Logic and Modelling

— Predicate Logic with Equality —

Jörg Endrullis

VU University Amsterdam

Russell's Barber Paradox

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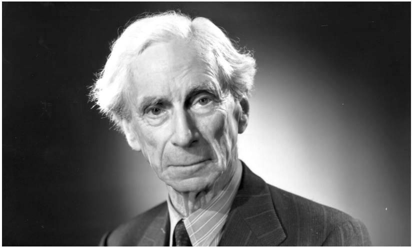

Bertrand Russell (1872–1970)

Russell’s Barber Paradox

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Actually this formula is unsatisfiable (not satisfiable).

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If the model $\mathcal{M}$ has domain A, then the interpretation of $=$ is

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Formulas may use $=$ like a predicate symbol (but $= \notin \mathcal{P}$).

Models in Predicate Logic with Equality

Let

- $\mathcal{F}$ be a set of function symbols,
- $\mathcal{P}$ a set of predicate symbols (not containing $=$).

A **model** $\mathcal{M}$ for $\langle \mathcal{F}, \mathcal{P} \rangle$ in predicate logic with equality consists of:

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- a non-empty set A, called **domain** or **universe**,
- an **interpretation operation** $(\cdot)^{\mathcal{M}}$ for the symbols in $\mathcal{F}$, $\mathcal{P}$
  (i) $f^{\mathcal{M}} : A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$
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- the **fixed interpretation** of $=$ in $\mathcal{M}$:

$$=^{\mathcal{M}} \; = \; \{ \langle a, a \rangle \mid a \in A \}$$

Models in Predicate Logic with Equality

Truth of a formula ϕ in a model $\mathcal{M}$ with universe A *with respect to environment* ℓ is defined by induction on the structure of ϕ:

Atomic formulas:

- $\mathcal{M} \models_\ell P(t_1, \ldots, t_n) \iff \langle t_1^{\mathcal{M},\ell}, \ldots, t_n^{\mathcal{M},\ell} \rangle \in P^{\mathcal{M}}$

Logic connectives:

- $\mathcal{M} \models_\ell \neg\phi \iff \mathcal{M} \not\models_\ell \phi$
- $\mathcal{M} \models_\ell \phi \wedge \psi \iff \mathcal{M} \models_\ell \phi$ and $\mathcal{M} \models_\ell \psi$
- $\mathcal{M} \models_\ell \phi \vee \psi \iff \mathcal{M} \models_\ell \phi$ or $\mathcal{M} \models_\ell \psi$
- $\mathcal{M} \models_\ell \phi \rightarrow \psi \iff$ (if $\mathcal{M} \models_\ell \phi$ then $\mathcal{M} \models_\ell \psi$)

Quantifiers:

- $\mathcal{M} \models_\ell \forall x\, \phi \iff$ for all $a \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto a]} \phi$
- $\mathcal{M} \models_\ell \exists x\, \phi \iff$ for some $a \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto a]} \phi$

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Truth of a formula ϕ in a model $\mathcal{M}$ with universe A *with respect to environment* ℓ is defined by induction on the structure of ϕ:

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- $\mathcal{M} \vDash_\ell t_1 = t_2 \iff \langle {t_1}^{\mathcal{M},\ell}, {t_2}^{\mathcal{M},\ell} \rangle \in =^{\mathcal{M}} \iff {t_1}^{\mathcal{M},\ell} = {t_2}^{\mathcal{M},\ell}$

Logic connectives:

- $\mathcal{M} \vDash_\ell \neg\phi \iff \mathcal{M} \nvDash_\ell \phi$
- $\mathcal{M} \vDash_\ell \phi \wedge \psi \iff \mathcal{M} \vDash_\ell \phi$ and $\mathcal{M} \vDash_\ell \psi$
- $\mathcal{M} \vDash_\ell \phi \vee \psi \iff \mathcal{M} \vDash_\ell \phi$ or $\mathcal{M} \vDash_\ell \psi$
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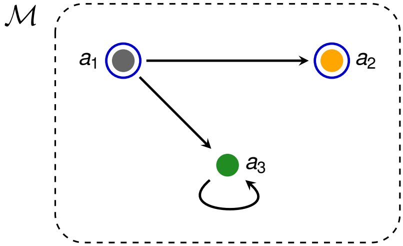
Another Simple Model



