Logic and Modelling

— Modal Logic —

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Modal Logic

In propositional logic and predicate logic, the **world is static.**

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Modal Logic allows to reason about **dynamics**:

- possible futures,
- knowledge and beliefs,
- different locations/worlds (with different properties),
- …

Modalities

Modal logic introduces **modalities**

- box $\Box$
- diamond $\Diamond$

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Modal logic introduces **modalities**

- box $\Box$
- diamond $\Diamond$

$\Box$	$\Diamond$
"Box"	"Diamond"
sure	*possibly*
always	*sometimes*
has to be	*maybe*
knows	*believes is possible*
guaranteed result	*possible result*

Modal Formulas

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Modal logic extends propositional logic with

$$\Box \qquad\qquad \text{and} \qquad\qquad \Diamond$$

as unary (having one argument) connectives.

Both $\Box$ and $\Diamond$ have the same binding strength as $\neg$.

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Example formulas

$$\Diamond p$$

$$\Box p \to p$$

$$\neg\Box\neg p \to \Diamond p$$

$$\Diamond p \land \Box\neg q$$

$$\Box(p \to q) \land \Diamond p$$

Modal Logic

Which of the following formulas are valid?

- $\Box p \leftrightarrow \neg \Diamond \neg p$
- $\Diamond p \wedge \Diamond q \rightarrow \Diamond (p \wedge q)$
- $\Box p \rightarrow \Diamond p$
- $\Box p \rightarrow p$
- $\Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
- $\Diamond \Diamond p \rightarrow \Diamond p$
- $\Box \neg \perp$

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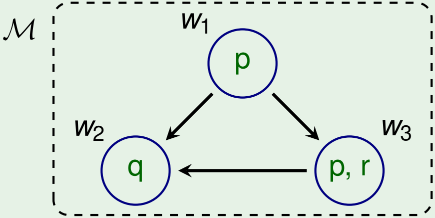
- $\Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
- $\Diamond \Diamond p \rightarrow \Diamond p$
- $\Box \neg \bot$

That depends on the interpretation of the modal operators!

Kripke Models

A **Kripke model** $\mathcal{M} = (W, R, L)$ consists of

- W, the **worlds**
- R, the **accessibility relation**
- L, the **labelling function**



Formally:

- $W = \{ w_1, w_2, w_3 \}$
- $R = \{ \langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_3, w_2 \rangle \}$
- $L(w_1) = \{ p \}$ $\quad L(w_2) = \{ q \}$ $\quad L(w_3) = \{ p, r \}$

Kripke Models: Truth in Worlds

The notation

$$\mathcal{M}, w \Vdash \phi$$

means: formula ϕ is true in the world w of Kripke model $\mathcal{M}$.

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We often abbreviate

$$\mathcal{M}, w \Vdash \phi$$

as

$$w \Vdash \phi$$

if the Kripke model $\mathcal{M}$ is clear from the context.

Kripke Models: Labelling Function

The **labelling function** L tells which propositional letters are true in which world:

$$w \Vdash p \iff p \in L(w)$$

$L(w)$ are the propositional letters that are true in world w!

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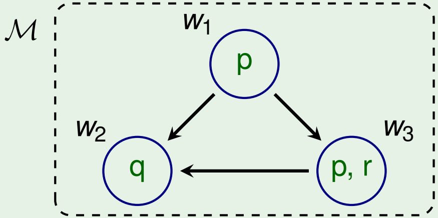

$L(w_1) = \{ p \}$
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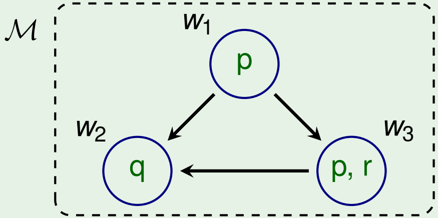

$L(w_1) = \{\ p\ \}$
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Hence

$w_1 \Vdash p \qquad w_2 \Vdash q \qquad w_3 \Vdash p \qquad w_3 \Vdash r$

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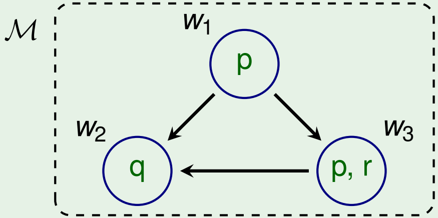

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Hence

$w_1 \Vdash p \qquad w_2 \Vdash q \qquad w_3 \Vdash p \qquad w_3 \Vdash r \qquad w_3 \nVdash q$

Truth in Worlds

Connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ behave as in propositional logic.

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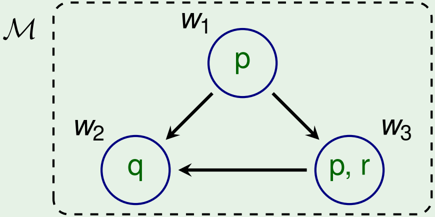

- $w_1 \quad \neg q$

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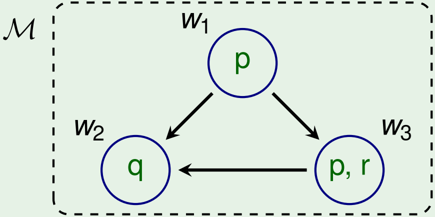

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$

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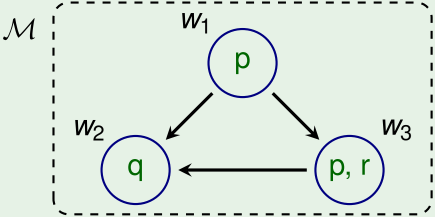

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
- $w_2 \quad p \vee q$

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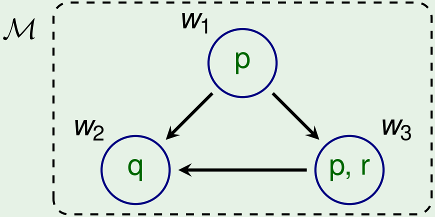

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
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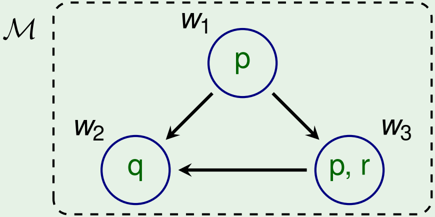

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
- $w_2 \Vdash p \vee q$ since $w_2 \Vdash q$
- $w_2 \quad q \rightarrow r$

