \wedge q)$ since $\neg\exists w \, \big(R(w_1, w) \wedge w \Vdash p \wedge q\big)$

Truth of Boxes: $\Box\phi$

$w \Vdash \Box\phi$

The formula $\Box\phi$ is true in world w if ϕ is true in all worlds w' with $R(w, w')$.

As a formula: $w \Vdash \Box\phi \iff \forall w' \left(R(w, w') \rightarrow w' \Vdash \phi\right)$

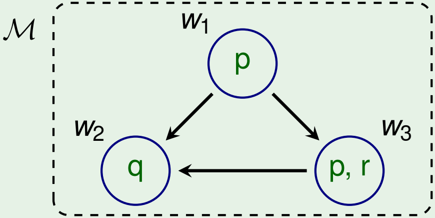


For example,

$w_1 \nVdash \Box p \qquad w_3 \Vdash \Box q \qquad w_1 \Vdash \Box(q \vee r) \qquad w_2 \Vdash \Box\bot$

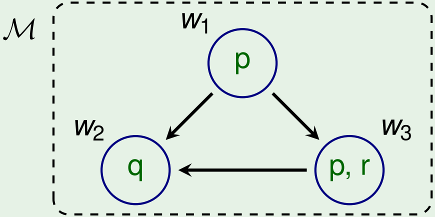
Note that $\Box\bot$ holds only in worlds without outgoing edges!

Truth of Boxes: $\Box\phi$



- $w_1 \not\Vdash \Box q$ since $R(w_1, w_3)$ and $w_3 \not\Vdash q$
- $w_1 \not\Vdash \Box p$ since $R(w_1, w_2)$ and $w_2 \not\Vdash p$
- $w_3 \Vdash \Box q$
- $w_1 \Vdash \Box(p \vee q)$
- $w_3 \not\Vdash \Box(q \wedge p)$
- $w_3 \Vdash \Box q \wedge p$
- $w_1 \Vdash p \wedge \Diamond p \wedge \neg\Box p$

Worlds without Outgoing Arrows



Note that w_2 has no outgoing arrows!

What can we say about truth of $\Diamond\phi$ and $\Box\phi$ in w_2?

- $w_2 \not\Vdash \Diamond\phi$
  $\Diamond\phi$ *never holds in worlds without outgoing arrows*
- $w_2 \Vdash \Box\phi$
  $\Box\phi$ *always holds in worlds without outgoing arrows*

This holds for whatever the formula ϕ is!

Formal Definition of Truth in Worlds

In Kripke model $\mathcal{M} = (W, R, L)$ we first define **truth per world**.

Definition of $\mathcal{M}, x \Vdash \phi$

$$
\begin{array}{rcl}
x \nVdash \bot & & \\
x \Vdash p & \Longleftrightarrow & p \in L(x) \\
x \Vdash \neg\phi & \Longleftrightarrow & x \nVdash \phi \\
x \Vdash \phi \wedge \psi & \Longleftrightarrow & x \Vdash \phi \ \text{ and } \ x \Vdash \psi \\
x \Vdash \phi \vee \psi & \Longleftrightarrow & x \Vdash \phi \ \text{ or } \ x \Vdash \psi \\
x \Vdash \phi \rightarrow \psi & \Longleftrightarrow & \text{if } \ x \Vdash \phi \ \text{ then also } \ x \Vdash \psi \\
x \Vdash \Diamond\phi & \Longleftrightarrow & \text{there exists } y \in W \text{ with } R(x, y) \text{ and } y \Vdash \phi \\
x \Vdash \Box\phi & \Longleftrightarrow & \text{for all } y \in W \text{ with } R(x, y) \text{ holds: } \ y \Vdash \phi
\end{array}
$$

Note the analogy with the truth definition in predicate logic.

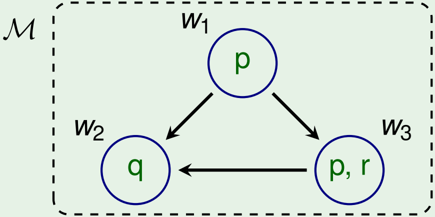
Truth in Kripke Modeles

Definition of Truth in Kripke Models

The formula ϕ is true in Kripke model $\mathcal{M} = (W, R, L)$, denoted

$$\mathcal{M} \models \phi \ ,$$

if and only if **for every world** $x \in W$ holds $x \Vdash \phi$.

