Logic and Modelling

— Predicate Logic with Equality —

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Russell's Barber Paradox

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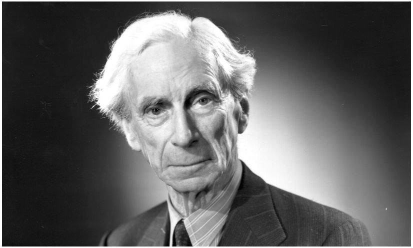

Bertrand Russell (1872–1970)

Russell's Barber Paradox

In a town with just one barber, who is male, all men are required by law to keep themselves clean-shaven.

Every man must do so by doing **exactly one** of two things:

(i) shaving himself; or

(ii) being shaved by the barber.

What does the law require of the barber? He has no option!

Case 1: The barber shaves himself. Then he does (i) and (ii). By doing both (i) and (ii), he violates the law that only permits exactly one of these options. $\times$

Case 2: The barber does not shave himself. Then he does not do (i). As he is the barber, he also is not shaved by the barber. So he does not do (ii). By doing neither (i) nor (ii), the law is again violated. $\times$

Russell's Barber Paradox

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Every man must do so by doing **exactly one** of two things:

(i) shaving himself; or

(ii) being shaved by the barber.

What does the law require of the barber? He has no option!

Is the following formula satisfiable?

$\exists x \big(man(x) \land \forall y \big(man(y) \rightarrow (shaves(x, y) \leftrightarrow \neg shaves(y, y))\big)\big)$

Or the following simplified version?

$$\exists x \forall y (shaves(x, y) \leftrightarrow \neg shaves(y, y))$$

Actually this formula is unsatisfiable (not satisfiable).

Predicate Logic with Equality

Predicate Logic with Equality

Fixed binary predicate symbol $=$ for equality.

Notation:

- infix notation $x = y$ instead of $=(x, y)$
- the notation $x \neq y$ is an abbreviation of $\neg x = y$

Uniform, fixed interpretation of $=$ in every model.

If the model $\mathcal{M}$ has domain A, then the interpretation of $=$ is

$$=^{\mathcal{M}} \;=\; \{\, \langle a, a \rangle \mid a \in A \,\}$$

Formulas may use $=$ like a predicate symbol (but $= \notin \mathcal{P}$).

Models in Predicate Logic with Equality

Let

- $\mathcal{F}$ be a set of function symbols,
- $\mathcal{P}$ a set of predicate symbols (not containing $=$).

A **model** $\mathcal{M}$ for $\langle \mathcal{F}, \mathcal{P} \rangle$ in predicate logic with equality consists of:

- a non-empty set A, called **domain** or **universe**,
- an **interpretation operation** $(\cdot)^{\mathcal{M}}$ for the symbols in $\mathcal{F}$, $\mathcal{P}$
  (i) $f^{\mathcal{M}} : A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$
  (ii) $P^{\mathcal{M}} \subseteq A^n$ for every n-ary predicate symbols $P \in \mathcal{P}$
- the **fixed interpretation** of $=$ in $\mathcal{M}$:

$$=^{\mathcal{M}} \;=\; \{\, \langle a, a \rangle \mid a \in A \,\}$$

Models in Predicate Logic with Equality

Truth of a formula ϕ in a model $\mathcal{M}$ with universe *A with respect to environment* ℓ is defined by induction on the structure of ϕ:

Atomic formulas:

- $\mathcal{M} \vDash_\ell P(t_1, \ldots, t_n) \iff \langle t_1{}^{\mathcal{M},\ell}, \ldots, t_n{}^{\mathcal{M},\ell} \rangle \in P^{\mathcal{M}}$
- $\mathcal{M} \vDash_\ell t_1 = t_2 \iff \langle t_1{}^{\mathcal{M},\ell}, t_2{}^{\mathcal{M},\ell} \rangle \in =^{\mathcal{M}} \iff t_1{}^{\mathcal{M},\ell} = t_2{}^{\mathcal{M},\ell}$

Logic connectives:

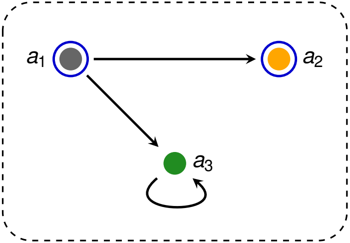
- $\mathcal{M} \vDash_\ell \neg\phi \iff \mathcal{M} \nvDash_\ell \phi$
- $\mathcal{M} \vDash_\ell \phi \wedge \psi \iff \mathcal{M} \vDash_\ell \phi$ and $\mathcal{M} \vDash_\ell \psi$
- $\mathcal{M} \vDash_\ell \phi \vee \psi \iff \mathcal{M} \vDash_\ell \phi$ or $\mathcal{M} \vDash_\ell \psi$
- $\mathcal{M} \vDash_\ell \phi \rightarrow \psi \iff$ (if $\mathcal{M} \vDash_\ell \phi$ then $\mathcal{M} \vDash_\ell \psi$)

