Logic and Modelling

— Semantics of Predicate Logic —

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Motivation of Semantics

Usefulness of Semantics

Can we prove the following by natural deduction?

$$p \lor q \to r, \;\; q \land r \to \neg p \quad \vdash \quad \neg(r \land \neg q)$$

What if it is not derivable? Might be difficult to show.

Using the **soundness and completeness theorem** we could concentrate on the **equivalent semantic entailment**:

$$p \lor q \to r, \;\; q \land r \to \neg p \quad \models \quad \neg(r \land \neg q)$$

and actually demonstrate:

$$p \lor q \to r, \;\; q \land r \to \neg p \quad \not\models \quad \neg(r \land \neg q)$$

That is, find a valuation v such that

$$v(p \lor q \to r) = \mathtt{T}$$
$$v(q \land r \to \neg p) = \mathtt{T}$$
$$v(\neg(r \land \neg q)) = \mathtt{F}$$

Usefulness of Semantics

Find a valuation v such that:

$$v(p \vee q \to r) = \mathtt{T} \tag{1}$$
$$v(q \wedge r \to \neg p) = \mathtt{T} \tag{2}$$
$$v(\neg(r \wedge \neg q)) = \mathtt{F} \tag{3}$$

For any such valuation v we find by **semantic reasoning**:

$$\begin{aligned} v(r \wedge \neg q) &= \mathtt{T} \\ v(r) &= \mathtt{T} \\ v(\neg q) &= \mathtt{T} \\ v(q) &= \mathtt{F} \end{aligned}$$

Every valuation with $v(r) = \mathtt{T}$ and $v(q) = \mathtt{F}$ fulfils (1), (2), (3)

Usefulness of Semantics

For example, the valuation

$$v(p) = \mathtt{T} \qquad v(q) = \mathtt{F} \qquad v(r) = \mathtt{T}$$

yields

$$v(p \vee q \to r) = \mathtt{T} \qquad v(\neg(r \wedge \neg q)) = \mathtt{F}$$

$$v(q \wedge r \to \neg p) = \mathtt{T}$$

This valuation v is a counter model to:

$$p \vee q \to r, \;\; q \wedge r \to \neg p \quad \models \quad \neg(r \wedge \neg q)$$

and hence justifies:

$$p \vee q \to r, \;\; q \wedge r \to \neg p \not\models \neg(r \wedge \neg q)$$

This implies, by the soundness (and completeness) theorem:

$$p \vee q \to r, \;\; q \wedge r \to \neg p \nvdash \neg(r \wedge \neg q)$$

Thus there is no natural-deduction derivation!

Usefulness of Semantics

Assume we have a logic with syntax and semantics and with a **soundness and completeness theorem**:

$$\phi_1, \ldots, \phi_n \vdash \phi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_n \vDash \phi$$

If we want to show

$$\phi_1, \ldots, \phi_n \vdash \phi \qquad (\star)$$

but don't succeed despite best efforts.

We might want to **change tactics**, and try to show:

$$\phi_1, \ldots, \phi_n \nvDash \phi$$

by a counter model to $\phi_1, \ldots, \phi_n \vDash \phi$.

If we succeed here, then $(\star)$ cannot hold.

Proof Theory ($\vdash$) versus Semantics ($\models$)

Proof theory with entailment $\vdash$

- rules prove operative explanation to logical symbols
- gives an existential characterisation of the formulas that are true in a logic:

$$\phi_1, \ldots, \phi_n \vdash \phi \iff \text{there exists a derivation of } \phi \text{ from premises } \phi_1, \ldots, \phi_n$$

- convenient for positive arguments: give a derivation

Semantics with entailment $\models$

- gives meaning to logical symbols
- gives a universal characterisation of the formulas that are true in a logic:

$$\phi_1, \ldots, \phi_n \models \phi \iff \text{all models that satisfy } \phi_1, \ldots, \phi_n, \text{ also satisfy } \phi$$

- convenient for negative arguments: give a counter model

Models

How to interpret formulas in predicate logic?