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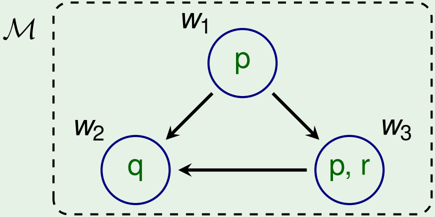

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
- $w_2 \Vdash p \vee q$ since $w_2 \Vdash q$
- $w_2 \nVdash q \rightarrow r$ since $w_2 \Vdash q$ and $w_2 \nVdash r$

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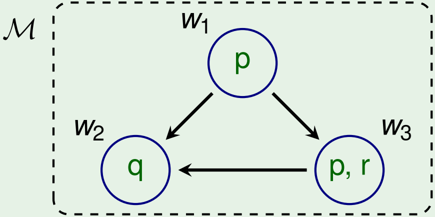

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- $w_2 \nVdash q \rightarrow r$ since $w_2 \Vdash q$ and $w_2 \nVdash r$
- $w_1 \quad q \rightarrow r$

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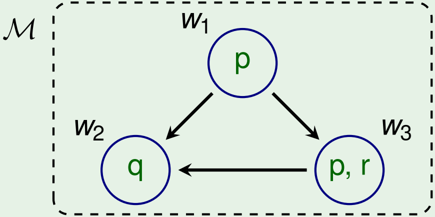

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
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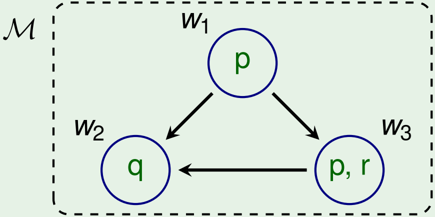

- $w_1 \Vdash \neg q$ since $w_1 \nVdash q$
- $w_2 \Vdash p \vee q$ since $w_2 \Vdash q$
- $w_2 \nVdash q \rightarrow r$ since $w_2 \Vdash q$ and $w_2 \nVdash r$
- $w_1 \Vdash q \rightarrow r$ since $w_1 \nVdash q$
- $w_3 \quad p \wedge r$

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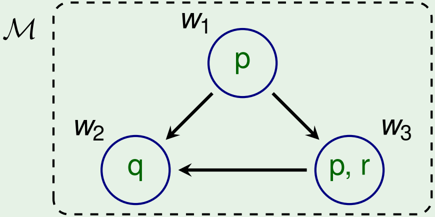

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- $w_3 \Vdash p \wedge r$ since $w_3 \Vdash p$ and $w_3 \Vdash r$

Truth of Diamonds: $\Diamond\phi$

$w \Vdash \Diamond\phi$

The formula $\Diamond\phi$ is true in world w if there exists a world w' such that $R(w, w')$ and ϕ is true in w'.

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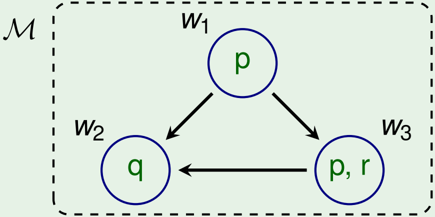

For example,

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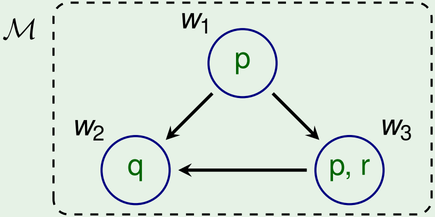

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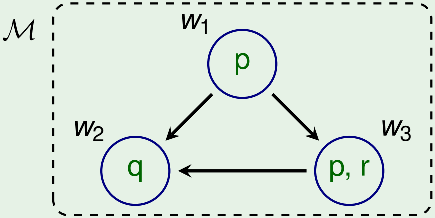

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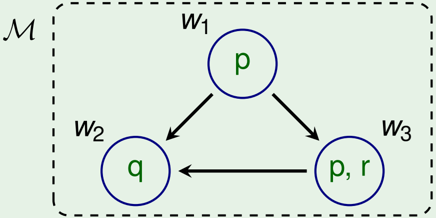

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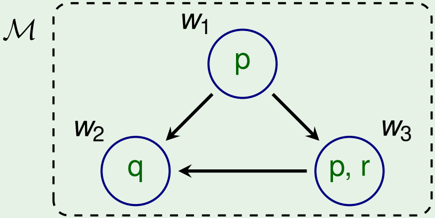

For example,

$w_1 \Vdash \Diamond p \qquad w_3 \not\Vdash \Diamond p$

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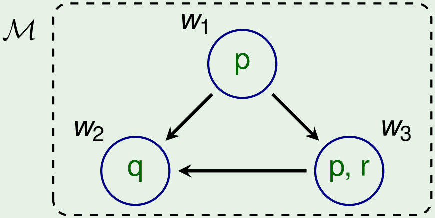

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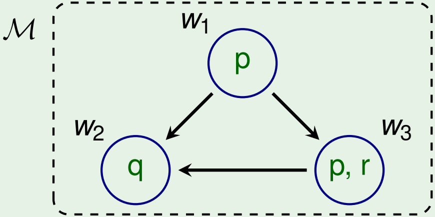

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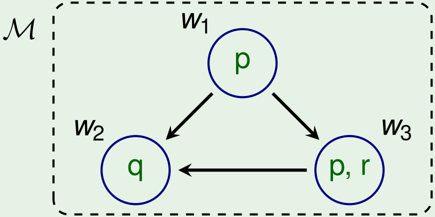

For example,

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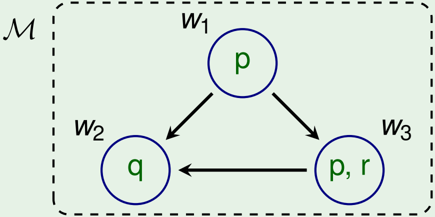
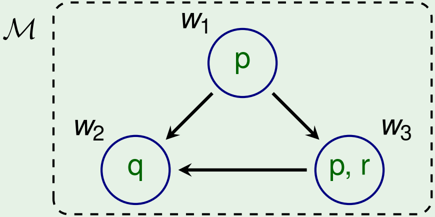


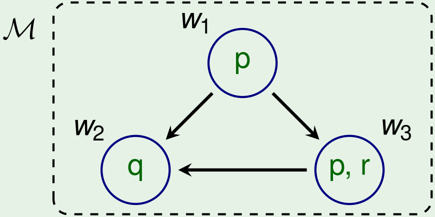
For example,

$w_1 \Vdash \Diamond p \qquad w_3 \nVdash \Diamond p \qquad w_1 \Vdash \Diamond q \qquad w_3 \Vdash \Diamond q$