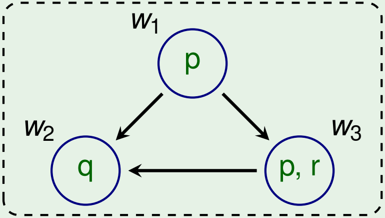


For example,

$$\mathcal{M} \not\models q \qquad \mathcal{M} \models p \vee q \qquad \mathcal{M} \models q \vee \Diamond q$$

Truth in Kripke Modeles

$\mathcal{M}$



- $\mathcal{M} \models p \rightarrow \Diamond q$
- $\mathcal{M} \models \Diamond r \rightarrow \Diamond q$
- $\mathcal{M} \not\models \Box q \rightarrow r$
- $\mathcal{M} \models p \rightarrow (q \rightarrow p)$ *
- $\mathcal{M} \models q \rightarrow \Box p$

* *all propositional tautologies also hold modal!*

Semantic Implication / Entailment

We define $\phi_1, \ldots, \phi_n \models \psi$ as

In **every world** w in **every Kripke model** $\mathcal{M}$ where

$$\mathcal{M}, w \models \phi_1 \text{ and } \ldots \text{ and } \mathcal{M}, w \models \phi_n$$

it holds also

$$\mathcal{M}, w \models \psi$$

$$\Box p \not\models \Diamond p$$

$$\Box p, \Diamond q \models \Diamond p$$

$$\Box \Diamond p \not\models \Diamond \Box p$$

$$\Diamond \Box p \not\models \Box \Diamond p$$

$$\Box (p \rightarrow q), \Diamond p \models \Diamond q$$

Modal Validity

Modal validity is semantic implication with zero premises.

Modal validity: $\models \psi$ as

In every world w in every Kripke model $\mathcal{M}$ holds $\mathcal{M}, w \models \psi$.

$$\models \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$$

$$\models \neg\Box\phi \rightarrow \Diamond\neg\phi$$

$$\models \Box\phi \vee \neg\Box\phi$$

Modal Logic Equivalence

We define $\phi \equiv \psi$ as

In every world w in every Kripke model $\mathcal{M}$

$$\mathcal{M}, w \models \phi \iff \mathcal{M}, w \models \psi$$

Alternative definition of modal equivalence

$$\phi \equiv \psi \quad \iff \quad \phi \models \psi \ \textbf{and} \ \psi \models \phi$$

$$\begin{array}{rcl}
\Box\phi & \equiv & \neg\Diamond\neg\phi \\
\Diamond\phi & \equiv & \neg\Box\neg\phi \\
\Diamond\neg\phi & \equiv & \neg\Box\phi \\
\Box(\phi \wedge \psi) & \equiv & \Box\phi \wedge \Box\psi \\
\Box(\phi \vee \psi) & \not\equiv & \Box\phi \vee \Box\psi \\
\phi \vee \psi & \equiv & \neg\phi \rightarrow \psi \ ^{\star}
\end{array}$$

*: all equivalences from propositional logic hold also modal !

Examples of Modal (Non-)Equivalence

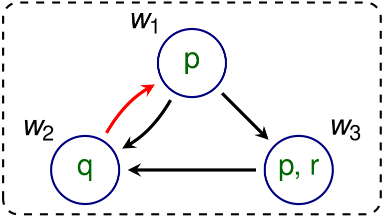
Are the following equivalences valid?

$$\neg\Diamond\phi \equiv \Diamond\neg\phi \quad \text{? No}$$

$$\Diamond(\phi \wedge \psi) \equiv \Diamond\phi \wedge \Diamond\psi \quad \text{? No}$$

$$\Diamond(\phi \vee \psi) \equiv \Diamond\phi \vee \Diamond\psi \quad \text{? Yes}$$

Exercises



- $W = \{w_1, w_2, w_3\}$
- $R = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle, \langle w_2, w_1\rangle, \langle w_3, w_2\rangle\}$
- $L(w_1) = \{p\}$
  $L(w_2) = \{q\}$
  $L(w_3) = \{p, r\}$

Check for yourself:

$w_2 \models \Box r \wedge \Box p$?