Simple formulas in predicate logic:

- $R(a, b)$
- $\forall x(P(x) \vee Q(x))$
- $\forall x\, (x \leq x \cdot e)$

What can we say about their **meaning** and **truth values**?

These depend on:

- the domain of quantification
- the interpretation of the predicate symbols R, P, Q, $\leq$
- the interpretation of the constants a, b, e
- the interpretation of the function symbol $\cdot$

Different interpretations can make these formulas true or false.

We need a concept of **model** for the interpretation of formulas.

Model Informally (1)

$$\exists x\, P(x)$$

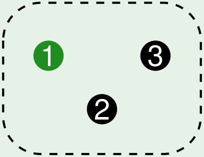
If we interpret $P(x)$ as 'x is green', this expresses:

There is something that is green.

This formula has a **model**:

- universe $\{1, 2, 3\}$
- $P(1),\ \neg P(2),\ \neg P(3)$

In a picture, this looks as follows:



A green dot indicates that P is T.

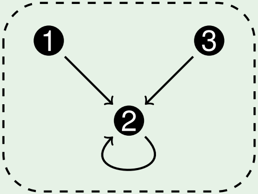
Model Informally (2)

$$\exists y \forall x\, R(x, y)$$

If we interpret $R(x, y)$ as 'x knows y', this expresses:

There is somebody who is known by everybody.

This formula has a model:



Here the formula is true since for $y = 2$, we have: $\forall x\, R(x, y)$

$R(1, 2)$ $\quad$ $R(2, 2)$ $\quad$ $R(3, 2)$

Models Formally

Let

- $\mathcal{F}$ be a set of function symbols,
- $\mathcal{P}$ a set of predicate symbols.

A **model** $\mathcal{M}$ for $\langle \mathcal{F}, \mathcal{P} \rangle$ consists of:

- a non-empty set A, called **domain** or **universe**,
- an **interpretation operation** $(\cdot)^{\mathcal{M}}$ for the symbols in $\mathcal{F}$, $\mathcal{P}$.
  - (i) $f^{\mathcal{M}} : A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$
  - (ii) $P^{\mathcal{M}} \subseteq A^n$ for every n-ary predicate symbols $P \in \mathcal{P}$

A symbol is n-ary if it has n arguments.

- $c^{\mathcal{M}} \in A$ for nullary function symbols (constants) $c \in \mathcal{F}$
- $f^{\mathcal{M}} : A \to A$ for 1-ary (unary) symbols $f \in \mathcal{F}$
- $f^{\mathcal{M}} : A \times A \to A$ for 2-ary (binary) symbols $f \in \mathcal{F}$
- $P^{\mathcal{M}} \subseteq A$ for 1-ary (unary) predicate symbols $P \in \mathcal{P}$
- $P^{\mathcal{M}} \subseteq A \times A$ for 2-ary (binary) predicate symbols $P \in \mathcal{P}$

Models Formally

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A symbol is n-ary if it has n arguments.

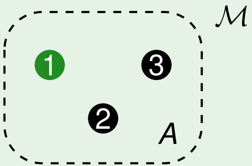
Note: concept of model is extremely liberal

- universe A can be **any non-empty set**
- **only one requirement** for interpretations:
  $f^{\mathcal{M}}$ and $P^{\mathcal{M}}$ have **same number of arguments** as f and P

Example (1)

$\mathcal{F} = \emptyset$, $\mathcal{P} = \{ P_{/1} \}$

Illustration of a model:



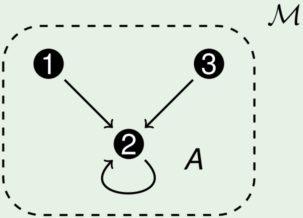
Formal definition of $\mathcal{M}$:

- domain $A = \{ 1, 2, 3 \}$
- interpretation operation $(\cdot)^{\mathcal{M}}$ on $\langle \mathcal{F}, \mathcal{P} \rangle$:
  - P is unary predicate:
    $P^{\mathcal{M}} = \{ 1 \} \subseteq A$.

