

Logic and Modelling

— Semantics of, and Translations into, Predicate Logic —

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Reminder: Models and Environments

Let

- ▶ \mathcal{F} be a set of function symbols,
- ▶ \mathcal{P} a set of predicate symbols.

A **model** \mathcal{M} for $\langle \mathcal{F}, \mathcal{P} \rangle$ consists of:

- ▶ a non-empty set A , called **domain** or **universe**,
- ▶ an **interpretation operation** $(\cdot)^{\mathcal{M}}$ for the symbols in \mathcal{F}, \mathcal{P} .
 - (i) $f^{\mathcal{M}} : A^n \rightarrow A$ for every n -ary function symbol $f \in \mathcal{F}$
 - (ii) $P^{\mathcal{M}} \subseteq A^n$ for every n -ary predicate symbols $P \in \mathcal{P}$

A symbol is **n -ary** if it has n arguments.

An **environment** (look-up function)

$$\ell : \mathbf{var} \rightarrow A$$

interprets **free** variables in the domain.

Reminder: Formula Truth in a Model

Truth of a formula ϕ in a model \mathcal{M} with universe A *with respect to environment* ℓ is defined by induction on the structure of ϕ :

- ▶ $\mathcal{M} \models_{\ell} \neg\phi \iff \mathcal{M} \not\models_{\ell} \phi$
- ▶ $\mathcal{M} \models_{\ell} \phi \wedge \psi \iff \mathcal{M} \models_{\ell} \phi$ and $\mathcal{M} \models_{\ell} \psi$
- ▶ $\mathcal{M} \models_{\ell} \phi \vee \psi \iff \mathcal{M} \models_{\ell} \phi$ or $\mathcal{M} \models_{\ell} \psi$
- ▶ $\mathcal{M} \models_{\ell} \phi \rightarrow \psi \iff$ (if $\mathcal{M} \models_{\ell} \phi$ then $\mathcal{M} \models_{\ell} \psi$)
- ▶ $\mathcal{M} \models_{\ell} \mathbf{P}(t_1, \dots, t_n) \iff \langle t_1^{\mathcal{M}, \ell}, \dots, t_n^{\mathcal{M}, \ell} \rangle \in \mathbf{P}^{\mathcal{M}}$

$$t^{\mathcal{M}, \ell} = \begin{cases} \ell(\mathbf{x}) & \text{if } t = \mathbf{x} \text{ for a variable } \mathbf{x} \\ \mathbf{c}^{\mathcal{M}} & \text{if } t = \mathbf{c} \text{ for a constant } \mathbf{c} \\ \mathbf{f}^{\mathcal{M}}(t_1^{\mathcal{M}, \ell}, \dots, t_n^{\mathcal{M}, \ell}) & \text{if } t = \mathbf{f}(t_1, \dots, t_n) \end{cases}$$

- ▶ $\mathcal{M} \models_{\ell} \forall \mathbf{x} \phi \iff$ for all $\mathbf{a} \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto \mathbf{a}]} \phi$
- ▶ $\mathcal{M} \models_{\ell} \exists \mathbf{x} \phi \iff$ for some $\mathbf{a} \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto \mathbf{a}]} \phi$

Reminder: Semantical Entailment

Semantic entailment in propositional logic

In **propositional logic**: $\phi_1, \dots, \phi_n \models \psi$ means:

For all valuations v :

$$v(\phi_1) = \top, \dots, v(\phi_n) = \top \implies v(\psi) = \top$$

Semantic entailment in predicate logic

In **predicate logic**: $\phi_1, \dots, \phi_n \models \psi$ means:

For all models \mathcal{M} , and all environments ℓ :

$$\mathcal{M} \models_{\ell} \phi_1, \dots, \mathcal{M} \models_{\ell} \phi_n \implies \mathcal{M} \models_{\ell} \psi$$

In words: for all models \mathcal{M} , and all environments ℓ such that

$$\mathcal{M} \models_{\ell} \phi_1 \text{ and } \dots \text{ and } \mathcal{M} \models_{\ell} \phi_n \text{ hold,}$$

it also holds that $\mathcal{M} \models_{\ell} \psi$.