Truth of Diamonds: $\Diamond\phi$

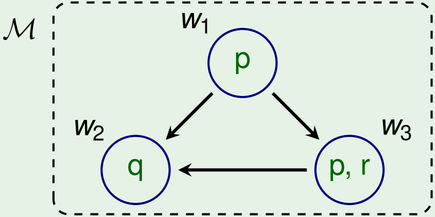


Truth of Diamonds: $\Diamond\phi$



- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$

Truth of Diamonds: $\Diamond\phi$



- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$
- $w_1 \Vdash \Diamond p$ since $R(w_1, w_3)$ and $w_3 \Vdash p$

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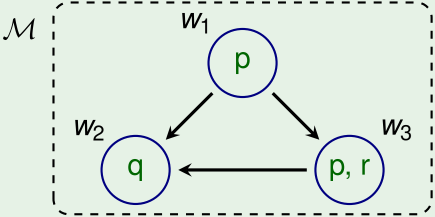
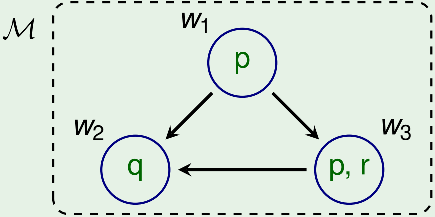


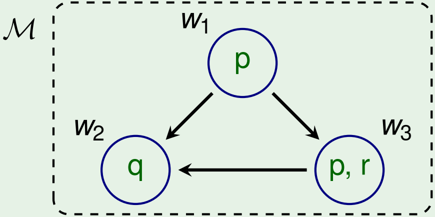
- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$
- $w_1 \Vdash \Diamond p$ since $R(w_1, w_3)$ and $w_3 \Vdash p$
- $w_1 \quad \Diamond p \wedge \Diamond q$

Truth of Diamonds: $\Diamond\phi$



- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$
- $w_1 \Vdash \Diamond p$ since $R(w_1, w_3)$ and $w_3 \Vdash p$
- $w_1 \Vdash \Diamond p \land \Diamond q$ since $w_1 \Vdash \Diamond p$ en $w_1 \Vdash \Diamond q$

Truth of Diamonds: $\Diamond\phi$



- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$
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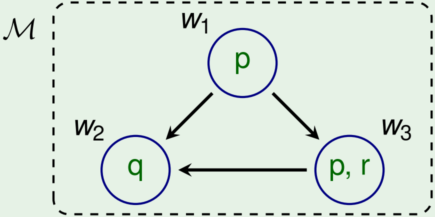
- $w_1 \Vdash \Diamond p \wedge \Diamond q$ since $w_1 \Vdash \Diamond p$ en $w_1 \Vdash \Diamond q$
- $w_1 \quad \Diamond(p \wedge q)$

Truth of Diamonds: $\Diamond\phi$



- $w_1 \Vdash \Diamond q$ since $R(w_1, w_2)$ and $w_2 \Vdash q$
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- $w_1 \Vdash \Diamond p \wedge \Diamond q$ since $w_1 \Vdash \Diamond p$ en $w_1 \Vdash \Diamond q$
- $w_1 \nVdash \Diamond(p \wedge q)$ since $\neg\exists w \, \big(R(w_1, w) \wedge w \Vdash p \wedge q\big)$

Truth of Boxes: $\Box\phi$

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The formula $\Box\phi$ is true in world w if ϕ is true in all worlds w' with $R(w, w')$.

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As a formula: $w \Vdash \Box\phi \iff \forall w' \left(R(w, w') \rightarrow w' \Vdash \phi\right)$

Truth of Boxes: $\Box\phi$

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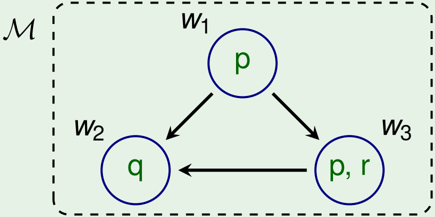
The formula $\Box\phi$ is true in world w if ϕ is true in all worlds w' with $R(w, w')$.

As a formula: $w \Vdash \Box\phi \iff \forall w' \left(R(w, w') \to w' \Vdash \phi\right)$



For example,

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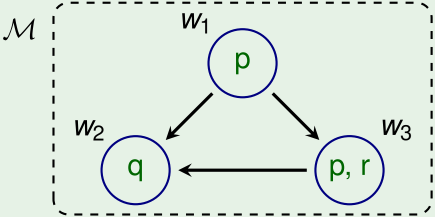

For example,

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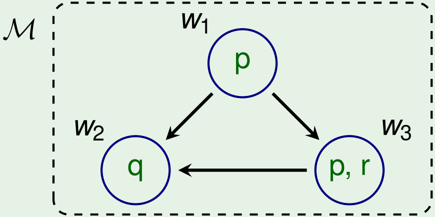

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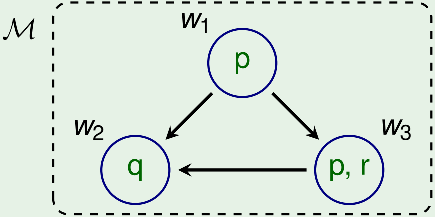

For example,

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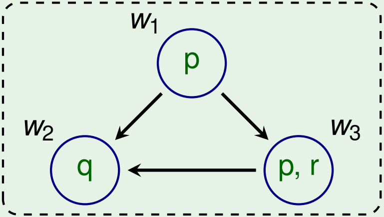
$\mathcal{M}$



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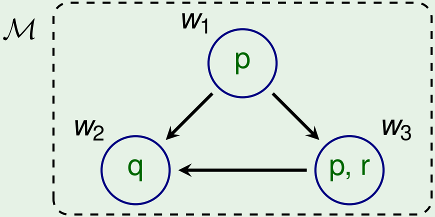

For example,

$w_1 \not\Vdash \Box p \qquad w_3 \Vdash \Box q \qquad w_1 \quad \Box(q \vee r)$

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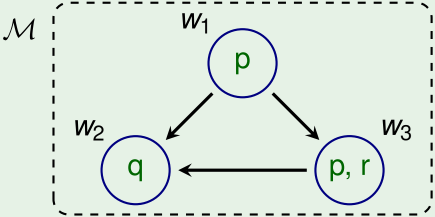

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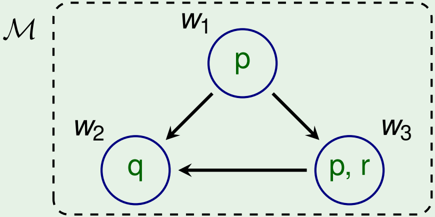