Quantifiers:

- $\mathcal{M} \vDash_\ell \forall x\, \phi \iff$ for all $a \in A$ it holds: $\mathcal{M} \vDash_{\ell[x \mapsto a]} \phi$
- $\mathcal{M} \vDash_\ell \exists x\, \phi \iff$ for some $a \in A$ it holds: $\mathcal{M} \vDash_{\ell[x \mapsto a]} \phi$

Another Simple Model

$\mathcal{M}$

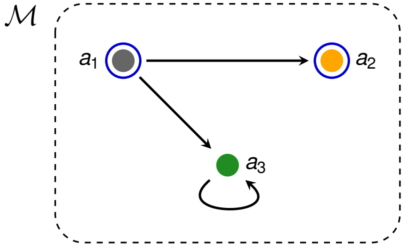


- $R^{\mathcal{M}}$: black arrows
- $P^{\mathcal{M}}$: blue circles
- $b^{\mathcal{M}}$: orange point
- $c^{\mathcal{M}}$: green point

Formal definition of the model $\mathcal{M}$:

- domain $A = \{ a_1, a_2, a_3 \}$
- $R^{\mathcal{M}} = \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_3 \rangle \}$
- $P^{\mathcal{M}} = \{ a_1, a_2 \}$
- $b^{\mathcal{M}} = a_2$
- $c^{\mathcal{M}} = a_3$.

Interpreting Formulas with Equality



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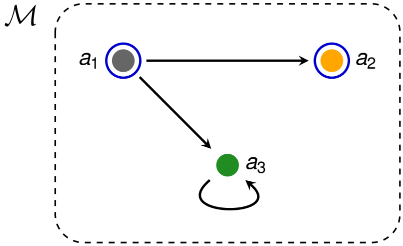
- $\mathcal{M} \not\models b = c$
- $\mathcal{M} \models b = b$
- $\mathcal{M} \models_{\ell[x \mapsto a_1]} x \neq c$
- $\mathcal{M} \models_{\ell[x \mapsto a_1]} x = x$
- $\mathcal{M} \models \forall x\, x = x$
- $\mathcal{M} \models \forall x\, \exists y\, x \neq y$

Interpreting Formulas with Equality



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- $\mathcal{M} \models \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z)$
- $\mathcal{M} \not\models \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$
- $\mathcal{M} \models \forall x \forall y (R(x,x) \wedge R(y,y) \rightarrow x = y)$
- $\mathcal{M} \not\models \forall x (R(x,b) \vee R(x,c))$
- $\mathcal{M} \models \forall x (x = b \vee R(x,c))$
- $\mathcal{M} \models \exists x \forall y (y = x \vee R(x,y))$

Model Cardinality

Constraining Model Cardinality (with At Least)

We consider the following sentences ϕ_n for $n \in \mathbb{N}$ with $n \geq 2$:

- $\phi_2 \;=\; \exists x_1 \exists x_2\; x_1 \neq x_2$
- $\phi_3 \;=\; \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$
- $\ldots$
- $\phi_n \;=\; \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$

Proposition

For all models $\mathcal{M}$ and all $n \geq 2$ the following statements hold:

(i) $\mathcal{M} \models \phi_n \iff A$ has at least n elements (i.e. $|A| \geq n$)

(ii) $\mathcal{M} \models \neg\phi_n \iff A$ has less than n elements (i.e. $|A| < n$)

(iii) $\mathcal{M} \models \phi_n \wedge \neg\phi_{n+1} \iff A$ has precisely n elements (i.e. $|A| = n$)

Constraining Model Cardinality (with At Most)

We consider the following sentences ψ_n for $n \in \mathbb{N}$ with $n \geq 1$:

- $\psi_1 \;=\; \forall x_1 \forall x_2 \; x_1 = x_2$
- $\psi_2 \;=\; \forall x_1 \forall x_2 \forall x_3 (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3)$
- ...
- $\psi_n \;=\; \forall x_1 \dots \forall x_{n+1} \bigvee_{1 \leq i < j \leq n} x_i = x_j$

Proposition

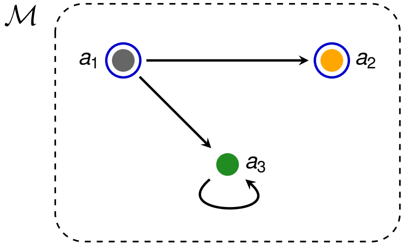
For all models $\mathcal{M}$ and all $n \geq 1$ the following statements hold:

(i) $\mathcal{M} \models \psi_n \iff A$ has at most n elements (i.e. $|A| \leq n$)

(ii) $\mathcal{M} \models \neg\psi_n \iff A$ has more than n elements (i.e.$|A| > n$)

(iii) $\mathcal{M} \models \neg\psi_n \wedge \psi_{n+1} \iff A$ has precisely $n+1$ elements (i.e. $|A| = n$)

At Most One



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At most one R-loop:

$$\mathcal{M} \models \forall x \forall y (R(x,x) \wedge R(y,y) \rightarrow x = y)$$

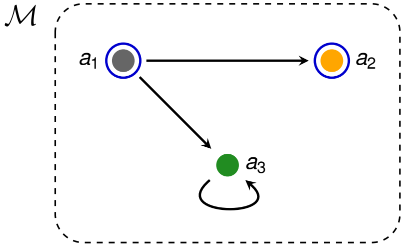
Not at least two R-loops:

$$\mathcal{M} \models \neg \exists x \exists y (R(x,x) \wedge R(y,y) \wedge x \neq y)$$

Both sentences are equivalent because:

at most one (≤ 1) $\iff$ not at least two (not ≥ 2)

At Least



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At least one P-value:

$$\mathcal{M} \models \exists x\, P(x)$$

At least two P-values:

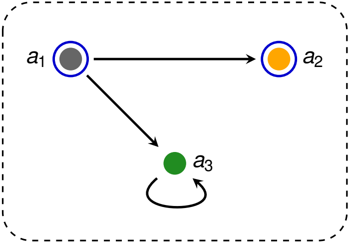
$$\mathcal{M} \models \exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$$

At least three P-values:

$$\mathcal{M} \not\models \exists x \exists y \exists z \,\big(\, P(x) \wedge P(y) \wedge P(z) \\ \wedge x \neq y \wedge x \neq z \wedge y \neq z \,\big)$$

At Most Two

$\mathcal{M}$



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At most two P-values:

$$\mathcal{M} \models \forall x \forall y \forall z (P(x) \land P(y) \land P(z) \to x = y \lor x = z \lor y = z)$$

Not at least three P-values:

$$\mathcal{M} \models \neg \exists x \exists y \exists z (P(x) \land P(y) \land P(z) \land x \neq y \land x \neq z \land y \neq z)$$

Both are equivalent because:

at most two (≤ 2) $\iff$ not: at least three (not ≥ 3)

Precisely One

There are different possibilities to express: precisely one.

Each of the following sentences expresses that

There is precisely one P-value.

There is at least one, and at most one P-value:

$$\exists x\, P(x) \land \forall x \forall y (P(x) \land P(y) \rightarrow x = y)$$

(Equivalently: there is a P-value, and all P-values are equal)

There is a P-value x, and all P-values are equal to x:

$$\exists x (\, P(x) \land \forall y (P(y) \rightarrow x = y))$$

There is a value x such that
an arbitrary value is a P-value if and only if it is x:

$$\exists x \forall y (P(y) \leftrightarrow x = y)$$

Translation into Predicate Logic with Equality

Translating into Predicate Logic with Equality

- Jan has more than one bicycle
- Jan has (precisely) two bicycles
- Jan has two bicycles, but he just uses one of them
- Everybody votes for at most one person
- Only Rutte en Samsom vote for themselves
- All members of parliament except for Rutte
- Apart from Mary, Jan also has other sisters that play chess

On the following slides we give the translations.

Thereby

- translation key,
- domains, and
- interpretations

will be self-explanatory (and left implicit).

Jan has . . . bicycles

Jan has more than one bicycle

$\exists x \exists y \big(B(x) \wedge B(y) \wedge H(j, x) \wedge H(j, y) \wedge x \neq y \big)$

Jan has (precisely) two bicycles

There are more solutions:

- $\exists x \exists y \big(B(x) \wedge B(y) \wedge x \neq y \wedge H(j, x) \wedge H(j, y)$
  $\wedge \; \forall z (B(z) \wedge H(j, z) \rightarrow z = x \vee z = y) \big)$
- $\exists x \exists y \big(x \neq y \wedge \forall z (B(z) \wedge H(j, z) \leftrightarrow z = x \vee z = y) \big)$

Jan has two bicycles, but he only uses one of them

$\exists x \exists y \big(x \neq y \wedge \forall z (B(z) \wedge H(j, z) \leftrightarrow z = x \vee z = y) \big)$