Logic Equivalence

Logical Equivalence

Definition (Logical equivalence \equiv)

Formulas ϕ and ψ are **logically equivalent**, denoted by

$$\phi \equiv \psi$$

if for all models \mathcal{M} and environments ℓ :

$$\mathcal{M} \models_{\ell} \phi \iff \mathcal{M} \models_{\ell} \psi$$

That is, ϕ and ψ are true in precisely the same models when interpreted with the same environments.

Logical Equivalence

Theorem

$$\phi \equiv \psi \iff \phi \vDash \psi \text{ and } \psi \vDash \phi$$

Therefore, logical equivalence is sometimes denoted $\vDash\!\!\vDash$.

Proof.

$$\phi \equiv \psi$$

$$\iff \text{for all } \mathcal{M} \text{ and } \ell:$$

$$(\mathcal{M} \vDash_{\ell} \phi \Rightarrow \mathcal{M} \vDash_{\ell} \psi) \text{ and } (\mathcal{M} \vDash_{\ell} \psi \Rightarrow \mathcal{M} \vDash_{\ell} \phi)$$

$$\iff \text{for all } \mathcal{M} \text{ and } \ell: \mathcal{M} \vDash_{\ell} \phi \Rightarrow \mathcal{M} \vDash_{\ell} \psi$$

$$\text{and for all } \mathcal{M} \text{ and } \ell: \mathcal{M} \vDash_{\ell} \psi \Rightarrow \mathcal{M} \vDash_{\ell} \phi)$$

$$\iff \phi \vDash \psi \text{ and } \psi \vDash \phi \text{ (that is, } \phi \vDash\!\!\vDash \psi)$$



Proving Logical Equivalence

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

For all models \mathcal{M} with domain A and environments ℓ we find:

$$\mathcal{M} \models_{\ell} \neg \forall x P(x)$$

$$\iff \text{not: } \mathcal{M} \models_{\ell} \forall x P(x)$$

$$\iff \text{not for all } a \in A: \mathcal{M} \models_{\ell[x \mapsto a]} P(x)$$

$$\iff \text{there exists } a \in A \text{ such that not: } \mathcal{M} \models_{\ell[x \mapsto a]} P(x)$$

$$\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \not\models_{\ell[x \mapsto a]} P(x)$$

$$\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \models_{\ell[x \mapsto a]} \neg P(x)$$

$$\iff \mathcal{M} \models_{\ell} \exists x \neg P(x)$$

Hence we can conclude: $\neg \forall x P(x) \equiv \exists x \neg P(x)$.

Note that each step is a bi-implication, so we have \Rightarrow and \Leftarrow .

Satisfiability, Validity and Consistency

Satisfiability, Validity, Consistency

Definition (Satisfiability, validity of formulas)

Let ϕ be a formula, and Γ be a set of formulas.

- ▶ ϕ is **satisfiable** \iff there is **some** model \mathcal{M} and **some** environment ℓ such that $\mathcal{M} \models_{\ell} \phi$.
- ▶ ϕ is **valid** $\iff \mathcal{M} \models_{\ell} \phi$ holds for **all** models \mathcal{M} and **all** environments ℓ in which ϕ can be checked.
- ▶ Γ is **consistent** or **satisfiable** \iff there is **some** model \mathcal{M} and **some** environment ℓ such that
$$\mathcal{M} \models_{\ell} \psi \quad \text{for all } \psi \in \Gamma.$$

Logical Equivalence, and Validity of Bi-implication

Proposition

For all formulas ϕ , ψ the following statements are equivalent:

- (i) $\phi \equiv \psi$
- (ii) $\phi \leftrightarrow \psi$ is valid.