For example,

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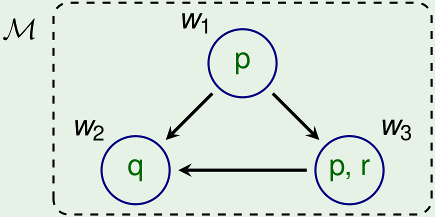

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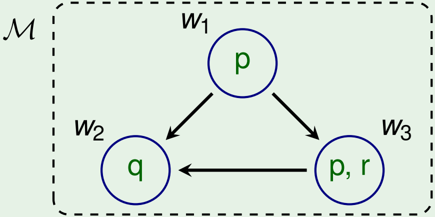
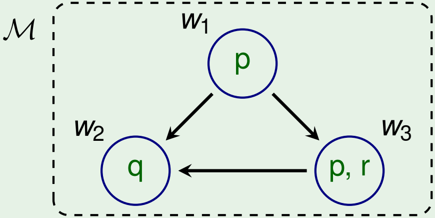


For example,

$w_1 \nVdash \Box p \qquad w_3 \Vdash \Box q \qquad w_1 \Vdash \Box(q \vee r) \qquad w_2 \Vdash \Box\bot$

Note that $\Box\bot$ holds only in worlds without outgoing edges!

Truth of Boxes: $\Box\phi$



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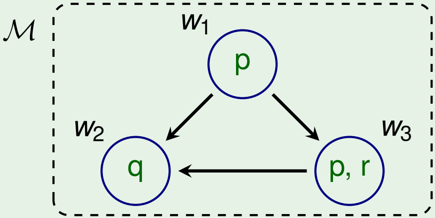

- $w_1 \not\Vdash \Box q$ since $R(w_1, w_3)$ and $w_3 \not\Vdash q$
- $w_1 \not\Vdash \Box p$ since $R(w_1, w_2)$ and $w_2 \not\Vdash p$
- $w_3 \Vdash \Box q$

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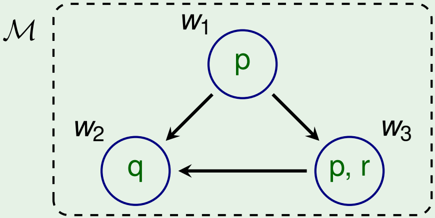

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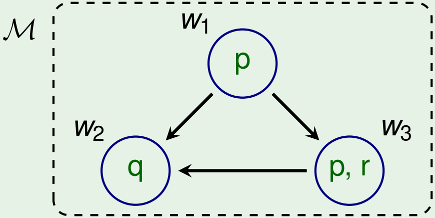

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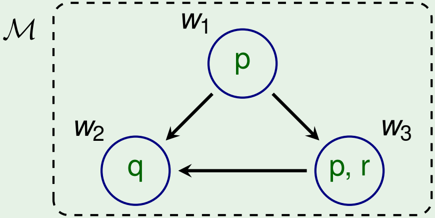

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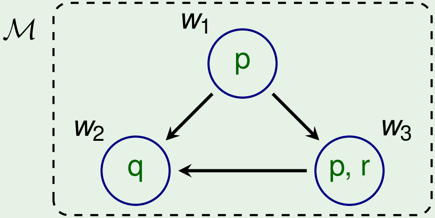
- $w_1 \Vdash \Box(p \vee q)$
- $w_3 \quad \Box(q \wedge p)$

Truth of Boxes: $\Box\phi$



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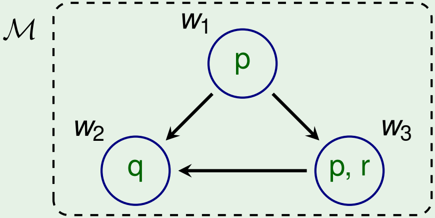

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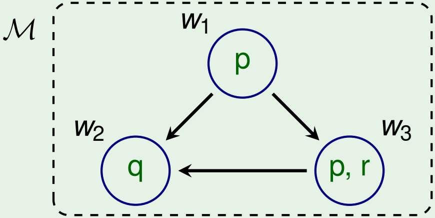

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- $w_3 \Vdash \Box q$
- $w_1 \Vdash \Box(p \vee q)$
- $w_3 \not\Vdash \Box(q \wedge p)$
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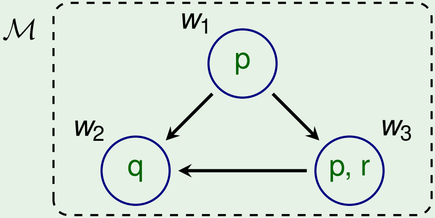
Truth of Boxes: $\Box\phi$



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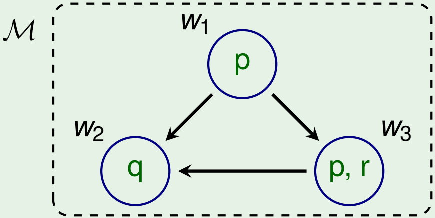
- $w_1 \Vdash \Box(p \vee q)$
- $w_3 \nVdash \Box(q \wedge p)$
- $w_3 \Vdash \Box q \wedge p$
- $w_1 \quad p \wedge \Diamond p \wedge \neg\Box p$

Truth of Boxes: $\Box\phi$



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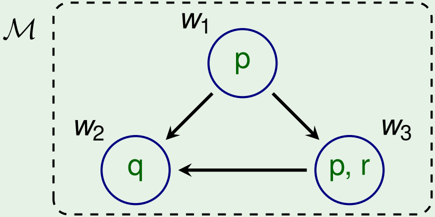
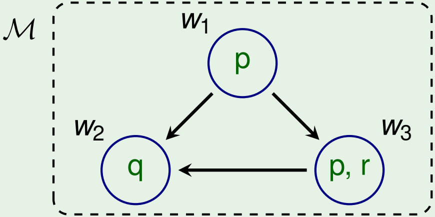
Worlds without Outgoing Arrows



Note that w_2 has no outgoing arrows!

Worlds without Outgoing Arrows



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What can we say about truth of $\Diamond\phi$ and $\Box\phi$ in w_2?