- $R^{\mathcal{M}}$: black arrows
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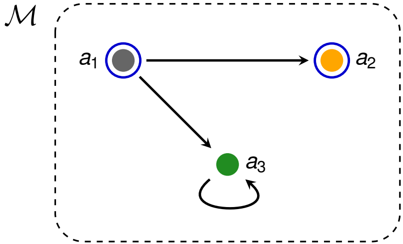
Formal definition of the model $\mathcal{M}$:

- domain $A = \{ a_1, a_2, a_3 \}$
- $R^{\mathcal{M}} = \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_3 \rangle \}$
- $P^{\mathcal{M}} = \{ a_1, a_2 \}$
- $b^{\mathcal{M}} = a_2$
- $c^{\mathcal{M}} = a_3$.

Interpreting Formulas with Equality



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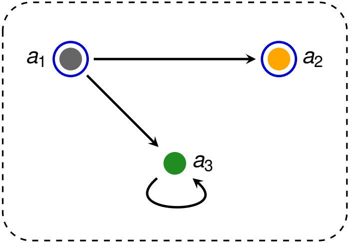

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- $\mathcal{M} \models b = c$

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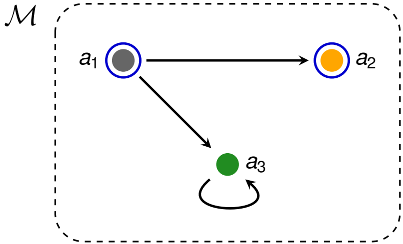

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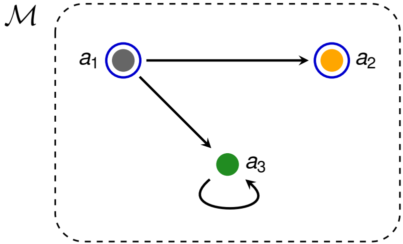

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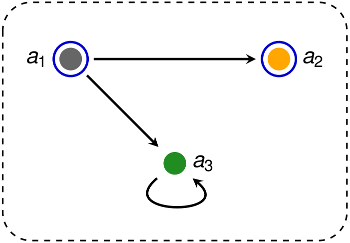

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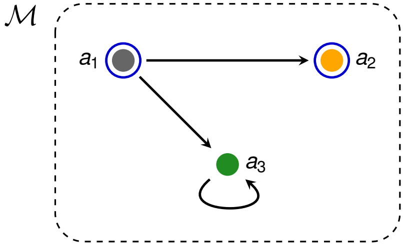

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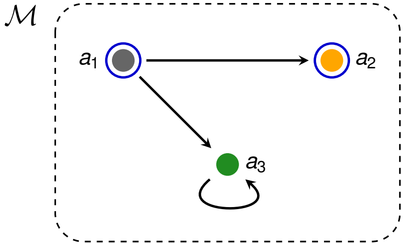

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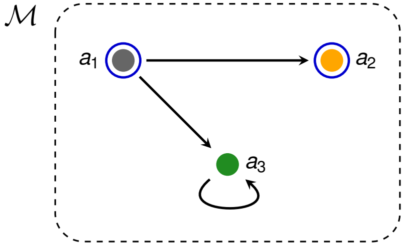

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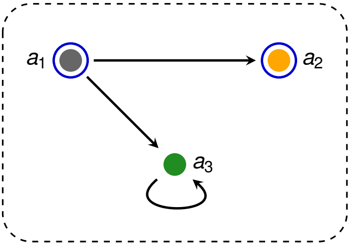

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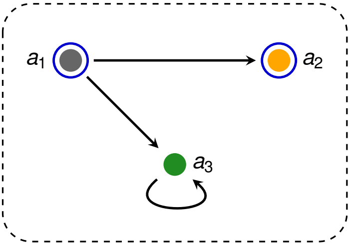

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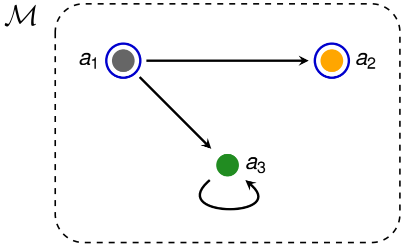