Proof.

For all models \mathcal{M} and environments ℓ it holds:

$$\begin{aligned} \mathcal{M} \models_{\ell} \phi \leftrightarrow \psi & \\ \iff \mathcal{M} \models_{\ell} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) & \\ \iff \mathcal{M} \models_{\ell} \phi \rightarrow \psi \text{ and } \mathcal{M} \models_{\ell} \psi \rightarrow \phi & \\ \iff \text{if } \mathcal{M} \models_{\ell} \phi, \text{ then } \mathcal{M} \models_{\ell} \psi, & \\ \text{and if } \mathcal{M} \models_{\ell} \psi, \text{ then } \mathcal{M} \models_{\ell} \phi & \\ \iff (\mathcal{M} \models_{\ell} \phi \iff \mathcal{M} \models_{\ell} \psi) & \end{aligned}$$

From this we conclude: $\phi \leftrightarrow \psi$ is valid if and only if $\phi \equiv \psi$. \square

Examples

$\mathcal{F} = \{ \textit{alma}/0 \}, \mathcal{P} = \{ \textit{loves}/2 \}$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$\forall x \forall y ((\textit{loves}(x, \textit{alma}) \wedge \textit{loves}(y, x)) \rightarrow \neg \textit{loves}(y, \textit{alma}))$



Alma Mahler/Werfel/Gropius (1879–1964)

Satisfiable Formula (Example)

$$\mathcal{F} = \{ \textit{alma}/0 \}, \mathcal{P} = \{ \textit{loves}/2 \}$$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((\textit{loves}(x, \textit{alma}) \wedge \textit{loves}(y, x)) \rightarrow \neg \textit{loves}(y, \textit{alma}))$$

The formula ϕ is **satisfiable**:

Consider model \mathcal{M}_1 with domain $A_1 = \{ a, g, f, w \}$ and:

$$\textit{alma}^{\mathcal{M}_1} = a \qquad \textit{loves}^{\mathcal{M}_1} = \{ \langle g, a \rangle, \langle f, a \rangle, \langle w, a \rangle \}$$

In this model there are not any lovers of lovers.

Hence: $\mathcal{M}_1 \models \phi$.

Consistent Set of Formulas (Example)

$$\mathcal{F} = \{ \textit{alma}/_0 \}, \mathcal{P} = \{ \textit{loves}/_2 \}$$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((\textit{loves}(x, \textit{alma}) \wedge \textit{loves}(y, x)) \rightarrow \neg \textit{loves}(y, \textit{alma}))$$

The formula ϕ is **satisfiable**:

Consider model \mathcal{M}_1 with domain $A_1 = \{ a, g, f, w \}$ and:

$$\textit{alma}^{\mathcal{M}_1} = a \quad \textit{loves}^{\mathcal{M}_1} = \{ \langle g, a \rangle, \langle f, a \rangle, \langle w, a \rangle \}$$

In this model there are not any lovers of lovers.

Hence: $\mathcal{M}_1 \models \phi$.

The set $\Gamma = \{ \phi, \neg \textit{loves}(\textit{alma}, \textit{alma}) \}$ is **consistent**:

since it also holds: $\mathcal{M}_1 \models \neg \textit{loves}(\textit{alma}, \textit{alma})$.

Not Valid Formula (Example)

$$\mathcal{F} = \{ \textit{alma}/0 \}, \mathcal{P} = \{ \textit{loves}/2 \}$$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((\textit{loves}(x, \textit{alma}) \wedge \textit{loves}(y, x)) \rightarrow \neg \textit{loves}(y, \textit{alma}))$$

The formula ϕ is **not valid**:

Let \mathcal{M}_2 have domain $A_2 = \{ a, g, f, w \}$, $\textit{alma}^{\mathcal{M}_1} = a$, and:

$$\textit{loves}^{\mathcal{M}_2} = \{ \langle a, g \rangle, \langle a, f \rangle, \langle a, w \rangle, \langle a, a \rangle, \langle g, a \rangle, \langle f, a \rangle, \langle w, a \rangle \}$$

In this model:

- ▶ Alma herself is a lover of a lover of herself.
- ▶ Alma loves herself.