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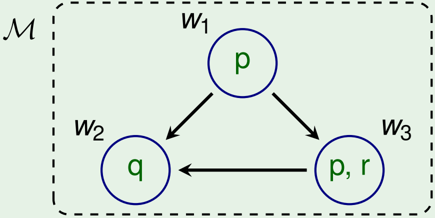

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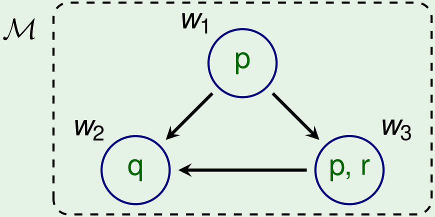

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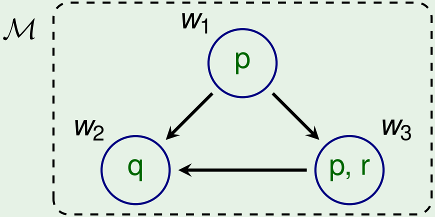

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What can we say about truth of $\Diamond\phi$ and $\Box\phi$ in w_2?

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  $\Diamond\phi$ never holds in worlds without outgoing arrows
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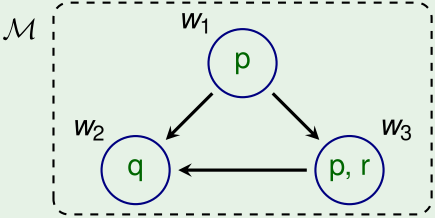
Worlds without Outgoing Arrows



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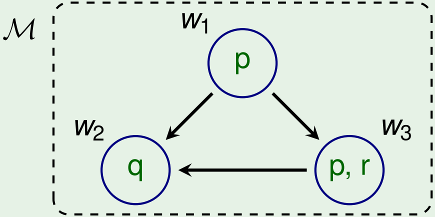
  $\Box\phi$ always holds in worlds without outgoing arrows

Worlds without Outgoing Arrows



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  $\Diamond\phi$ *never holds in worlds without outgoing arrows*

- $w_2 \Vdash \Box\phi$

  $\Box\phi$ *always holds in worlds without outgoing arrows*

This holds for whatever the formula ϕ is!

Formal Definition of Truth in Worlds

In Kripke model $\mathcal{M} = (W, R, L)$ we first define **truth per world**.

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Definition of $\mathcal{M}, x \Vdash \phi$

$x \nVdash \perp$

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Definition of $\mathcal{M}, x \Vdash \phi$

$$x \nVdash \bot$$

$$x \Vdash p \quad \Longleftrightarrow \quad p \in L(x)$$

Formal Definition of Truth in Worlds

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Definition of $\mathcal{M}, x \Vdash \phi$

$$
\begin{aligned}
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x \Vdash \phi \wedge \psi &\iff x \Vdash \phi \text{ and } x \Vdash \psi
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x \Vdash \phi \rightarrow \psi & \Longleftrightarrow & \text{if } \ x \Vdash \phi \ \text{ then also } \ x \Vdash \psi
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x \Vdash \phi \to \psi & \iff & \text{if } \ x \Vdash \phi \ \text{ then also } \ x \Vdash \psi \\
x \Vdash \Diamond\phi & \iff & \text{there exists } y \in W \text{ with } R(x, y) \text{ and } y \Vdash \phi
\end{array}
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x \Vdash \Box\phi & \iff & \text{for all } y \in W \text{ with } R(x, y) \text{ holds: } y \Vdash \phi
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\end{array}
$$

Note the analogy with the truth definition in predicate logic.

Truth in Kripke Modeles

Definition of Truth in Kripke Models

The formula ϕ is true in Kripke model $\mathcal{M} = (W, R, L)$, denoted

$$\mathcal{M} \models \phi \; ,$$

if and only if **for every world** $x \in W$ holds $x \Vdash \phi$.

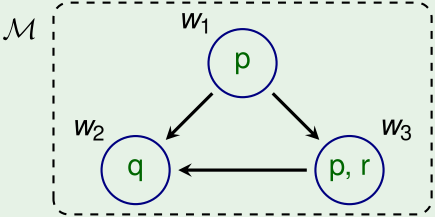
Truth in Kripke Modeles

Definition of Truth in Kripke Models

The formula ϕ is true in Kripke model $\mathcal{M} = (W, R, L)$, denoted

$$\mathcal{M} \models \phi \, ,$$

if and only if **for every world** $x \in W$ holds $x \Vdash \phi$.



For example,

$\mathcal{M} \quad q$

Truth in Kripke Modeles

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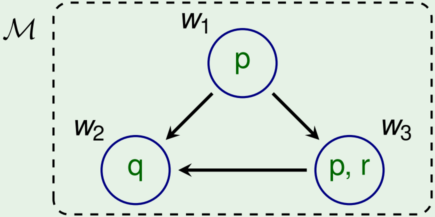

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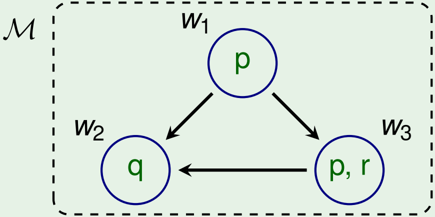

For example,

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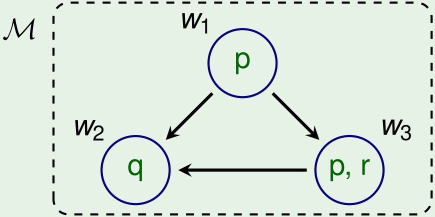

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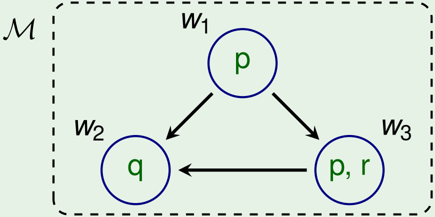

For example,

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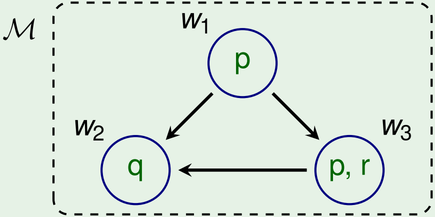

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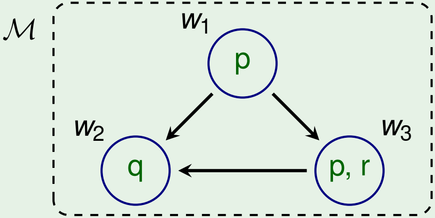

- $\mathcal{M} \quad p \to \Diamond q$

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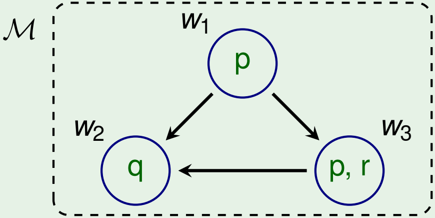

