Logic and Modelling

— Semantics of, and Translations into, Predicate Logic —

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Reminder: Models and Environments

Let

- $\mathcal{F}$ be a set of function symbols,
- $\mathcal{P}$ a set of predicate symbols.

A **model** $\mathcal{M}$ for $\langle \mathcal{F}, \mathcal{P} \rangle$ consists of:

- a non-empty set A, called **domain** or **universe**,
- an **interpretation operation** $(\cdot)^{\mathcal{M}}$ for the symbols in $\mathcal{F}$, $\mathcal{P}$.
  - (i) $f^{\mathcal{M}} : A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$
  - (ii) $P^{\mathcal{M}} \subseteq A^n$ for every n-ary predicate symbols $P \in \mathcal{P}$

A symbol is n-ary if it has n arguments.

An **environment** (look-up function)

$$\ell : \textbf{var} \to A$$

interprets free variables in the domain.

Reminder: Formula Truth in a Model

Truth of a formula ϕ in a model $\mathcal{M}$ with universe *A with respect to environment* ℓ is defined by induction on the structure of ϕ:

- $\mathcal{M} \models_\ell \neg\phi \iff \mathcal{M} \not\models_\ell \phi$
- $\mathcal{M} \models_\ell \phi \wedge \psi \iff \mathcal{M} \models_\ell \phi$ and $\mathcal{M} \models_\ell \psi$
- $\mathcal{M} \models_\ell \phi \vee \psi \iff \mathcal{M} \models_\ell \phi$ or $\mathcal{M} \models_\ell \psi$
- $\mathcal{M} \models_\ell \phi \to \psi \iff$ (if $\mathcal{M} \models_\ell \phi$ then $\mathcal{M} \models_\ell \psi$)
- $\mathcal{M} \models_\ell \boldsymbol{P}(t_1, \ldots, t_n) \iff \langle {t_1}^{\mathcal{M},\ell}, \ldots, {t_n}^{\mathcal{M},\ell} \rangle \in \boldsymbol{P}^{\mathcal{M}}$

$$t^{\mathcal{M},\ell} = \begin{cases} \ell(\boldsymbol{x}) & \text{if } t = \boldsymbol{x} \text{ for a variable } \boldsymbol{x} \\ \boldsymbol{c}^{\mathcal{M}} & \text{if } t = \boldsymbol{c} \text{ for a constant } \boldsymbol{c} \\ \boldsymbol{f}^{\mathcal{M}}({t_1}^{\mathcal{M},\ell}, \ldots, {t_n}^{\mathcal{M},\ell}) & \text{if } t = \boldsymbol{f}(t_1, \ldots, t_n) \end{cases}$$

- $\mathcal{M} \models_\ell \forall x\, \phi \iff$ for all $a \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto a]} \phi$
- $\mathcal{M} \models_\ell \exists x\, \phi \iff$ for some $a \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto a]} \phi$

Reminder: Semantical Entailment

Semantic entailment in propositional logic

In **propositional logic**: $\phi_1, \ldots, \phi_n \vDash \psi$ means:

For all valuations v:

$$v(\phi_1) = \mathtt{T}, \ldots, v(\phi_n) = \mathtt{T} \implies v(\psi) = \mathtt{T}$$

Semantic entailment in predicate logic

In **predicate logic**: $\phi_1, \ldots, \phi_n \vDash \psi$ means:

For all models $\mathcal{M}$, and all environments ℓ :

$$\mathcal{M} \vDash_\ell \phi_1, \ldots, \mathcal{M} \vDash_\ell \phi_n \implies \mathcal{M} \vDash_\ell \psi$$

In words: for all models $\mathcal{M}$, and all environments ℓ such that

$$\mathcal{M} \vDash_\ell \phi_1 \quad \text{and} \quad \ldots \text{ and } \quad \mathcal{M} \vDash_\ell \phi_n \quad \text{hold,}$$

it also holds that $\mathcal{M} \vDash_\ell \psi$.