Hence: $\mathcal{M}_2 \not\models \phi$.

Inconsistent Set of Formulas (Example)

$$\mathcal{F} = \{ \textit{alma}/0 \}, \mathcal{P} = \{ \textit{loves}/2 \}$$

None of Alma's lovers' lovers loves her.

translated to the formula ϕ :

$$\forall x \forall y ((\textit{loves}(x, \textit{alma}) \wedge \textit{loves}(y, x)) \rightarrow \neg \textit{loves}(y, \textit{alma}))$$

The set $\Gamma = \{ \phi, \textit{loves}(\textit{alma}, \textit{alma}) \}$ is **inconsistent**:

Suppose that \mathcal{M} is a model with $\mathcal{M} \models \textit{loves}(\textit{alma}, \textit{alma})$.

Then in \mathcal{M} :

- ▶ Alma is a lover of Alma.
- ▶ Alma is a lover of a lover of Alma.
- ▶ Alma is a lover's lover of Alma who loves Alma.

Consequently $\mathcal{M} \not\models \phi$.

Hence there is no model satisfying both formulas in the set.

Translation into Predicate Logic and the Interplay of Quantifiers

Translating into Predicate Logic

Marie and Jan are clever $C(m) \wedge C(j)$

Not everybody is clever $\neg \forall x C(x)$

Somebody has learned logic $\exists x LL(x)$

Not everybody has learned logic,
but Marie and Jan have $\neg \forall x LL(x) \wedge LL(m) \wedge LL(j)$

Specification and model used

$C(x)$: x is clever

m : Marie

$LL(x)$: x has learned logic

j : Jan

So we use the following model \mathcal{M} :

- ▶ domain A = the set of all humans
- ▶ $C^{\mathcal{M}} = \{ x \in A \mid x \text{ is clever} \}$
- ▶ $LL^{\mathcal{M}} = \{ x \in A \mid x \text{ has learned logic} \}$
- ▶ $j^{\mathcal{M}} = \text{Jan}$, and $m^{\mathcal{M}} = \text{Marie}$.

The Combination of \forall and \rightarrow

What does $\forall x(LL(x) \rightarrow C(x))$ mean? Translating step by step:

- ▶ for all x , if x has learned logic, then x is clever
- ▶ every x who has learned logic is clever
- ▶ everyone who has learned logic is clever

Don't confuse $\forall x(LL(x) \rightarrow C(x))$ with $\forall xLL(x) \rightarrow \forall xC(x)$

$\forall x LL(x)$ everybody has learned logic

$\forall x C(x)$ everybody is clever

$\forall x LL(x) \rightarrow \forall x C(x)$ if everybody has learned logic,
everybody is clever

Question: How can we make precise that $\forall x(LL(x) \rightarrow C(x))$ and $\forall x LL(x) \rightarrow \forall x C(x)$ have a different meaning?

$$\forall x(LL(x) \rightarrow C(x)) \not\equiv \forall x LL(x) \rightarrow \forall x C(x)$$

Model \mathcal{M} with $A = \{1, 2\}$, $LL^{\mathcal{M}} = \{1\}$, $C^{\mathcal{M}} = \{2\}$

$$\mathcal{M} \not\models \forall x(LL(x) \rightarrow C(x))$$

$$\mathcal{M} \not\models_{\ell} \forall x(LL(x) \rightarrow C(x)) \quad \times$$

$$\iff \text{for all } a \in A: \mathcal{M} \not\models_{\ell[x \mapsto a]} LL(x) \rightarrow C(x)$$