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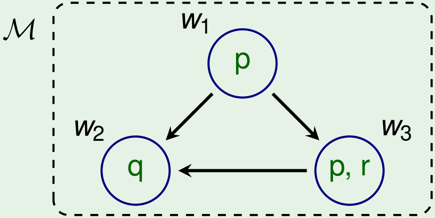

- $\mathcal{M} \models p \to \Diamond q$
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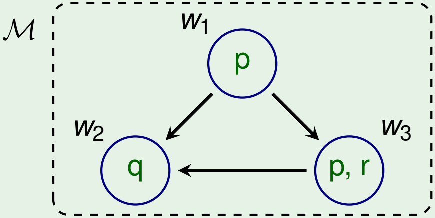
Truth in Kripke Modeles



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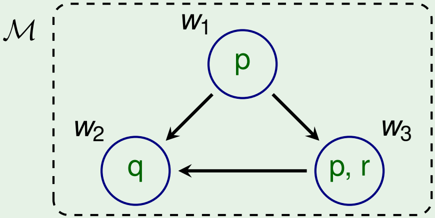

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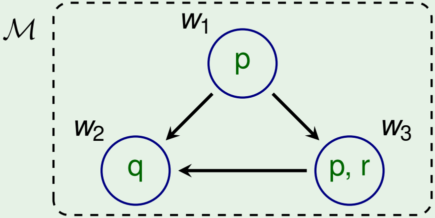
Truth in Kripke Modeles



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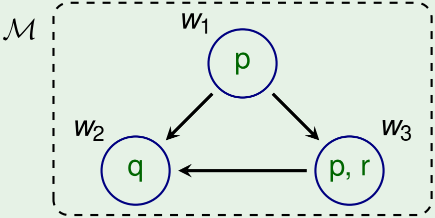

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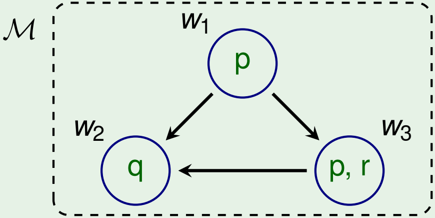

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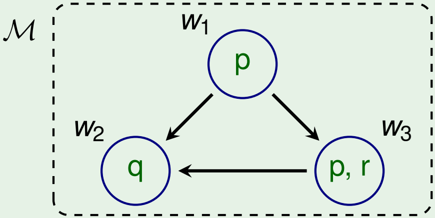

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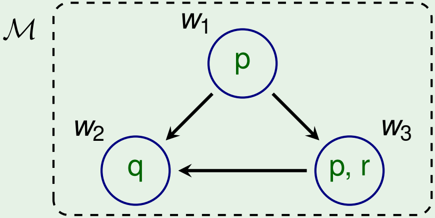

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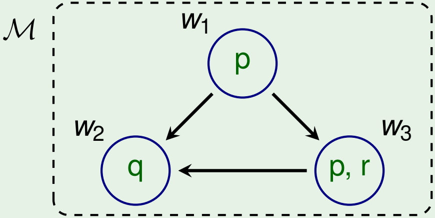

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* *all propositional tautologies also hold modal!*

Semantic Implication / Entailment

We define $\phi_1, \ldots, \phi_n \models \psi$ as

In **every world** w in **every Kripke model** $\mathcal{M}$ where

$$\mathcal{M}, w \models \phi_1 \text{ and } \ldots \text{ and } \mathcal{M}, w \models \phi_n$$

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$$\Diamond\Box p \not\models \Box\Diamond p$$

$$\Box(p \to q), \Diamond p \qquad \Diamond q$$

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Modal Validity

Modal validity is semantic implication with zero premises.

Modal validity: $\models \psi$ as

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We define $\phi \equiv \psi$ as

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In every world *w* in every Kripke model $\mathcal{M}$

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Alternative definition of modal equivalence

$$\phi \equiv \psi \quad \iff \quad \phi \models \psi \quad \textbf{and} \quad \psi \models \phi$$

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$$\begin{aligned} \Box\phi &\equiv \neg\Diamond\neg\phi \\ \Diamond\phi &\equiv \neg\Box\neg\phi \\ \Diamond\neg\phi &\equiv \neg\Box\phi \end{aligned}$$

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\phi \vee \psi &\equiv \neg\phi \rightarrow \psi \quad ^\star
\end{aligned}$$

*: all equivalences from propositional logic hold also modal !

Examples of Modal (Non-)Equivalence

Are the following equivalences valid?

$$\neg\Diamond\phi \quad \equiv \quad \Diamond\neg\phi \quad ?$$

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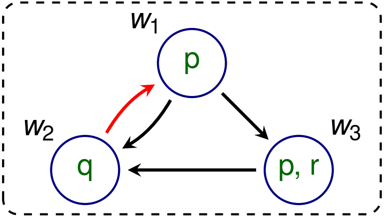
Exercises



- $W = \{w_1, w_2, w_3\}$
- $R = \{\langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_2, w_1 \rangle, \langle w_3, w_2 \rangle\}$
- $L(w_1) = \{p\}$
  $L(w_2) = \{q\}$
  $L(w_3) = \{p, r\}$

Check for yourself:

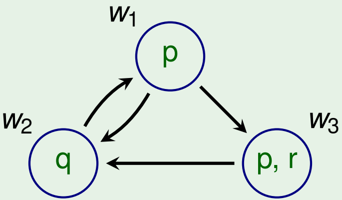
$w_2 \models \Box r \wedge \Box p$?

$w_1 \models \Box p$?

$w_1 \models \Diamond \Box p$?

$w_1 \models \Box \Diamond p$?

Exercises



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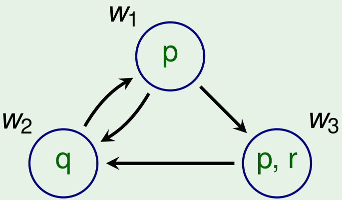
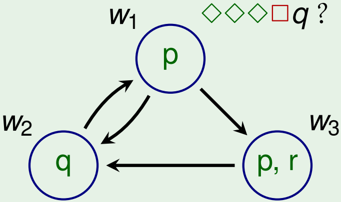


How to evaluate complex formulas?

$$w_1 \models \Diamond\Diamond\Diamond\Box q \qquad ?$$

Exercises

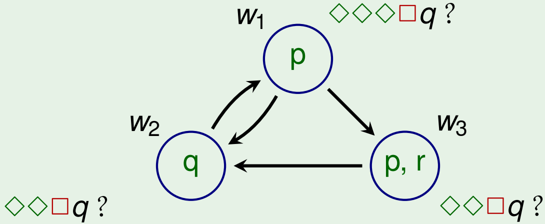


How to evaluate complex formulas?

$$w_1 \models \Diamond\Diamond\Diamond\Box q \qquad ?$$

Often it is helps to annotate the models!