Jan has . . . bicycles

Jan has more than one bicycle

$$\exists x \exists y \big(B(x) \wedge B(y) \wedge H(j,x) \wedge H(j,y) \wedge x \neq y \big)$$

Jan has (precisely) two bicycles

There are more solutions:

- $\exists x \exists y \big(B(x) \wedge B(y) \wedge x \neq y \wedge H(j,x) \wedge H(j,y)$
  $\wedge\ \forall z (B(z) \wedge H(j,z) \to z = x \vee z = y) \big)$
- $\exists x \exists y \big(x \neq y \wedge \forall z (B(z) \wedge H(j,z) \leftrightarrow z = x \vee z = y) \big)$

Jan has two bicycles, but he only uses one of them

$$\exists x \exists y \big(x \neq y \wedge \forall z (B(z) \wedge H(j,z) \leftrightarrow z = x \vee z = y) \wedge U(j,x) \wedge \neg U(j,y) \big)$$

Everybody Votes for At Most One Person

We translate in a few steps.

Jan votes for at most one person

$\forall x \forall y(V(j, x) \wedge V(j, y) \rightarrow x = y)$

Now a property:

z votes for at most one person:

$\forall x \forall y(V(z, x) \wedge V(z, y) \rightarrow x = y)$

The sentence 'Everybody votes for at most one person.' states that this property is shared by everyone ($\forall z$):

Everybody votes for at most one person:

$\forall z \forall x \forall y(V(z, x) \wedge V(z, y) \rightarrow x = y)$

Rutte, Samsom, and the Parliament

Only Rutte and Samsom vote for themselves

Two solutions:

- $V(r,r) \wedge V(s,s) \wedge \forall x\big(\, V(x,x) \;\rightarrow\; x = r \vee x = s\,\big)$
- $\forall x\big(\, V(x,x) \;\leftrightarrow\; x = r \vee x = s\,\big)$

All members of parliament except for one vote for Rutte

Again two solutions:

- There is a member who does not vote for Rutte, but all others do:

  $\exists x\big(\, MP(x) \wedge \neg V(x,r) \wedge \forall y(MP(y) \wedge y \neq x \rightarrow V(y,r))\,\big)$

- Precisely one human being is a member of parliament and does not vote for Rutte:

  $\exists x \forall y\big(\, MP(y) \wedge \neg V(y,r) \leftrightarrow y = x\,\big)$

Jan's Chess Playing Sisters

The sentence:

Apart from Mary, Jan has other sisters who play chess

may have two readings:

- . . . has at least one other sister . . .
- . . . at least two other sisters . . .

. . . has at least one other sister . . .

$$S(m,j) \wedge C(m) \wedge \exists x\big(\, x \neq m \wedge S(x,j) \wedge C(x)\,\big)$$

. . . at least two other sisters . . .

$$S(m,j) \wedge C(m) \wedge \exists x \exists y \,\big(\, x \neq m \wedge y \neq m \wedge x \neq y \\ \wedge S(x,j) \wedge C(x) \wedge S(y,j) \wedge C(y)\,\big)$$

Natural Deduction with Equality

Natural Deduction Rules for Equality

There are two rules for equality, introduction and elimination.

Equality introduction $=_i$

$$\frac{}{t = t}\ =_i$$

Equality elimination $=_e$

$$\frac{t_1 = t_2 \qquad \phi[t_1/x]}{\phi[t_2/x]}\ =_e$$

Reflexivity of Equality

We can derive

$$\vdash \; \forall x \; x = x$$

as follows:

	y	
1	$y = y$	$=_i$
2	$\forall x \; x = x$	$\forall_i$ 1–1

Symmetry of Equality

We show that

$$t_1 = t_2 \vdash t_2 = t_1$$

1	$t_1 = t_2$	premise
2	$t_1 = t_1$	$=_i$
3	$t_2 = t_1$	$=_e$ 1,2

The rule $=_e$ in step 3 is applied with the formula $\phi = x = t_1$.

Then $\phi[t_1/x] \ = \ t_1 = t_1$ and $\phi[t_2/x] \ = \ t_2 = t_1$.

Recall the $=_e$-rule and the instance with $\phi = x = t_1$:

$$\frac{t_1 = t_2 \qquad \phi[t_1/x]}{\phi[t_2/x]} \; =_e \qquad\qquad \frac{t_1 = t_2 \qquad t_1 = t_1}{t_2 = t_1} \; =_e$$

Example: $P(c), \neg P(d) \vdash \neg\, c = d$

1	$P(c)$	premise
2	$\neg P(d)$	premise
3	$c = d$	assumption
4	$P(d)$	$=_e$ 3,1
5	$\bot$	$\neg_e$ 2,4
6	$\neg\, c = d$	$\neg_i$ 3–5