Logic Equivalence

Logical Equivalence

Definition (Logical equivalence $\equiv$)

Formulas ϕ and ψ are **logically equivalent**, denoted by

$$\phi \equiv \psi$$

if for all models $\mathcal{M}$ and environments ℓ:

$$\mathcal{M} \vDash_{\ell} \phi \iff \mathcal{M} \vDash_{\ell} \psi$$

That is, ϕ and ψ are true in precisely the same models when interpreted with the same environments.

Logical Equivalence

Theorem

$$\phi \equiv \psi \iff \phi \vDash \psi \text{ and } \psi \vDash \phi$$

Therefore, logical equivalence is sometimes denoted $\Dashv\vDash$.

Proof.

$\phi \equiv \psi$

$\iff$ for all $\mathcal{M}$ and ℓ:
$(\mathcal{M} \vDash_\ell \phi \Rightarrow \mathcal{M} \vDash_\ell \psi)$ and $(\mathcal{M} \vDash_\ell \psi \Rightarrow \mathcal{M} \vDash_\ell \phi)$

$\iff$ for all $\mathcal{M}$ and ℓ: $\mathcal{M} \vDash_\ell \phi \Rightarrow \mathcal{M} \vDash_\ell \psi)$
and for all $\mathcal{M}$ and ℓ: $\mathcal{M} \vDash_\ell \psi \Rightarrow \mathcal{M} \vDash_\ell \phi)$

$\iff$ $\phi \vDash \psi$ and $\psi \vDash \phi$ (that is, $\phi \Dashv\vDash \psi$)

$\square$

Proving Logical Equivalence

$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

For all models $\mathcal{M}$ with domain A and environments ℓ we find:

$$\begin{aligned}
&\mathcal{M} \models_\ell \neg\forall x\, P(x) \\
&\quad\iff \text{not: } \mathcal{M} \models_\ell \forall x\, P(x) \\
&\quad\iff \text{not for all } a \in A\text{: } \mathcal{M} \models_{\ell[x\mapsto a]} P(x) \\
&\quad\iff \text{there exists } a \in A \text{ such that not: } \mathcal{M} \models_{\ell[x\mapsto a]} P(x) \\
&\quad\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \not\models_{\ell[x\mapsto a]} P(x) \\
&\quad\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \models_{\ell[x\mapsto a]} \neg P(x) \\
&\quad\iff \mathcal{M} \models_\ell \exists x\, \neg P(x)
\end{aligned}$$

Hence we can conclude: $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$.

Note that each step is a bi-implication, so we have $\dashv$ and $\models$.

Satisfiability, Validity and Consistency

Satisfiability, Validity, Consistency

Definition (Satisfiability, validity of formulas)

Let ϕ be a formula, and Γ be a set of formulas.

- ϕ is **satisfiable** $\iff$ there is some model $\mathcal{M}$ and some environment ℓ such that $\mathcal{M} \vDash_\ell \phi$.
- ϕ is **valid** $\iff$ $\mathcal{M} \vDash_\ell \phi$ holds for all models $\mathcal{M}$ and all environments ℓ in which ϕ can be checked.
- Γ is **consistent** or **satisfiable** $\iff$ there is some model $\mathcal{M}$ and some environment ℓ such that

$$\mathcal{M} \vDash_\ell \psi \quad \text{for all } \psi \in \Gamma.$$

Logical Equivalence, and Validity of Bi-implication

Proposition

For all formulas ϕ, ψ the following statements are equivalent:

(i) $\phi \equiv \psi$

(ii) $\phi \leftrightarrow \psi$ is valid.

Proof.