Exercises



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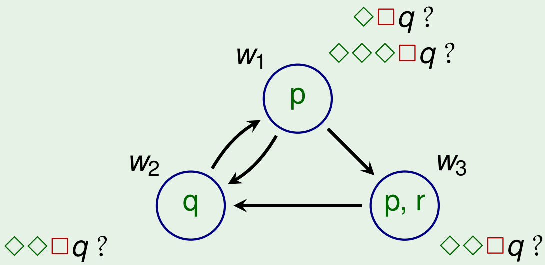

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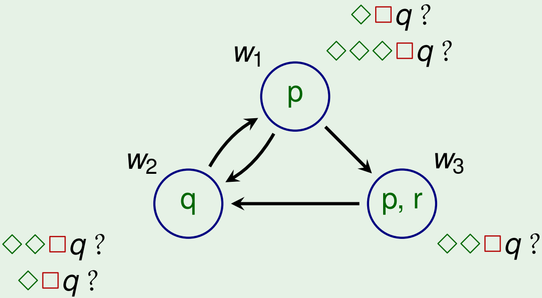

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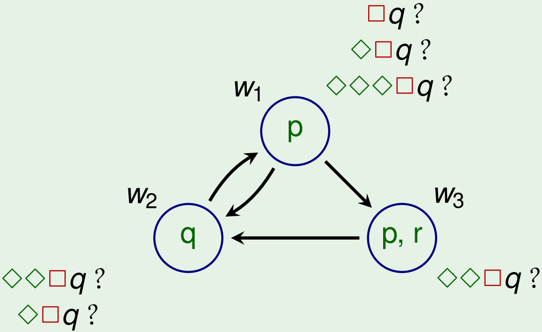

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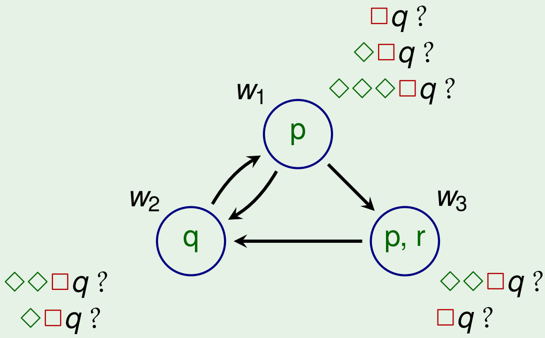

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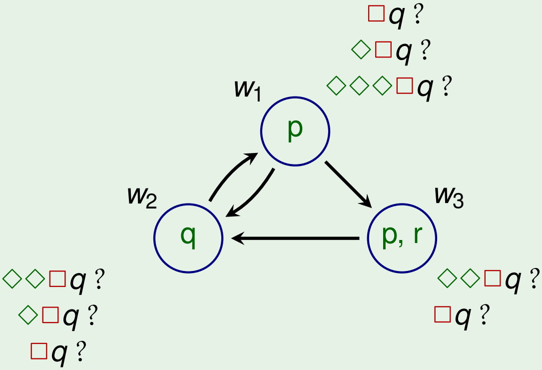

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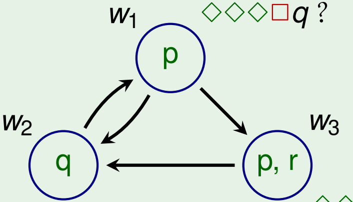
Exercises

q ?
$\Box q$?
$\Diamond\Box q$?
$\Diamond\Diamond\Diamond\Box q$?



$\Diamond\Diamond\Box q$?
$\Diamond\Box q$?
$\Box q$?
q ?

$\Diamond\Diamond\Box q$?
$\Box q$?
q ?

How to evaluate complex formulas?

$$w_1 \models \Diamond\Diamond\Diamond\Box q \qquad ?$$

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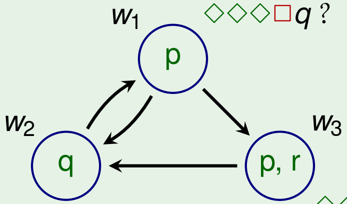
Exercises

q ? No

$\Box q$?

$\Diamond\Box q$?

$\Diamond\Diamond\Diamond\Box q$?



$\Diamond\Diamond\Box q$?

$\Diamond\Box q$?

$\Box q$?

q ? Yes

$\Diamond\Diamond\Box q$?

$\Box q$?

q ? No

How to evaluate complex formulas?

$$w_1 \models \Diamond\Diamond\Diamond\Box q \qquad ?$$

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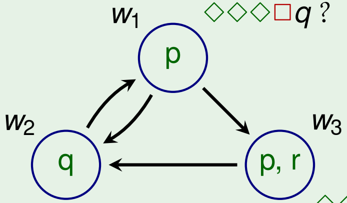
Exercises

q ? No

$\Box q$? No

$\Diamond \Box q$?

$\Diamond\Diamond\Diamond \Box q$?



$\Diamond\Diamond \Box q$?

$\Diamond \Box q$?

$\Box q$? No

q ? Yes

$\Diamond\Diamond \Box q$?

$\Box q$?

q ? No

How to evaluate complex formulas?

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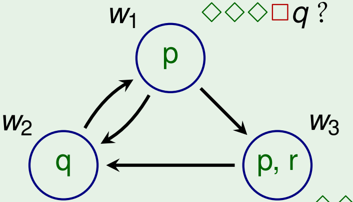
q ? No

$\Box q$? No

$\Diamond\Box q$?

$\Diamond\Diamond\Diamond\Box q$?



$\Diamond\Diamond\Box q$?

$\Diamond\Box q$?

$\Box q$? No

q ? Yes

$\Diamond\Diamond\Box q$?

$\Box q$? Yes

q ? No

How to evaluate complex formulas?

$$w_1 \models \Diamond\Diamond\Diamond\Box q \qquad ?$$

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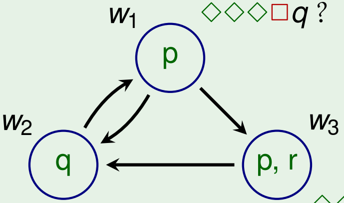
Exercises

q ? No
$\Box q$? No
$\Diamond\Box q$? Yes
$\Diamond\Diamond\Diamond\Box q$?