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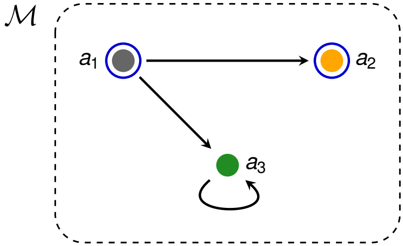

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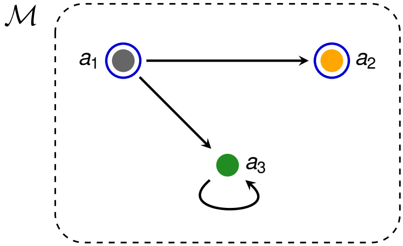

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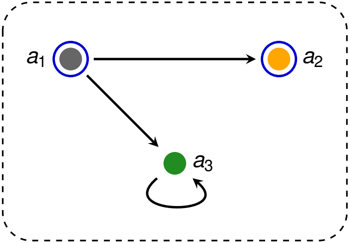

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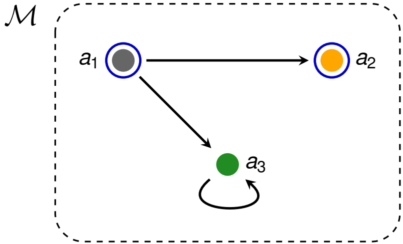

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Interpreting Formulas with Equality

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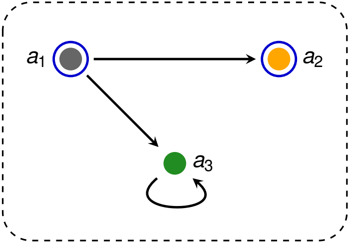

- $R^{\mathcal{M}}$: black arrows
- $P^{\mathcal{M}}$: blue circles
- $b^{\mathcal{M}}$: orange point
- $c^{\mathcal{M}}$: green point

- $\mathcal{M} \models \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z)$
- $\mathcal{M} \models \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$

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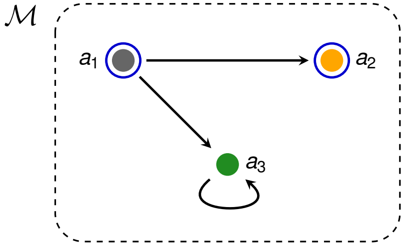

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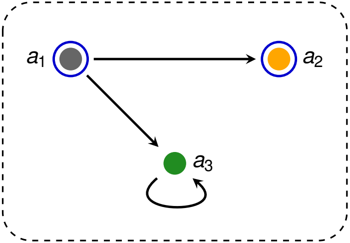

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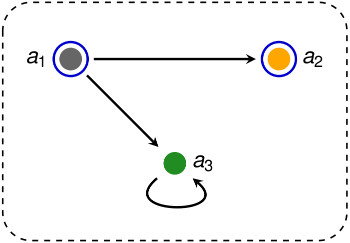

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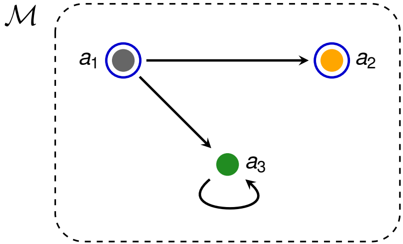

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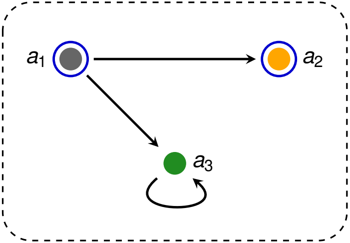

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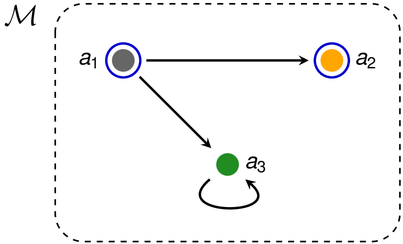

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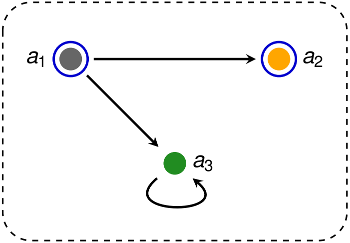

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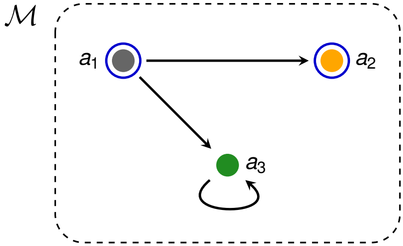