Example (2)

$\mathcal{F} = \emptyset, \ \mathcal{P} = \{ R_{/2} \}$

Illustration of a model:



Formal definition of $\mathcal{M}$:

- domain $A = \{ 1, 2, 3 \}$
- interpretation operation $(\cdot)^{\mathcal{M}}$ on $\langle \mathcal{F}, \mathcal{P} \rangle$:
  - R is binary predicate:
    $R^{\mathcal{M}} = \{ \langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle \} \subseteq A \times A$.

Example (3)

$\mathcal{F} = \{\ e/0,\ \cdot/2\ \}$, $\mathcal{P} = \{\ \leq/2\ \}$

We consider a model $\mathcal{M}$ with:

- universe $B = \{\ s \mid s \text{ is binary string}\ \}$
- interpretation operation $(\cdot)^{\mathcal{M}}$:
  - e is interpreted as the empty string ϵ
    - $e^{\mathcal{M}} = \text{empty string } \epsilon$
  - $\cdot$ is interpreted as concatenation of strings
    - $\cdot^{\mathcal{M}} : B \times B \to B,\quad a_0 \cdots a_n \cdot^{\mathcal{M}} b_0 \cdots b_m = a_0 \cdots a_n b_0 \cdots b_m$
      e.g.: $01 \cdot^{\mathcal{M}} 100 = 01100$
  - $s_1 \leq s_2$ is interpreted as: s_1 is a prefix of s_2
    - $\leq^{\mathcal{M}} = \{\ \langle s_1, s_2 \rangle \mid s_1 \text{ is a prefix of } s_2\ \}$
      e.g.: $01 \leq^{\mathcal{M}} 01100$

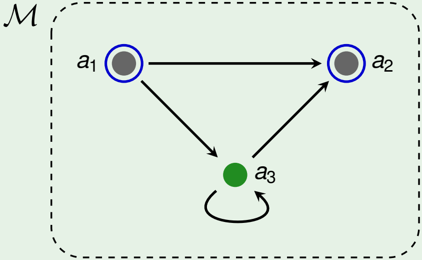
We will be interested in interpreting in $\mathcal{M}$ formulas like:

- $\exists y \forall x\, (y \leq x)$: there is a binary string that is prefix of every binary string
- $\forall x\, (x \leq x \cdot e)$: every binary string is a prefix of ...

Interpretation

A Simple Model

$\mathcal{F} = \{\ c_{/0}\ \}$, $\mathcal{P} = \{\ P_{/1},\ R_{/2}\ \}$



- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

is a model $\mathcal{M}$ with:

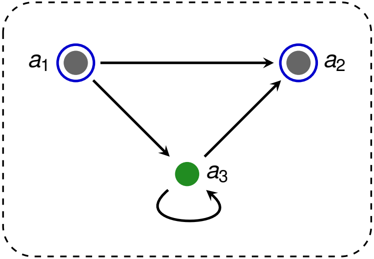
- domain $A = \{a_1, a_2, a_3\}$
- $c^{\mathcal{M}} = a_3$
- $P^{\mathcal{M}} = \{a_1, a_2\}$
- $R^{\mathcal{M}} = \{\langle a_1, a_2\rangle, \langle a_1, a_3\rangle, \langle a_3, a_2\rangle, \langle a_3, a_3\rangle\}$

Interpretation

of formulas without quantifiers and free variables

Example

$\mathcal{M}$



- $\mathcal{M} \not\models P(c)$
- $\mathcal{M} \models \neg P(c)$
- $\mathcal{M} \models R(c, c)$
- $\mathcal{M} \models R(c, c) \vee P(c)$

- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

Interpreting Formulas Without Quantifiers

Truth definition for a formula ϕ without quantifiers and free variables in a model $\mathcal{M}$ by induction on the structure of ϕ :