$\Diamond\Diamond\Box q$?
$\Diamond\Box q$?
$\Box q$? No
q ? Yes

$\Diamond\Diamond\Box q$?
$\Box q$? Yes
q ? No

How to evaluate complex formulas?

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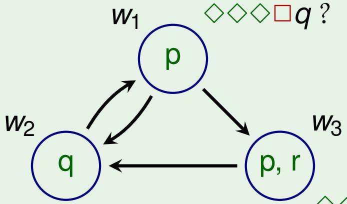
Exercises

q ? No
$\Box q$? No
$\Diamond\Box q$? Yes
$\Diamond\Diamond\Diamond\Box q$?



$\Diamond\Diamond\Box q$?
$\Diamond\Box q$? No
$\Box q$? No
q ? Yes

$\Diamond\Diamond\Box q$?
$\Box q$? Yes
q ? No

How to evaluate complex formulas?

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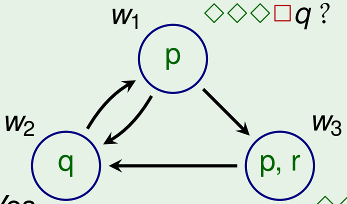
Exercises

q ? No
$\Box q$? No
$\Diamond\Box q$? Yes
$\Diamond\Diamond\Diamond\Box q$?



$\Diamond\Diamond\Box q$? Yes
$\Diamond\Box q$? No
$\Box q$? No
q ? Yes

$\Diamond\Diamond\Box q$?
$\Box q$? Yes
q ? No

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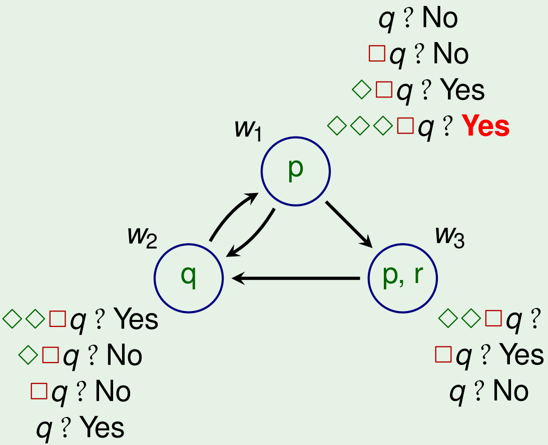
Exercises



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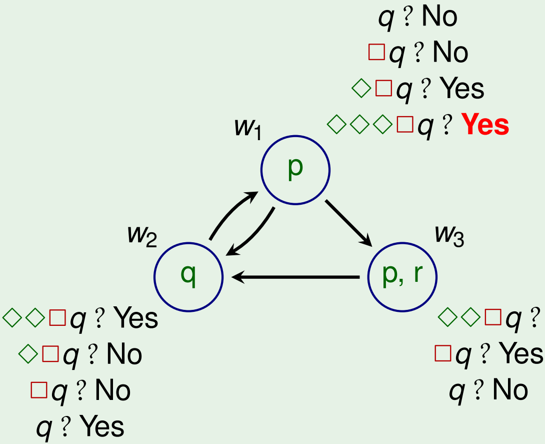

How to evaluate complex formulas?

$$w_1 \models \Diamond\Diamond\Diamond\Box q \qquad ?\ \text{Yes}$$

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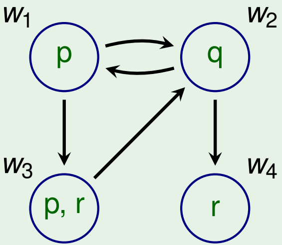
Exam Preparation Exercises

Example

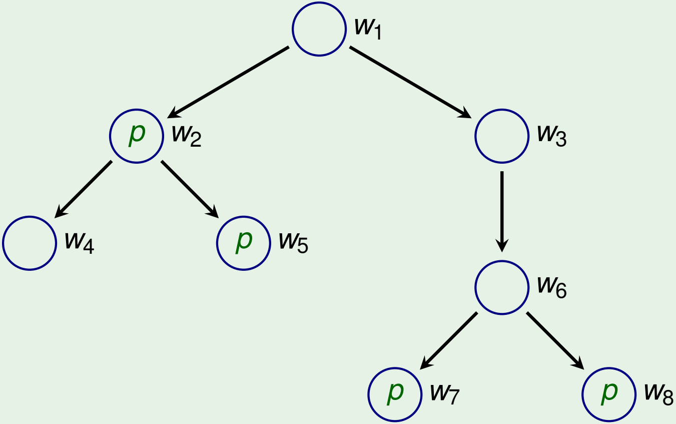


Determine the truth value of every formula in every world:

$\Diamond\Box q$	?
$\Diamond\Diamond\Box q$	?
$\Box\Diamond\Box(q \vee r)$	?
$\Diamond(\Box(q \vee r) \to p)$	?
$\Box(\Diamond p \to \Diamond\Diamond r)$	?

Exam Preparation Exercises

Example

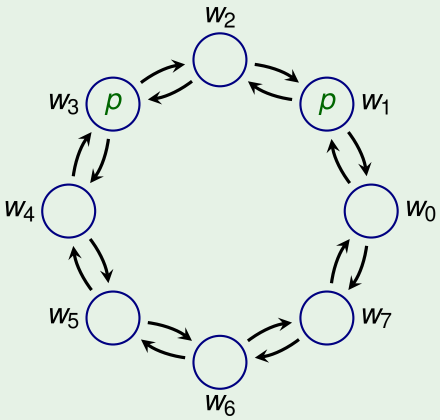


Determine in which worlds the following formula holds:

$\Box\Diamond\Box\Diamond p$?

Exam Preparation Exercises

Example

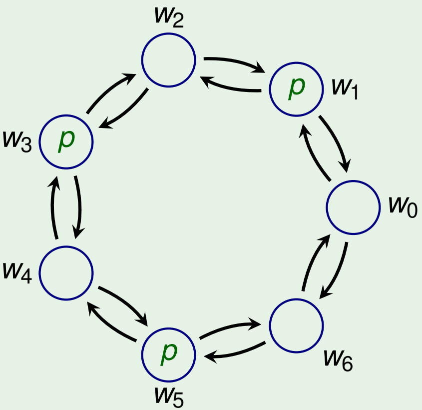


Determine in which worlds the following formula holds:

$$\Diamond\Diamond\Diamond\Diamond\Box p \qquad ?$$

Exam Preparation Exercises

Example



Determine in which worlds the following formula holds:

$\Diamond\Diamond\Diamond\Box\Box p$?