$$\iff \text{for all } a \in A: \text{if } \mathcal{M} \models_{\ell[x \mapsto a]} LL(x), \text{ then } \mathcal{M} \not\models_{\ell[x \mapsto a]} C(x)$$

$$\iff \text{for all } a \in A: \text{if } a \in LL^{\mathcal{M}}, \text{ then } a \in C^{\mathcal{M}}$$

$$\implies \text{if } 1 \in LL^{\mathcal{M}}, \text{ then } 1 \in C^{\mathcal{M}} \quad \times$$

$$\mathcal{M} \models \forall x LL(x) \rightarrow \forall x C(x)$$

$$\mathcal{M} \models_{\ell} \forall x LL(x) \rightarrow \forall x C(x) \quad \checkmark$$

$$\iff \text{if } \mathcal{M} \models_{\ell} \forall x LL(x) \text{ then } \mathcal{M} \models_{\ell} \forall x C(x)$$

$$\iff \text{not: } \mathcal{M} \models_{\ell} \forall x LL(x)$$

$$\iff \text{not for all } a \in A: \mathcal{M} \models_{\ell[x \mapsto a]} LL(x)$$

$$\iff \text{not for all } a \in A: a \in LL^{\mathcal{M}} \quad \checkmark$$

Distribution of \forall over \wedge and \vee ?

Does it work for \wedge and \vee ?

▶ $\forall x(LL(x) \wedge C(x)) \equiv \forall x LL(x) \wedge \forall x C(x)$?

▶ $\forall x(LL(x) \vee C(x)) \equiv \forall x LL(x) \vee \forall x C(x)$?

Are there counter models?

In the first case, no, in the second case, yes!

$$\forall x(LL(x) \wedge C(x)) \equiv \forall x LL(x) \wedge \forall x C(x)$$

▶ For every model \mathcal{M} with domain A it holds that:

$$\mathcal{M} \models \forall x(LL(x) \wedge C(x)) \iff LL^{\mathcal{M}} = C^{\mathcal{M}} = A$$

▶ The same is the case for $\mathcal{M} \models \forall x LL(x) \wedge \forall x C(x)$

▶ Thus the two formulas hold in precisely the same models.

But: $\forall x(LL(x) \vee C(x)) \not\equiv \forall x LL(x) \vee \forall x C(x)$

Define \mathcal{M} as follows:

$$\text{domain } A = \{1, 2\} \quad LL^{\mathcal{M}} = \{1\} \quad C^{\mathcal{M}} = \{2\}$$

Then:

- ▶ $\mathcal{M} \models_e \forall x(LL(x) \vee C(x))$, because:
 - ▶ $\mathcal{M} \models_{e[x \mapsto 1]} LL(x)$ and thus $\mathcal{M} \models_{e[x \mapsto 1]} LL(x) \vee C(x)$
 - ▶ $\mathcal{M} \models_{e[x \mapsto 2]} C(x)$ and thus $\mathcal{M} \models_{e[x \mapsto 2]} LL(x) \vee C(x)$
 - ▶ Hence: $\mathcal{M} \models_e \forall x(LL(x) \vee C(x))$
- ▶ But $\mathcal{M} \not\models_e \forall x LL(x) \vee \forall x C(x)$, because:
 - ▶ $\mathcal{M} \not\models_{e[x \mapsto 2]} LL(x)$ and thus $\mathcal{M} \not\models_e \forall x LL(x)$
 - ▶ $\mathcal{M} \not\models_{e[x \mapsto 1]} C(x)$ and thus $\mathcal{M} \not\models_e \forall x C(x)$

Follow-up question:

What about distribution of \exists over \wedge and \vee ?

The Combination of \exists and \wedge

Another frequent combination is that of \exists and \wedge , as in

$$\exists x(L(x) \wedge C(x))$$

We specify meanings for ...