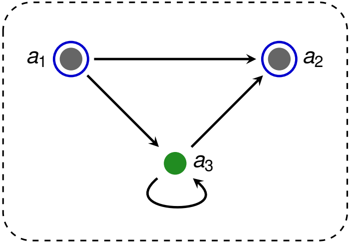
- $\mathcal{M} \models \neg\phi \iff$ not: $\mathcal{M} \models \phi \iff \mathcal{M} \not\models \phi$
- $\mathcal{M} \models \phi \wedge \psi \iff \mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$
- $\mathcal{M} \models \phi \vee \psi \iff \mathcal{M} \models \phi$ or $\mathcal{M} \models \psi$
- $\mathcal{M} \models \phi \to \psi \iff ((\mathcal{M} \models \phi) \implies (\mathcal{M} \models \psi))$
  $\iff$ (if $\mathcal{M} \models \phi$ then $\mathcal{M} \models \psi$)
  $\iff$ not ($\mathcal{M} \models \phi$ and $\mathcal{M} \not\models \psi$)
- $\mathcal{M} \models \boldsymbol{P}(t_1, \ldots, t_n) \iff \langle {t_1}^{\mathcal{M}}, \ldots, {t_n}^{\mathcal{M}} \rangle \in \boldsymbol{P}^{\mathcal{M}}$

The last clause uses the **interpretation of terms** $t^{\mathcal{M}}$:

- if $t = \boldsymbol{c}$ for a constant $\boldsymbol{c}$, then $t^{\mathcal{M}} = \boldsymbol{c}^{\mathcal{M}}$
- if $t = \boldsymbol{f}(t_1, \ldots, t_n)$, then $t^{\mathcal{M}} = \boldsymbol{f}^{\mathcal{M}}({t_1}^{\mathcal{M}}, \ldots, {t_n}^{\mathcal{M}})$

Example (1)

$\mathcal{M}$



- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

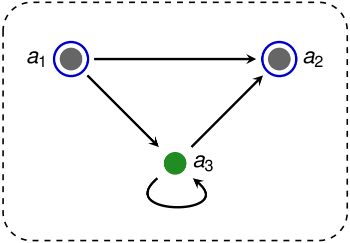
We use the formal definition to check $\mathcal{M} \models P(c)$ or $\mathcal{M} \not\models P(c)$:

$$\begin{aligned} &\mathcal{M} \models P(c) \quad \times \\ &\quad \iff c^{\mathcal{M}} \in P^{\mathcal{M}} && \text{(by definition of } \models\text{)} \\ &\quad \iff a_3 \in \{ a_1, a_2 \} \quad \times && \text{(by definition of } \mathcal{M}\text{)} \end{aligned}$$

Hence we indeed conclude: $\mathcal{M} \not\models P(c)$.

Example (2)

$\mathcal{M}$



- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models \neg P(c)$:

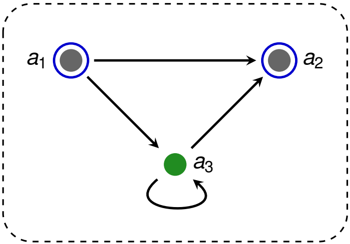
$$\mathcal{M} \models \neg P(c) \checkmark$$

$$\begin{aligned}
&\iff \text{not: } \mathcal{M} \models P(c) && \text{(by definition of } \models\text{)}\\
&\iff \text{not: } c^{\mathcal{M}} \in P^{\mathcal{M}} && \text{(by definition of } \models\text{)}\\
&\iff c^{\mathcal{M}} \notin P^{\mathcal{M}} &&\\
&\iff a_3 \notin \{ a_1, a_2 \} \checkmark && \text{(by definition of } \mathcal{M}\text{)}
\end{aligned}$$

Hence we conclude: $\mathcal{M} \models \neg P(c)$.

Examle (3)

$\mathcal{M}$



- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c)$:

$\mathcal{M} \models R(c, c)$ ✓

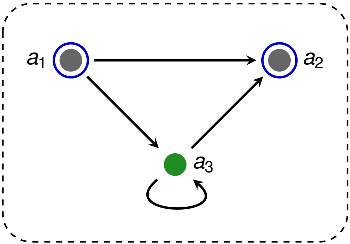
$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}}$ (by definition of $\models$)

$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \}$ ✓ (by definition of $\mathcal{M}$)

Hence we conclude: $\mathcal{M} \models R(c, c)$.