For all models $\mathcal{M}$ and environments ℓ it holds:

$$
\begin{aligned}
& \mathcal{M} \vDash_\ell \phi \leftrightarrow \psi \\
\iff\ & \mathcal{M} \vDash_\ell (\phi \to \psi) \wedge (\psi \to \phi) \\
\iff\ & \mathcal{M} \vDash_\ell \phi \to \psi \text{ and } \mathcal{M} \vDash_\ell \psi \to \phi \\
\iff\ & \qquad \text{if } \mathcal{M} \vDash_\ell \phi, \text{ then } \mathcal{M} \vDash_\ell \psi, \\
& \text{and } \text{if } \mathcal{M} \vDash_\ell \psi, \text{ then } \mathcal{M} \vDash_\ell \phi \\
\iff\ & (\mathcal{M} \vDash_\ell \phi \iff \mathcal{M} \vDash_\ell \psi)
\end{aligned}
$$

From this we conclude: $\phi \leftrightarrow \psi$ is valid if and only if $\phi \equiv \psi$. $\square$

Examples

$\mathcal{F} = \{ \, alma_{/0} \, \}, \ \mathcal{P} = \{ \, loves_{/2} \, \}$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((loves(x, alma) \wedge loves(y, x)) \rightarrow \neg loves(y, alma))$$

Alma Mahler/Werfel/Gropius (1879 – 1964)

Satisfiable Formula (Example)

$\mathcal{F} = \{ \mathit{alma}_{/0} \}$, $\mathcal{P} = \{ \mathit{loves}_{/2} \}$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((\mathit{loves}(x, \mathit{alma}) \land \mathit{loves}(y, x)) \rightarrow \neg \mathit{loves}(y, \mathit{alma}))$$

The formula ϕ is **satisfiable**:

Consider model $\mathcal{M}_1$ with domain $A_1 = \{ a, g, f, w \}$ and:

$$\mathit{alma}^{\mathcal{M}_1} = a \qquad \mathit{loves}^{\mathcal{M}_1} = \{ \langle g, a \rangle, \langle f, a \rangle, \langle w, a \rangle \}$$

In this model there are not any lovers of lovers.

Hence: $\mathcal{M}_1 \models \phi$.

Consistent Set of Formulas (Example)

$\mathcal{F} = \{ \, alma_{/0} \, \}, \;\; \mathcal{P} = \{ \, loves_{/2} \, \}$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((loves(x, alma) \wedge loves(y, x)) \rightarrow \neg loves(y, alma))$$

The formula ϕ is **satisfiable**:

Consider model $\mathcal{M}_1$ with domain $A_1 = \{ \, a, g, f, w \, \}$ and:

$$alma^{\mathcal{M}_1} = a \qquad\qquad loves^{\mathcal{M}_1} = \{ \, \langle g, a \rangle, \langle f, a \rangle, \langle w, a \rangle \, \}$$

In this model there are not any lovers of lovers.

Hence: $\mathcal{M}_1 \models \phi$.

The set $\Gamma = \{ \, \phi, \neg loves(alma, alma) \, \}$ is **consistent**:
since it also holds: $\mathcal{M}_1 \models \neg loves(alma, alma)$.

Not Valid Formula (Example)

$\mathcal{F} = \{ \, alma_{/0} \, \}$, $\mathcal{P} = \{ \, loves_{/2} \, \}$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((loves(x, alma) \wedge loves(y, x)) \rightarrow \neg loves(y, alma))$$

The formula ϕ is **not valid**:

Let $\mathcal{M}_2$ have domain $A_2 = \{ \, a, g, f, w \, \}$, $alma^{\mathcal{M}_1} = a$, and:

$$loves^{\mathcal{M}_2} = \{ \, \langle a, g \rangle, \langle a, f \rangle, \langle a, w \rangle, \langle a, a \rangle, \langle g, a \rangle, \langle f, a \rangle, \langle w, a \rangle \, \}$$

In this model:

- Alma herself is a lover of a lover of herself.
- Alma loves herself.

Hence: $\mathcal{M}_2 \not\models \phi$.