$L(x)$: x is a logician

r : Rosalie

$K(x, y)$: x knows y

j : Jan

We translate the formula $\exists x(L(x) \wedge C(x))$ step by step:

- ▶ for some x it holds that x is a logician, and that x is clever
- ▶ there is an x that is logician and clever
- ▶ there is a clever logician
- ▶ *some* logicians are clever

Some Logicians versus All Logicians

Recapitulating:

$\forall x(L(x) \rightarrow C(x))$ *all logicians are clever*

$\exists x(L(x) \wedge C(x))$ *some logicians are clever*

Note: $\forall x(L(x) \wedge C(x))$ means something quite different

- ▶ for all x , x is a logician, and x is clever
- ▶ every x is logician and clever
- ▶ everybody is logician and clever

What does $\exists x(L(x) \rightarrow C(x))$ mean? **Hint: again very different.**

Exercise

Specify a model such that:

$$\forall x(L(x) \rightarrow C(x)) \neq \forall x(L(x) \wedge C(x))$$

Free Variables and Properties

Formulas with free variables express properties and relations.

$L(x)$: x is a logician

r : Rosalie

$K(x, y)$: x knows y

j : Jan

$L(x)$ x is a logician

$L(x) \wedge C(x)$ x is a clever logician

$K(j, y) \wedge L(y)$ Jan knows y , and y is a logician

$K(x, y) \wedge L(y)$ x knows y , and y is a logician

$\exists x(K(j, x) \wedge L(x))$ Jan knows a logician

$\exists x(K(y, x) \wedge L(x))$ y knows a logician

$\forall y \exists x(K(y, x) \wedge L(x))$ Everybody knows a logician

Free Variables and Environments

For a formula with free variables such as

$$L(x),$$

its validity depends on the interpretation of x :

$$\mathcal{M} \models_{\ell[x \mapsto m]} L(x) \iff m \in L^{\mathcal{M}}$$

Since we chose $L^{\mathcal{M}}$ to be the set of logicians, $L(x)$ thus says:

x is a logician.

A formula such as

$$\exists y (K(x, y) \wedge L(y))$$

with free variable x expresses a property:

x knows a logician

Formulas with Free Variables

... express properties and relations

- ▶ formula **without** free variables: a **sentence**
- ▶ formula **with one** free variable: an **property**
- ▶ formula **with two or more** free variables: a **relation**

$L(r)$ Roos is a logician (sentence)

$L(y)$ y is a logician (property)

$K(x, y)$ x knows y (relation)

$K(j, r) \wedge L(r)$ { Jan knows Roos,
and Roos is a logician (sentence)

$K(j, y) \wedge L(y)$ Jan knows y , and y is a logician (property)

$K(x, y) \wedge L(y)$ x knows y , and y is a logician (relation)

$\exists x(K(y, x) \wedge L(x))$ y knows a logician (property)

$\forall y \exists x(K(y, x) \wedge L(x))$ Everybody knows a logician (sentence)

Alternative Translations

We can translate the sentence:

Nobody is perfect.

in two ways:

There does not exist anybody who is perfect. : $\neg\exists x P(x)$

Everybody is not perfect. : $\forall x \neg P(x)$

These three English sentences have the same meaning.

The two formulas are **logically equivalent**:

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

It holds in general, for every formula ϕ :

$$\neg\exists x \phi \equiv \forall x \neg\phi$$

Alternative Translations

Another example:

Jan does not know a logician.

Again two possibilities:

There is not anybody Jan knows who is a logician. : $\neg\exists x(K(j, x) \wedge L(x))$

Everybody Jan knows is not a logician.; $\forall x(K(j, x) \rightarrow \neg L(x))$

Using the equivalence $\neg\exists x \equiv \forall x\neg$, we can transform

$\neg\exists x(K(j, x) \wedge L(x))$ into $\forall x\neg(K(j, x) \wedge L(x))$.