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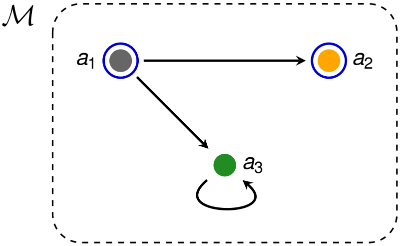

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Model Cardinality

Constraining Model Cardinality (with At Least)

We consider the following sentences ϕ_n for $n \in \mathbb{N}$ with $n \geq 2$:

Constraining Model Cardinality (with At Least)

We consider the following sentences ϕ_n for $n \in \mathbb{N}$ with $n \geq 2$:

- $\phi_2 \; = \; \exists x_1 \exists x_2 \; x_1 \neq x_2$

Constraining Model Cardinality (with At Least)

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- ...
- $\phi_n \;=\; \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$

Proposition

For all models $\mathcal{M}$ and all $n \geq 2$ the following statements hold:

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For all models $\mathcal{M}$ and all $n \geq 2$ the following statements hold:

(i) $\mathcal{M} \models \phi_n \iff$

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For all models $\mathcal{M}$ and all $n \geq 2$ the following statements hold:

(i) $\mathcal{M} \models \phi_n \iff A$ has at least n elements (i.e. $|A| \geq n$)

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For all models $\mathcal{M}$ and all $n \geq 2$ the following statements hold:

(i) $\mathcal{M} \models \phi_n \iff A$ has at least n elements (i.e. $|A| \geq n$)

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(iii) $\mathcal{M} \models \phi_n \wedge \neg\phi_{n+1} \iff A$ has precisely n elements (i.e. $|A| = n$)

Constraining Model Cardinality (with At Most)

We consider the following sentences ψ_n for $n \in \mathbb{N}$ with $n \geq 1$:

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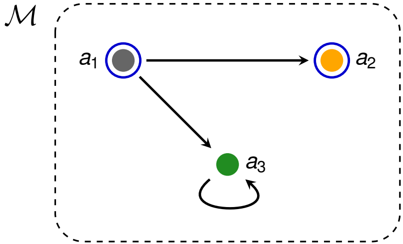
(iii) $\mathcal{M} \models \neg\psi_n \wedge \psi_{n+1} \iff A$ has precisely $n + 1$ elements (i.e. $|A| = n$)

At Most One



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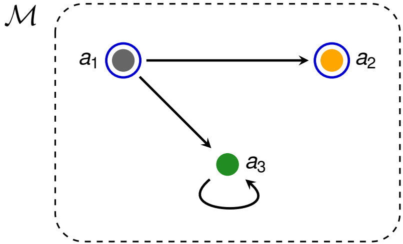

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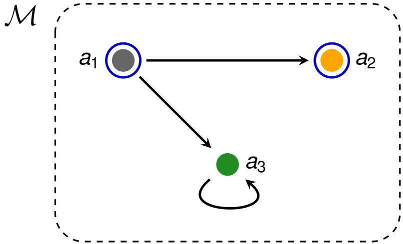
At most one R-loop:

At Most One



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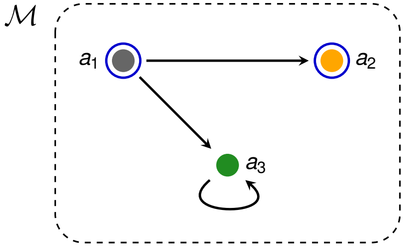
At most one R-loop:

$$\mathcal{M} \models \forall x \forall y (R(x,x) \wedge R(y,y) \rightarrow x = y)$$

At Most One



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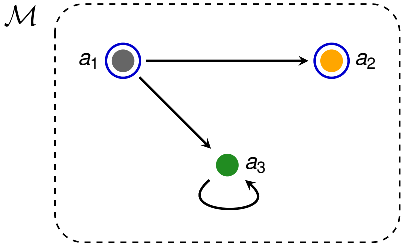
At most one R-loop:

$$\mathcal{M} \models \forall x \forall y (R(x,x) \wedge R(y,y) \rightarrow x = y)$$

Not at least two R-loops:

At Most One



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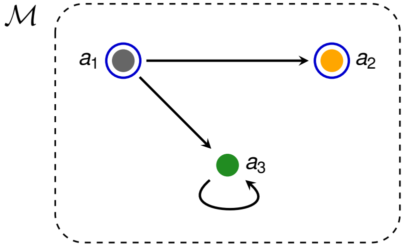

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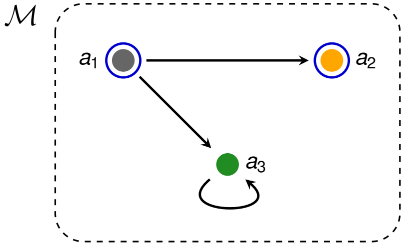
Both sentences are equivalent because:

at most one (≤ 1) $\iff$ not at least two (not ≥ 2)

At Least



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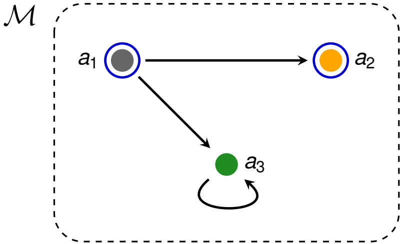

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At Least

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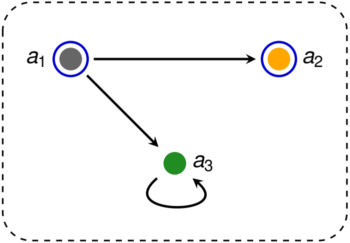

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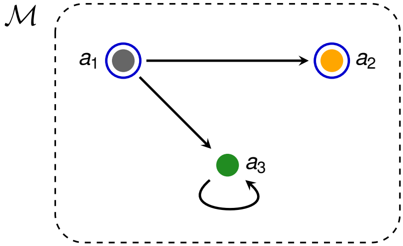
At least one P-value:

$$\mathcal{M} \models \exists x\, P(x)$$

At Least



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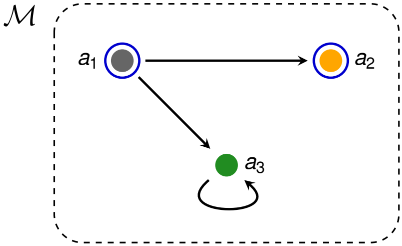

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At Least

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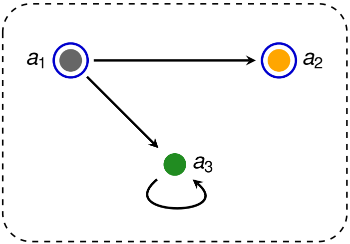

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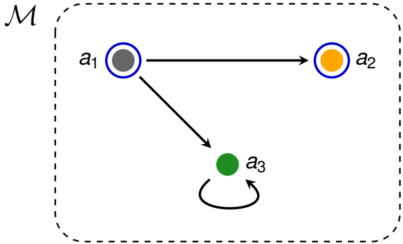

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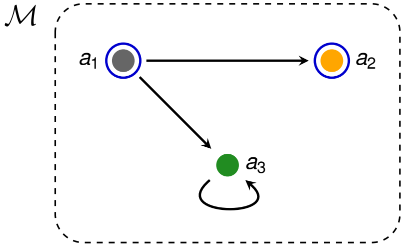

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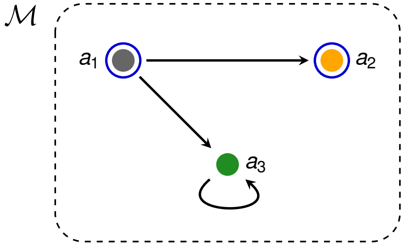
At Least



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$$\mathcal{M} \models \exists x \exists y \exists z \,\big(P(x) \wedge P(y) \wedge P(z) \wedge x \neq y \wedge x \neq z \wedge y \neq z \,\big)$$

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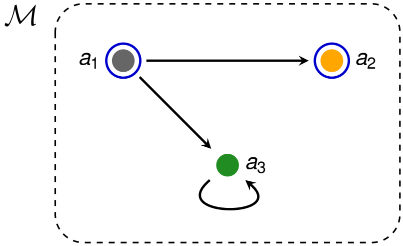