Inconsistent Set of Formulas (Example)

$\mathcal{F} = \{ \mathit{alma}_{/0} \}, \ \mathcal{P} = \{ \mathit{loves}_{/2} \}$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((\mathit{loves}(x, \mathit{alma}) \wedge \mathit{loves}(y, x)) \rightarrow \neg \mathit{loves}(y, \mathit{alma}))$$

The set $\Gamma = \{ \phi, \mathit{loves}(\mathit{alma}, \mathit{alma}) \}$ is **inconsistent**:

Suppose that $\mathcal{M}$ is a model with $\mathcal{M} \models \mathit{loves}(\mathit{alma}, \mathit{alma})$.

Then in $\mathcal{M}$:

- Alma is a lover of Alma.
- Alma is a lover of a lover of Alma.
- Alma is a lover's lover of Alma who loves Alma.

Consequently $\mathcal{M} \not\models \phi$.

Hence there is no model satisfying both formulas in the set.

Translation into Predicate Logic and the Interplay of Quantifiers

Translating into Predicate Logic

Marie and Jan are clever	$C(m) \wedge C(j)$
Not everybody is clever	$\neg\forall x\, C(x)$
Somebody has learned logic	$\exists x\, LL(x)$
Not everybody has learned logic, but Marie and Jan have	$\neg\forall x\, LL(x) \wedge LL(m) \wedge LL(j)$

Specification and model used

$C(x)$: x is clever $\qquad m$: Marie

$LL(x)$: x has learned logic $\qquad j$: Jan

So we use the following model $\mathcal{M}$:

- domain $A =$ the set of all humans
- $C^{\mathcal{M}} = \{\, x \in A \mid x \text{ is clever} \,\}$
- $LL^{\mathcal{M}} = \{\, x \in A \mid x \text{ has learned logic} \,\}$
- $j^{\mathcal{M}} =$ Jan, and $m^{\mathcal{M}} =$ Marie.

The Combination of $\forall$ and $\rightarrow$

What does $\forall x(LL(x) \rightarrow C(x))$ mean? Translating step by step:

- for all x, if x has learned logic, then x is clever
- every x who has learned logic is clever
- everyone who has learned logic is clever

Don't confuse $\forall x(LL(x) \rightarrow C(x))$ with $\forall x LL(x) \rightarrow \forall x C(x)$

$\forall x\, LL(x)$	everybody has learned logic
$\forall x\, C(x)$	everybody is clever
$\forall x\, LL(x) \rightarrow \forall x\, C(x)$	if everybody has learned logic, everybody is clever

Question: How can we make precise that $\forall x(LL(x) \rightarrow C(x))$ and $\forall x\, LL(x) \rightarrow \forall x\, C(x)$ have a different meaning?

$\forall x(LL(x) \to C(x)) \;\not\equiv\; \forall x\, LL(x) \to \forall x\, C(x)$

Model $\mathcal{M}$ with $A = \{\,1, 2\,\}$, $LL^{\mathcal{M}} = \{\,1\,\}$, $C^{\mathcal{M}} = \{\,2\,\}$

$\mathcal{M} \not\models \forall x(LL(x) \to C(x))$

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{M} \models_{\ell} \forall x(LL(x) \to C(x)) \quad \times} \\
\iff & \text{for all } a \in A\text{: } \mathcal{M} \models_{\ell[x \mapsto a]} LL(x) \to C(x) \\
\iff & \text{for all } a \in A\text{: if } \mathcal{M} \models_{\ell[x \mapsto a]} LL(x)\text{, then } \mathcal{M} \models_{\ell[x \mapsto a]} C(x) \\
\iff & \text{for all } a \in A\text{: if } a \in LL^{\mathcal{M}}\text{, then } a \in C^{\mathcal{M}} \\
\implies & \text{if } 1 \in LL^{\mathcal{M}}\text{, then } 1 \in C^{\mathcal{M}} \quad \times
\end{array}
$$