From propositional logic we know that

$$\neg(K(j, x) \wedge L(x)) \equiv K(j, x) \rightarrow \neg L(x)$$

Combining these two transformations we obtain in fact:

$$\neg\exists x(K(j, x) \rightarrow L(x)) \equiv \forall x(K(j, x) \rightarrow \neg L(x))$$

Interplay between Quantifiers and Connectives

\neg versus \exists , \forall

Proposition

For all variables \mathbf{x} and formulas ϕ it holds:

$$(i) \quad \neg \forall \mathbf{x} \phi \equiv \exists \mathbf{x} \neg \phi$$

$$(ii) \quad \neg \exists \mathbf{x} \phi \equiv \forall \mathbf{x} \neg \phi$$

$$(iii) \quad \forall \mathbf{x} \phi \equiv \neg \exists \mathbf{x} \neg \phi$$

$$(iv) \quad \exists \mathbf{x} \phi \equiv \neg \forall \mathbf{x} \neg \phi$$

Equivalently, the following formulas are valid:

$$(i) \quad \neg \forall \mathbf{x} \phi \leftrightarrow \exists \mathbf{x} \neg \phi$$

$$(ii) \quad \neg \exists \mathbf{x} \phi \leftrightarrow \forall \mathbf{x} \neg \phi$$

$$(iii) \quad \forall \mathbf{x} \phi \leftrightarrow \neg \exists \mathbf{x} \neg \phi$$

$$(iv) \quad \exists \mathbf{x} \phi \leftrightarrow \neg \forall \mathbf{x} \neg \phi$$

\wedge, \vee versus \exists, \forall

Proposition

For all variables \mathbf{x} and formulas ϕ it holds:

- (i) $\forall \mathbf{x}(\phi \wedge \psi) \equiv \forall \mathbf{x} \phi \wedge \forall \mathbf{x} \psi$
- (ii) $\exists \mathbf{x}(\phi \vee \psi) \equiv \exists \mathbf{x} \phi \vee \exists \mathbf{x} \psi$

However, **in general**:

- ▶ $\forall \mathbf{x}(\phi \vee \psi) \not\equiv \forall \mathbf{x} \phi \vee \forall \mathbf{x} \psi$
- ▶ $\exists \mathbf{x}(\phi \wedge \psi) \not\equiv \exists \mathbf{x} \phi \wedge \exists \mathbf{x} \psi$

But note: for some specific ϕ and ψ :

- ▶ $\forall \mathbf{x}(\phi \vee \psi) \equiv \forall \mathbf{x} \phi \vee \forall \mathbf{x} \psi$
- ▶ $\exists \mathbf{x}(\phi \wedge \psi) \equiv \exists \mathbf{x} \phi \wedge \exists \mathbf{x} \psi$

can hold (e.g. if ϕ and ψ are the same formula).

\wedge, \vee versus \exists, \forall

Proposition

For all variables \mathbf{x} and formulas ϕ it holds:

- (i) $\forall \mathbf{x}(\phi \wedge \psi) \equiv \forall \mathbf{x} \phi \wedge \forall \mathbf{x} \psi$
- (ii) $\exists \mathbf{x}(\phi \vee \psi) \equiv \exists \mathbf{x} \phi \vee \exists \mathbf{x} \psi$

However, **in general**:

- ▶ $\forall \mathbf{x}(\phi \vee \psi) \not\equiv \forall \mathbf{x} \phi \vee \forall \mathbf{x} \psi$
- ▶ $\exists \mathbf{x}(\phi \wedge \psi) \not\equiv \exists \mathbf{x} \phi \wedge \exists \mathbf{x} \psi$

Yet:

Proposition

For all variables \mathbf{x} and formulas ϕ these formulas are **valid**:

- (i) $(\forall \mathbf{x} \phi \vee \forall \mathbf{x} \psi) \rightarrow \forall \mathbf{x}(\phi \vee \psi)$
- (ii) $\exists \mathbf{x}(\phi \wedge \psi) \rightarrow \exists \mathbf{x} \phi \wedge \exists \mathbf{x} \psi$