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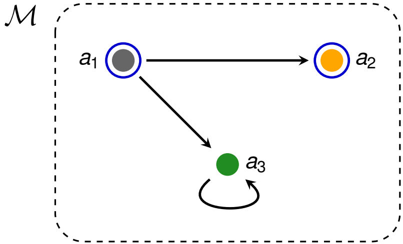
At Most Two



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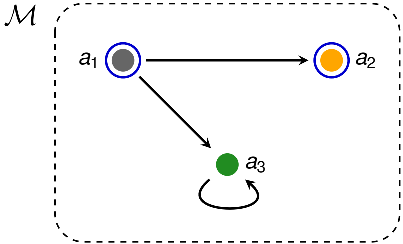

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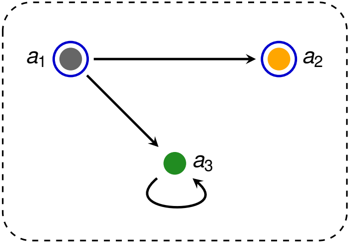
At Most Two

$\mathcal{M}$



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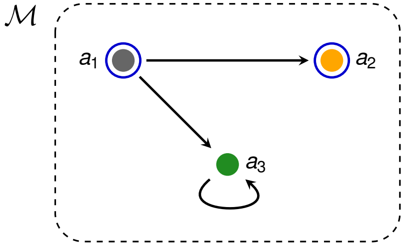

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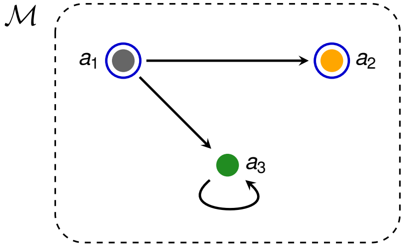

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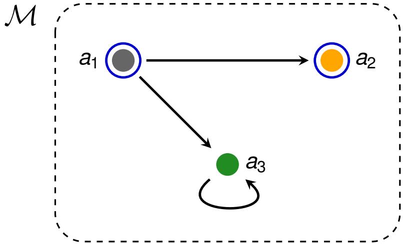

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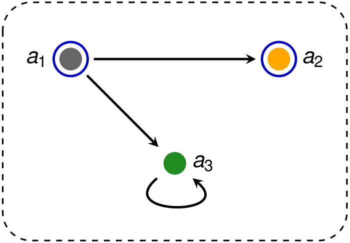

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Both are equivalent because:

at most two (≤ 2) $\iff$ not: at least three (not ≥ 3)

Precisely One

There are different possibilities to express: precisely one.

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Each of the following sentences expresses that

There is precisely one P-value.

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(Equivalently: there is a P-value, and all P-values are equal)

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There is a *P*-value x, and all *P*-values are equal to x:

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There is a value x such that
an arbitrary value is a P-value if and only if it is x:

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$$\exists x \forall y (P(y) \leftrightarrow x = y)$$

Translation into Predicate Logic with Equality

Translating into Predicate Logic with Equality

- Jan has more than one bicycle
- Jan has (precisely) two bicycles
- Jan has two bicycles, but he just uses one of them
- Everybody votes for at most one person
- Only Rutte en Samsom vote for themselves
- All members of parliament except for Rutte
- Apart from Mary, Jan also has other sisters that play chess

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On the following slides we give the translations.

Thereby

- translation key,
- domains, and
- interpretations

will be self-explanatory (and left implicit).

Jan has . . . bicycles

Jan has more than one bicycle

Jan has … bicycles

Jan has more than one bicycle

$$\exists x \exists y \big(B(x) \wedge B(y) \wedge H(j, x) \wedge H(j, y) \wedge x \neq y \big)$$

Jan has . . . bicycles

Jan has more than one bicycle

$$\exists x \exists y \big(B(x) \wedge B(y) \wedge H(j, x) \wedge H(j, y) \wedge x \neq y \big)$$

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Jan has (precisely) two bicycles

There are more solutions:

- $\exists x \exists y \big(B(x) \wedge B(y) \wedge x \neq y \wedge H(j,x) \wedge H(j,y)$
  $\wedge\ \forall z (B(z) \wedge H(j,z) \rightarrow z = x \vee z = y) \big)$

Jan has ... bicycles

Jan has more than one bicycle

$\exists x \exists y \big(B(x) \wedge B(y) \wedge H(j,x) \wedge H(j,y) \wedge x \neq y \big)$

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Jan has two bicycles, but he only uses one of them

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Jan has two bicycles, but he only uses one of them

$$\exists x \exists y \big(x \neq y \wedge \forall z (B(z) \wedge H(j,z) \leftrightarrow z = x \vee z = y) \\ \wedge U(j,x) \wedge \neg U(j,y) \big)$$

Everybody Votes for At Most One Person

We translate in a few steps.