Example (4)

$\mathcal{M}$



- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c,c) \vee P(c)$:

$$\begin{aligned}
&\mathcal{M} \models R(c,c) \vee P(c) \quad \checkmark \\
&\iff \mathcal{M} \models R(c,c) \text{ or } \mathcal{M} \models P(c) && \text{(by definition of } \models\text{)} \\
&\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \text{ or } c^{\mathcal{M}} \in P^{\mathcal{M}} && \text{(by definition of } \models\text{)} \\
&\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \quad \checkmark \\
&\qquad \text{or } a_3 \in \{ a_1, a_2 \} \quad \times \qquad \checkmark && \text{(by definition of } \mathcal{M}\text{)}
\end{aligned}$$

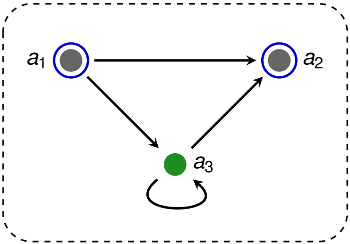
Hence we conclude: $\mathcal{M} \models R(c,c) \vee P(c)$.

Interpretation

of formulas with quantifiers and free variables

Example (Interpretation of Quantifiers)

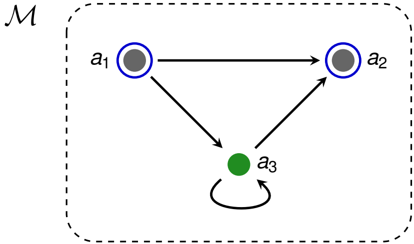
$\mathcal{M}$



- $c^{\mathcal{M}}$: green point
- $P^{\mathcal{M}}$: blue circles
- $R^{\mathcal{M}}$: arrows

- $\mathcal{M} \not\models \forall x\, P(x)$
- $\mathcal{M} \models \neg\forall x\, P(x)$
- $\mathcal{M} \models \exists x P(x)$
- $\mathcal{M} \not\models \forall x(P(x) \to \exists y\, R(x,y))$
- $\mathcal{M} \models \forall x(P(x) \vee R(x,x))$
- $\mathcal{M} \not\models \exists x \forall y\, R(x,y)$
- $\mathcal{M} \models \forall x(\exists y R(x,y) \to \exists y(R(x,y) \wedge P(y)))$

Example (Interpretation of Free Variables)



What about:

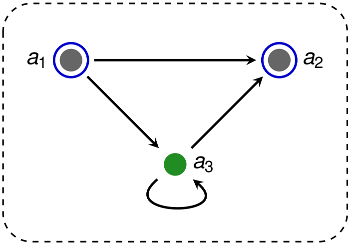
- $\mathcal{M} \models P(x)$?
- $\mathcal{M} \models R(x, x)$?
- $\mathcal{M} \models R(x, y)$?
- $\mathcal{M} \models \exists y\, R(x, y)$?

This depends on the interpretation of the free variables x and y:

- $\mathcal{M} \models_{[x \mapsto a_1]} P(x)$ $\quad\mathcal{M} \not\models_{[x \mapsto a_3]} P(x)$
- $\mathcal{M} \not\models_{[x \mapsto a_1]} R(x, x)$ $\quad\mathcal{M} \models_{[x \mapsto a_3]} R(x, x)$
- $\mathcal{M} \not\models_{[x \mapsto a_2][y \mapsto a_3]} R(x, y)$
- $\mathcal{M} \models_{[x \mapsto a_3]} \exists y\, R(x, y)$ $\quad\mathcal{M} \not\models_{[x \mapsto a_2]} \exists y\, R(x, y)$

Model plus Environments

$\mathcal{M}$



An environment

$$\ell : \textbf{var} \to A$$

(look-up function) interprets free variables in the domain.