$\mathcal{M} \models \forall x\, LL(x) \to \forall x\, C(x)$

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{M} \models_{\ell} \forall x\, LL(x) \to \forall x\, C(x) \quad \checkmark} \\
\iff & \text{if } \mathcal{M} \models_{\ell} \forall x\, LL(x) \text{ then } \mathcal{M} \models_{\ell} \forall x\, C(x) \\
\Longleftarrow & \text{not: } \mathcal{M} \models_{\ell} \forall x\, LL(x) \\
\iff & \text{not for all } a \in A\text{: } \mathcal{M} \models_{\ell[x \mapsto a]} LL(x) \\
\iff & \text{not for all } a \in A\text{: } a \in LL^{\mathcal{M}} \quad \checkmark
\end{array}
$$

Distribution of $\forall$ over $\wedge$ and $\vee$?

Does it work for $\wedge$ and $\vee$?

- $\forall x(LL(x) \wedge C(x)) \equiv \forall x\, LL(x) \wedge \forall x\, C(x)$?
- $\forall x(LL(x) \vee C(x)) \equiv \forall x\, LL(x) \vee \forall x\, C(x)$?

Are there counter models?

In the first case, no, in the second case, yes!

$\forall x(LL(x) \wedge C(x)) \equiv \forall x\, LL(x) \wedge \forall x\, C(x)$

- For every model $\mathcal{M}$ with domain A it holds that:

$$\mathcal{M} \models \forall x(LL(x) \wedge C(x)) \iff LL^{\mathcal{M}} = C^{\mathcal{M}} = A$$

- The same is the case for $\mathcal{M} \models \forall x\, LL(x) \wedge \forall x\, C(x)$
- Thus the two formulas hold in precisely the same models.

But: $\forall x(LL(x) \vee C(x)) \not\equiv \forall x\, LL(x) \vee \forall x\, C(x)$

Define $\mathcal{M}$ as follows:

$$\text{domain } A = \{1,2\} \qquad LL^{\mathcal{M}} = \{\ 1\ \} \qquad C^{\mathcal{M}} = \{\ 2\ \}$$

Then:

- $\mathcal{M} \models_{\ell} \forall x(LL(x) \vee C(x))$, because:
  - $\mathcal{M} \models_{\ell[x\mapsto 1]} LL(x)$ and thus $\mathcal{M} \models_{\ell[x\mapsto 1]} LL(x) \vee C(x)$
  - $\mathcal{M} \models_{\ell[x\mapsto 2]} C(x)$ and thus $\mathcal{M} \models_{\ell[x\mapsto 2]} LL(x) \vee C(x)$
  - Hence: $\mathcal{M} \models_{\ell} \forall x(LL(x) \vee C(x))$
- But $\mathcal{M} \not\models_{\ell} \forall x\, LL(x) \vee \forall x\, C(x)$, because:
  - $\mathcal{M} \not\models_{\ell[x\mapsto 2]} LL(x)$ and thus $\mathcal{M} \not\models_{\ell} \forall x\, LL(x)$
  - $\mathcal{M} \not\models_{\ell[x\mapsto 1]} C(x)$ and thus $\mathcal{M} \not\models_{\ell} \forall x\, C(x)$

Follow-up question:

What about distribution of $\exists$ over $\wedge$ and $\vee$?

The Combination of $\exists$ and $\wedge$

Another frequent combination is that of $\exists$ and $\wedge$, as in

$$\exists x(L(x) \wedge C(x))$$

We specify meanings for . . .

$L(x)$: x is a logician	r : Rosalie
$K(x, y)$: x knows y	j : Jan

We translate the formula $\exists x(L(x) \wedge C(x))$ step by step:

- for some x it holds that x is a logician, and that x is clever
- there is an x that is logician and clever
- there is a clever logician
- *some* logicians are clever

Some Logicians versus All Logicians

Recapitulating:

$\forall x(L(x) \rightarrow C(x))$ *all* logicians are clever

$\exists x(L(x) \wedge C(x))$ *some* logicians are clever

Note: $\forall x(L(x) \wedge C(x))$ means something quite different

- for all x, x is a logician, and x is clever
- every x is logician and clever
- everybody is logician and clever

What does $\exists x(L(x) \rightarrow C(x))$ mean? Hint: again very different.