Everybody Votes for At Most One Person

We translate in a few steps.

Jan votes for at most one person

Everybody Votes for At Most One Person

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$$\forall x \forall y (V(j, x) \land V(j, y) \to x = y)$$

Everybody Votes for At Most One Person

We translate in a few steps.

Jan votes for at most one person

$\forall x \forall y (V(j, x) \wedge V(j, y) \rightarrow x = y)$

Now a property:

z votes for at most one person:

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The sentence 'Everybody votes for at most one person.' states that this property is shared by everyone ($\forall z$):

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Rutte, Samsom, and the Parliament

Only Rutte and Samsom vote for themselves

Rutte, Samsom, and the Parliament

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Two solutions:

Rutte, Samsom, and the Parliament

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Two solutions:

- $V(r, r) \wedge V(s, s) \wedge \forall x\big(\, V(x, x) \;\rightarrow\; x = r \vee x = s \,\big)$
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Rutte, Samsom, and the Parliament

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All members of parliament except for one vote for Rutte

Rutte, Samsom, and the Parliament

Only Rutte and Samsom vote for themselves

Two solutions:

- $V(r,r) \wedge V(s,s) \wedge \forall x\big(\ V(x,x)\ \rightarrow\ x = r \vee x = s\ \big)$
- $\forall x\big(\ V(x,x)\ \leftrightarrow\ x = r \vee x = s\ \big)$

All members of parliament except for one vote for Rutte

Again two solutions:

Rutte, Samsom, and the Parliament

Only Rutte and Samsom vote for themselves

Two solutions:

- $V(r,r) \land V(s,s) \land \forall x\big(\, V(x,x) \ \rightarrow\ x = r \lor x = s \,\big)$
- $\forall x\big(\, V(x,x) \ \leftrightarrow\ x = r \lor x = s \,\big)$

All members of parliament except for one vote for Rutte

Again two solutions:

- There is a member who does not vote for Rutte,
  but all others do:

Rutte, Samsom, and the Parliament

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- $V(r,r) \wedge V(s,s) \wedge \forall x\big(\, V(x,x) \;\rightarrow\; x = r \vee x = s \,\big)$
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All members of parliament except for one vote for Rutte

Again two solutions:

- There is a member who does not vote for Rutte,
  but all others do:

  $\exists x\big(\, MP(x) \wedge \neg V(x,r) \wedge \forall y(MP(y) \wedge y \neq x \rightarrow V(y,r)) \,\big)$

Rutte, Samsom, and the Parliament

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Two solutions:

- $V(r, r) \wedge V(s, s) \wedge \forall x\big(\, V(x, x) \;\rightarrow\; x = r \vee x = s\,\big)$
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All members of parliament except for one vote for Rutte

Again two solutions:

- There is a member who does not vote for Rutte, but all others do:

  $$\exists x\big(\, MP(x) \wedge \neg V(x, r) \wedge \forall y(MP(y) \wedge y \neq x \rightarrow V(y, r))\,\big)$$

- Precisely one human being is a member of parliament and does not vote for Rutte:

Rutte, Samsom, and the Parliament

Only Rutte and Samsom vote for themselves

Two solutions:

- $V(r,r) \wedge V(s,s) \wedge \forall x\big(\, V(x,x) \;\rightarrow\; x = r \vee x = s \,\big)$
- $\forall x\big(\, V(x,x) \;\leftrightarrow\; x = r \vee x = s \,\big)$

All members of parliament except for one vote for Rutte

Again two solutions:

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- Precisely one human being is a member of parliament and does not vote for Rutte:

  $$\exists x \forall y\big(\, MP(y) \wedge \neg V(y,r) \leftrightarrow y = x \,\big)$$

Jan's Chess Playing Sisters

The sentence:

Apart from Mary, Jan has other sisters who play chess

may have two readings:

Jan's Chess Playing Sisters

The sentence:

Apart from Mary, Jan has other sisters who play chess

may have two readings:

- … has at least one other sister …

Jan's Chess Playing Sisters

The sentence:

> Apart from Mary, Jan has other sisters who play chess

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- … at least two other sisters …

Jan's Chess Playing Sisters

The sentence:

Apart from Mary, Jan has other sisters who play chess

may have two readings:

- . . . has at least one other sister . . .
- . . . at least two other sisters . . .