$w_1 \models \Box p$?

$w_1 \models \Diamond \Box p$?

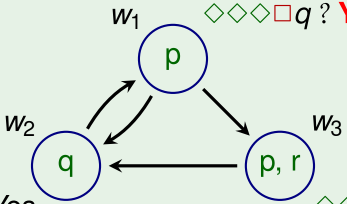
$w_1 \models \Box \Diamond p$?

q ? No

$\Box q$? No

$\Diamond\Box q$? Yes

$\Diamond\Diamond\Diamond\Box q$? **Yes**



$\Diamond\Diamond\Box q$? Yes

$\Diamond\Box q$? No

$\Box q$? No

q ? Yes

$\Diamond\Diamond\Box q$?

$\Box q$? Yes

q ? No

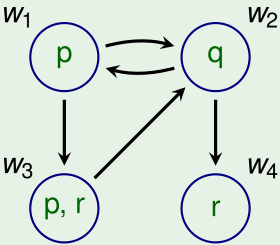
How to evaluate complex formulas?

$w_1 \models \Diamond\Diamond\Diamond\Box q$? Yes

Often it is helps to annotate the models!

Exam Preparation Exercises

Example

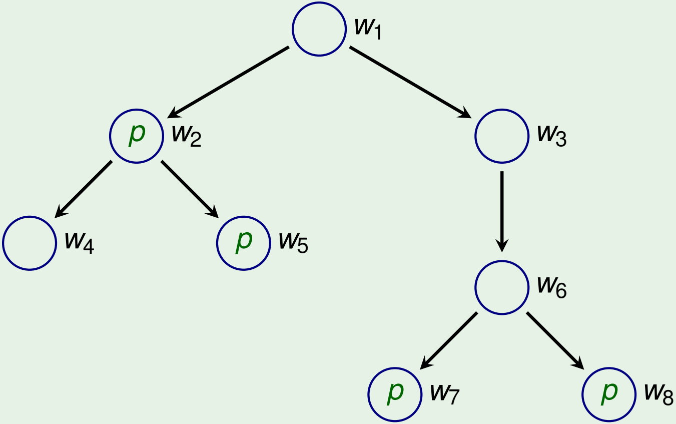


Determine the truth value of every formula in every world:

$\Diamond\Box q$	?
$\Diamond\Diamond\Box q$	?
$\Box\Diamond\Box(q \vee r)$	?
$\Diamond(\Box(q \vee r) \rightarrow p)$	?
$\Box(\Diamond p \rightarrow \Diamond\Diamond r)$	?

Exam Preparation Exercises

Example

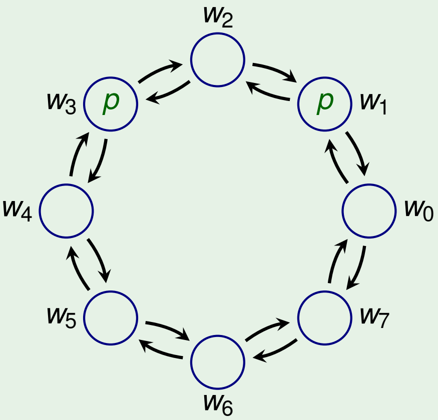


Determine in which worlds the following formula holds:

$$\Box\Diamond\Box\Diamond p \qquad ?$$

Exam Preparation Exercises

Example

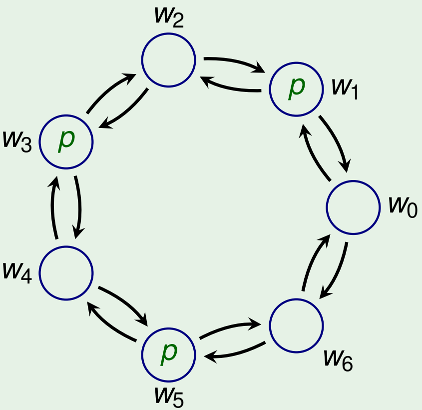


Determine in which worlds the following formula holds:

$\Diamond\Diamond\Diamond\Diamond\Box p$?

Exam Preparation Exercises

Example



Determine in which worlds the following formula holds:

$\Diamond\Diamond\Diamond\Box\Box p$?