→ versus \exists , \forall

In general:

- ▶ $\forall \mathbf{x}(\phi \rightarrow \psi) \not\equiv \forall \mathbf{x} \phi \rightarrow \forall \mathbf{x} \psi$
- ▶ $\exists \mathbf{x}(\phi \rightarrow \psi) \not\equiv \exists \mathbf{x} \phi \rightarrow \exists \mathbf{x} \psi$

Note again: for specific ϕ and ψ it is possible:

- ▶ $\forall \mathbf{x}(\phi \rightarrow \psi) \equiv \forall \mathbf{x} \phi \rightarrow \forall \mathbf{x} \psi$
- ▶ $\exists \mathbf{x}(\phi \rightarrow \psi) \equiv \exists \mathbf{x} \phi \rightarrow \exists \mathbf{x} \psi$

(e.g. if ϕ and ψ are the same formula).

Yet, we have:

Proposition

For all variables \mathbf{x} and formulas ϕ these formulas are **valid**:

- $\forall \mathbf{x}(\phi \rightarrow \psi) \rightarrow (\forall \mathbf{x} \phi \rightarrow \forall \mathbf{x} \psi)$
- $(\exists \mathbf{x} \phi \rightarrow \exists \mathbf{x} \psi) \rightarrow \exists \mathbf{x}(\phi \rightarrow \psi)$

Quantifier Operations

Proposition

For all variables \mathbf{x} , \mathbf{y} and formulas ϕ it holds:

- (i) $\forall \mathbf{x} \forall \mathbf{y} \phi \equiv \forall \mathbf{y} \forall \mathbf{x} \phi$
- (ii) $\exists \mathbf{x} \exists \mathbf{y} \phi \equiv \exists \mathbf{y} \exists \mathbf{x} \phi$
- (iii) $\forall \mathbf{x} \phi \equiv \phi$ if \mathbf{x} not free in ϕ
- (iv) $\exists \mathbf{x} \phi \equiv \phi$ if \mathbf{x} not free in ϕ

However, **in general**:

$$\exists \mathbf{x} \forall \mathbf{y} \phi \not\equiv \forall \mathbf{y} \exists \mathbf{x} \phi$$

Change of Bound Variables

Proposition

For all variables \mathbf{x} and formulas ϕ it holds:

- (i) $\exists \mathbf{x} \phi \equiv \exists \mathbf{z} \phi[\mathbf{z}/\mathbf{x}]$ if \mathbf{z} not free in ϕ
- (ii) $\forall \mathbf{x} \phi \equiv \forall \mathbf{z} \phi[\mathbf{z}/\mathbf{x}]$ if \mathbf{z} not free in ϕ

Recall that

$$\phi[t/\mathbf{x}]$$

is the result of replacing all free occurrences of \mathbf{x} in ϕ by t if **no capture of free variables happens**, otherwise *undefined*.

$$\exists y R(\mathbf{x}, y) [y/\mathbf{x}] \text{ is undefined}$$

since in the replacement result $\exists y R(y, y)$ **capture of** the inserted variable y has happened.

Quantifier Operations (Advanced)

Proposition

For all variables \mathbf{x} , and formulas ϕ , ψ such that \mathbf{x} does not occur free in ψ it holds:

$$(i) \quad \forall \mathbf{x} \phi \rightarrow \psi \equiv \exists \mathbf{x} (\phi \rightarrow \psi)$$

$$(ii) \quad \exists \mathbf{x} \phi \rightarrow \psi \equiv \forall \mathbf{x} (\phi \rightarrow \psi)$$

$$(iii) \quad (\psi \rightarrow \exists \mathbf{x} \phi) \equiv \exists \mathbf{x} (\psi \rightarrow \phi)$$

$$(iv) \quad (\psi \rightarrow \forall \mathbf{x} \phi) \equiv \forall \mathbf{x} (\psi \rightarrow \phi)$$