. . . has at least one other sister . . .

Jan's Chess Playing Sisters

The sentence:

Apart from Mary, Jan has other sisters who play chess

may have two readings:

- … has at least one other sister …
- … at least two other sisters …

… has at least one other sister …

$$S(m,j) \wedge C(m) \wedge \exists x\big(\, x \neq m \wedge S(x,j) \wedge C(x)\,\big)$$

Jan's Chess Playing Sisters

The sentence:

Apart from Mary, Jan has other sisters who play chess

may have two readings:

- ... has at least one other sister ...
- ... at least two other sisters ...

... has at least one other sister ...

$$S(m,j) \wedge C(m) \wedge \exists x \big(x \neq m \wedge S(x,j) \wedge C(x) \big)$$

... at least two other sisters ...

Jan's Chess Playing Sisters

The sentence:

> Apart from Mary, Jan has other sisters who play chess

may have two readings:

- … has at least one other sister …
- … at least two other sisters …

… has at least one other sister …

$$S(m,j) \wedge C(m) \wedge \exists x \big(x \neq m \wedge S(x,j) \wedge C(x) \big)$$

… at least two other sisters …

$$\begin{aligned} S(m,j) \wedge C(m) \wedge \exists x \exists y \big(& x \neq m \wedge y \neq m \wedge x \neq y \\ & \wedge S(x,j) \wedge C(x) \wedge S(y,j) \wedge C(y) \big) \end{aligned}$$

Natural Deduction with Equality

Natural Deduction Rules for Equality

There are two rules for equality, introduction and elimination.

Equality introduction $=_i$

$$\frac{}{t = t} \; =_i$$

Equality elimination $=_e$

$$\frac{t_1 = t_2 \qquad \phi[t_1/x]}{\phi[t_2/x]} \; =_e$$

Reflexivity of Equality

We can derive

$$\vdash \ \forall x\ x = x$$

as follows:

	y	
1	$y = y$	$=_i$
2	$\forall x\ x = x$	$\forall_i$ 1–1

Symmetry of Equality

We show that

$$t_1 = t_2 \vdash \; t_2 = t_1$$

1	$t_1 = t_2$	premise
2	$t_1 = t_1$	$=_i$
3	$t_2 = t_1$	$=_e$ 1,2

The rule $=_e$ in step 3 is applied with the formula $\phi = x = t_1$.

Then $\phi[t_1/x] \; = \; t_1 = t_1$ and $\phi[t_2/x] \; = \; t_2 = t_1$.

Recall the $=_e$-rule and the instance with $\phi = x = t_1$:

$$\frac{t_1 = t_2 \qquad \phi[t_1/x]}{\phi[t_2/x]} \; =_e \qquad\qquad \frac{t_1 = t_2 \qquad t_1 = t_1}{t_2 = t_1} \; =_e$$

Example: $P(c), \neg P(d) \vdash \neg\, c = d$

1	$P(c)$	premise
2	$\neg P(d)$	premise

Example: $P(c), \neg P(d) \vdash \neg\, c = d$

1	$P(c)$	premise
2	$\neg P(d)$	premise
3	$c = d$	assumption

Example: $P(c), \neg P(d) \vdash \neg\, c = d$

1	$P(c)$	premise
2	$\neg P(d)$	premise
3	$c = d$	assumption
4	$P(d)$	$=_e$ 3,1

Example: $P(c), \neg P(d) \vdash \neg\, c = d$

1	$P(c)$	premise
2	$\neg P(d)$	premise
3	$c = d$	assumption
4	$P(d)$	$=_e$ 3,1
5	$\perp$	$\neg_e$ 2,4

Example: $P(c), \neg P(d) \vdash \neg\, c = d$

1	$P(c)$	premise
2	$\neg P(d)$	premise
3	$\boxed{c = d}$	assumption
4	$P(d)$	$=_e$ 3,1
5	$\bot$	$\neg_e$ 2,4
6	$\neg\, c = d$	$\neg_i$ 3–5