Exercise

Specify a model such that:

$$\forall x(L(x) \rightarrow C(x)) \not\equiv \forall x(L(x) \wedge C(x))$$

Free Variables and Properties

Formulas with free variables express properties and relations.

$L(x)$: x is a logician	r : Rosalie
$K(x, y)$: x knows y	j : Jan

$L(x)$	x is a logician
$L(x) \wedge C(x)$	x is a clever logician
$K(j, y) \wedge L(y)$	Jan knows y, and y is a logician
$K(x, y) \wedge L(y)$	x knows y, and y is a logician
$\exists x(K(j, x) \wedge L(x))$	Jan knows a logician
$\exists x(K(y, x) \wedge L(x))$	y knows a logician
$\forall y \exists x(K(y, x) \wedge L(x))$	Everybody knows a logician

Free Variables and Environments

For a formula with free variables such as

$$L(x)\ ,$$

its validity depends on the interpretation of x:

$$\mathcal{M} \ \models_{\ell[x \mapsto m]} \ L(x) \quad \Longleftrightarrow \quad m \in L^{\mathcal{M}}$$

Since we chose $L^{\mathcal{M}}$ to be the set of logicians, $L(x)$ thus says:

x is a logician.

A formula such as

$$\exists y(K(x, y) \wedge L(y))$$

with free variable x expresses a property:

x knows a logician

Formulas with Free Variables

. . . express properties and relations

- formula without free variables: a **sentence**
- formula with one free variable: an **property**
- formula with two or more free variables: a **relation**

$L(r)$	Roos is a logician	(sentence)
$L(y)$	y is a logician	(property)
$K(x, y)$	x knows y	(relation)
$K(j, r) \wedge L(r)$	Jan knows Roos, and Roos is a logician	(sentence)
$K(j, y) \wedge L(y)$	Jan knows y, and y is a logician	(property)
$K(x, y) \wedge L(y)$	x knows y, and y is a logician	(relation)
$\exists x(K(y, x) \wedge L(x))$	y knows a logician	(property)
$\forall y \exists x(K(y, x) \wedge L(x))$	Everybody knows a logician	(sentence)

Alternative Translations

We can translate the sentence:

Nobody is perfect.

in two ways:

There does not exist anybody who is perfect. : $\neg\exists x\, P(x)$

Everybody is not perfect. : $\forall x\, \neg P(x)$

These three English sentences have the same meaning.

The two formulas are **logically equivalent**:

$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

It holds in general, for every formula ϕ :

$$\neg\exists x\, \phi \equiv \forall x\, \neg\phi$$

Alternative Translations

Another example:

Jan does not know a logician.

Again two possibilities:

There is not anybody Jan knows who is a logician. : $\neg\exists x(K(j,x) \wedge L(x))$

Everybody Jan knows is not a logician.; $\forall x(K(j,x) \rightarrow \neg L(x))$

Using the equivalence $\neg\exists x \equiv \forall x \neg$, we can transform

$$\neg\exists x(K(j,x) \wedge L(x)) \quad \text{into} \quad \forall x \neg(K(j,x) \wedge L(x)).$$

From propositional logic we know that

$$\neg(K(j,x) \wedge L(x)) \equiv K(j,x) \rightarrow \neg L(x)$$

Combining these two transformations we obtain in fact:

$$\neg\exists x(K(j,x) \rightarrow L(x)) \equiv \forall x(K(j,x) \rightarrow \neg L(x))$$

Interplay between Quantifiers and Connectives

$\neg$ versus $\exists$, $\forall$

Proposition

For all variables $\boldsymbol{x}$ and formulas ϕ it holds:

(i) $\neg\forall \boldsymbol{x}\, \phi \equiv \exists \boldsymbol{x} \neg\phi$

(ii) $\neg\exists \boldsymbol{x}\, \phi \equiv \forall \boldsymbol{x} \neg\phi$

(iii) $\forall \boldsymbol{x}\, \phi \equiv \neg\exists \boldsymbol{x} \neg\phi$

(iv) $\exists \boldsymbol{x}\, \phi \equiv \neg\forall \boldsymbol{x} \neg\phi$

Equivalently, the following formulas are valid:

(i) $\neg\forall \boldsymbol{x}\, \phi \leftrightarrow \exists \boldsymbol{x} \neg\phi$

(ii) $\neg\exists \boldsymbol{x}\, \phi \leftrightarrow \forall \boldsymbol{x} \neg\phi$

(iii) $\forall \boldsymbol{x}\, \phi \leftrightarrow \neg\exists \boldsymbol{x} \neg\phi$

(iv) $\exists \boldsymbol{x}\, \phi \leftrightarrow \neg\forall \boldsymbol{x} \neg\phi$

$\wedge, \vee$ versus $\exists, \forall$

Proposition

For all variables $\boldsymbol{x}$ and formulas ϕ it holds:

(i) $\forall \boldsymbol{x}(\phi \wedge \psi) \equiv \forall \boldsymbol{x}\, \phi \wedge \forall \boldsymbol{x}\, \psi$

(ii) $\exists \boldsymbol{x}(\phi \vee \psi) \equiv \exists \boldsymbol{x}\, \phi \vee \exists \boldsymbol{x}\, \psi$

However, in general:

- $\forall \boldsymbol{x}(\phi \vee \psi) \not\equiv \forall \boldsymbol{x}\, \phi \vee \forall \boldsymbol{x}\, \psi$
- $\exists \boldsymbol{x}(\phi \wedge \psi) \not\equiv \exists \boldsymbol{x}\, \phi \wedge \exists \boldsymbol{x}\, \psi$

But note: for some specific ϕ and ψ :

- $\forall \boldsymbol{x}(\phi \vee \psi) \equiv \forall \boldsymbol{x}\, \phi \vee \forall \boldsymbol{x}\, \psi$
- $\exists \boldsymbol{x}(\phi \wedge \psi) \equiv \exists \boldsymbol{x}\, \phi \wedge \exists \boldsymbol{x}\, \psi$

can hold (e.g. if ϕ and ψ are the same formula).

$\wedge$, $\vee$ versus $\exists$, $\forall$

Proposition

For all variables $\boldsymbol{x}$ and formulas ϕ it holds:

(i) $\forall \boldsymbol{x}(\phi \wedge \psi) \equiv \forall \boldsymbol{x}\, \phi \wedge \forall \boldsymbol{x}\, \psi$

(ii) $\exists \boldsymbol{x}(\phi \vee \psi) \equiv \exists \boldsymbol{x}\, \phi \vee \exists \boldsymbol{x}\, \psi$

However, in general:

- $\forall \boldsymbol{x}(\phi \vee \psi) \not\equiv \forall \boldsymbol{x}\, \phi \vee \forall \boldsymbol{x}\, \psi$
- $\exists \boldsymbol{x}(\phi \wedge \psi) \not\equiv \exists \boldsymbol{x}\, \phi \wedge \exists \boldsymbol{x}\, \psi$

Yet:

Proposition

For all variables $\boldsymbol{x}$ and formulas ϕ these formulas are **valid**:

(i) $(\forall \boldsymbol{x}\, \phi \vee \forall \boldsymbol{x}\, \psi) \to \forall \boldsymbol{x}(\phi \vee \psi)$

(ii) $\exists \boldsymbol{x}(\phi \wedge \psi) \to \exists \boldsymbol{x}\, \phi \wedge \exists \boldsymbol{x}\, \psi$

$\to$ versus $\exists$, $\forall$

In general:

- $\forall \boldsymbol{x}(\phi \to \psi) \not\equiv \forall \boldsymbol{x}\,\phi \to \forall \boldsymbol{x}\,\psi$
- $\exists \boldsymbol{x}(\phi \to \psi) \not\equiv \exists \boldsymbol{x}\,\phi \to \exists \boldsymbol{x}\,\psi$

Note again: for specific ϕ and ψ it is possible:

- $\forall \boldsymbol{x}(\phi \to \psi) \equiv \forall \boldsymbol{x}\,\phi \to \forall \boldsymbol{x}\,\psi$
- $\exists \boldsymbol{x}(\phi \to \psi) \equiv \exists \boldsymbol{x}\,\phi \to \exists \boldsymbol{x}\,\psi$

(e.g. if ϕ and ψ are the same formula).

Yet, we have:

Proposition

For all variables $\boldsymbol{x}$ and formulas ϕ these formulas are **valid**:

(i) $\forall \boldsymbol{x}(\phi \to \psi) \to (\forall \boldsymbol{x}\,\phi \to \forall \boldsymbol{x}\,\psi)$

(ii) $(\exists \boldsymbol{x}\,\phi \to \exists \boldsymbol{x}\,\psi) \to \exists \boldsymbol{x}(\phi \to \psi)$

Quantifier Operations

Proposition

For all variables $\boldsymbol{x}$, $\boldsymbol{y}$ and formulas ϕ it holds:

(i) $\forall \boldsymbol{x} \forall \boldsymbol{y}\, \phi \equiv \forall \boldsymbol{y} \forall \boldsymbol{x}\, \phi$

(ii) $\exists \boldsymbol{x} \exists \boldsymbol{y}\, \phi \equiv \exists \boldsymbol{y} \exists \boldsymbol{x}\, \phi$

(iii) $\forall \boldsymbol{x}\, \phi \equiv \phi$ if $\boldsymbol{x}$ not free in ϕ

(iv) $\exists \boldsymbol{x}\, \phi \equiv \phi$ if $\boldsymbol{x}$ not free in ϕ

However, in general:

$$\exists \boldsymbol{x} \forall \boldsymbol{y}\, \phi \not\equiv \forall \boldsymbol{y} \exists \boldsymbol{x}\, \phi$$

Change of Bound Variables

Proposition

For all variables $\boldsymbol{x}$ and formulas ϕ it holds:

(i) $\exists \boldsymbol{x}\, \phi \equiv \exists \boldsymbol{z}\, \phi[\boldsymbol{z}/\boldsymbol{x}]$ if $\boldsymbol{z}$ not free in ϕ

(ii) $\forall \boldsymbol{x}\, \phi \equiv \forall \boldsymbol{z}\, \phi[\boldsymbol{z}/\boldsymbol{x}]$ if $\boldsymbol{z}$ not free in ϕ

Recall that

$$\phi[t/\boldsymbol{x}]$$

is the result of replacing all free occurrences of $\boldsymbol{x}$ in ϕ by t if no capture of free variables happens, otherwise *undefined*.

$$\exists y\; R(x, y)\; [y/x] \text{ is undefined}$$

since in the replacement result $\exists y\; R(y, y)$ capture of the inserted variable y has happened.

Quantifier Operations (Advanced)

Proposition

For all variables $\boldsymbol{x}$, and formulas ϕ, ψ such that $\boldsymbol{x}$ does not occur free in ψ it holds:

(i) $\forall \boldsymbol{x} \phi \rightarrow \psi \equiv \exists \boldsymbol{x}(\phi \rightarrow \psi)$

(ii) $\exists \boldsymbol{x} \phi \rightarrow \psi \equiv \forall \boldsymbol{x}(\phi \rightarrow \psi)$

(iii) $(\psi \rightarrow \exists \boldsymbol{x} \phi) \equiv \exists \boldsymbol{x}(\psi \rightarrow \phi)$

(iv) $(\psi \rightarrow \forall \boldsymbol{x} \phi) \equiv \forall \boldsymbol{x}(\psi \rightarrow \phi)$