Example environment:

$$\ell(x) = a_1 \qquad \ell(y) = a_3$$

Let us determine whether

- $\mathcal{M} \models_\ell P(x)$ $\qquad$ $\mathcal{M} \not\models_\ell P(y)$
- $\mathcal{M} \not\models_\ell R(x, x)$ $\qquad$ $\mathcal{M} \models_\ell R(y, y)$
- $\mathcal{M} \models_\ell R(x, y)$
- $\mathcal{M} \not\models_\ell R(y, x)$
- $\mathcal{M} \models_{\ell[x \mapsto a_2]} R(y, x)$ $\qquad$ $\mathcal{M} \models_\ell \exists x\, R(y, x)$

Modification of Environments

In order to determine the truth values for $\forall x$- en $\exists x$-formulas we must be able to modify the interpretation of the variable x.

Definition (Modified Environment)

Let $\ell : \mathbf{var} \to A$ be an environment, and $\boldsymbol{x}$ a variable.

By $\ell[\boldsymbol{x} \mapsto a]$ we denote the environment that interprets $\boldsymbol{x}$ as a, and that for all other variables acts in the same way as ℓ:

$$\ell[\boldsymbol{x} \mapsto a](\boldsymbol{y}) = \begin{cases} a & \text{if } \boldsymbol{y} = \boldsymbol{x} \\ \ell(\boldsymbol{y}) & \text{if } \boldsymbol{y} \neq \boldsymbol{x} \end{cases}$$

More modifications are possible.

For $\ell' = \ell[x \mapsto a][z \mapsto b]$ we have:

$$\ell'(\boldsymbol{y}) = \ell[x \mapsto a][z \mapsto b](\boldsymbol{y}) = \begin{cases} a & \text{if } \boldsymbol{y} = x \\ b & \text{if } \boldsymbol{y} = z \\ \ell(\boldsymbol{y}) & \text{if } \boldsymbol{y} \neq x \text{ and } \boldsymbol{y} \neq z \end{cases}$$

Interpretation of Terms in Model with Environment

Terms are built from variables, constants, and function symbols:

- variables are interpreted according to the environment ℓ
- constants are interpreted according to $(\cdot)^{\mathcal{M}}$
- function symbols are interpreted according to $(\cdot)^{\mathcal{M}}$

Let $\mathcal{M}$ be a model and ℓ an environment.

Interpretation of terms

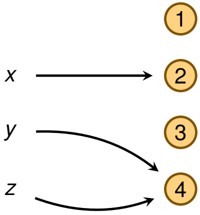
The interpretation $t^{\mathcal{M},\ell}$ of a term t is defined as:

$$t^{\mathcal{M},\ell} = \begin{cases} \ell(\boldsymbol{x}) & \text{if } t = \boldsymbol{x} \text{ for a variable } \boldsymbol{x} \\ \boldsymbol{c}^{\mathcal{M}} & \text{if } t = \boldsymbol{c} \text{ for a constant } \boldsymbol{c} \\ \boldsymbol{f}^{\mathcal{M}}({t_1}^{\mathcal{M},\ell}, \ldots, {t_n}^{\mathcal{M},\ell}) & \text{if } t = \boldsymbol{f}(t_1, \ldots, t_n) \end{cases}$$

by induction on the term structure.

Quantification by Varying the Environment

Model and environment:

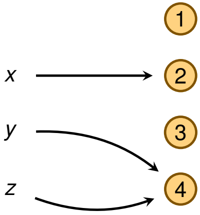


- domain: $A = \{1, 2, 3, 4\}$
- $L^{\mathcal{M}} = <$
- environment ℓ:
  - $\ell(x) = 2$
  - $\ell(y) = \ell(z) = 4$

- Now $\mathcal{M} \not\models_{\ell} L(y, x)$

Quantification by Varying the Environment

Model and environment:



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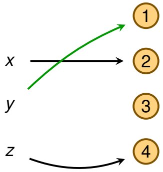
- environment ℓ:
  - $\ell(x) = 2$
  - $\ell(y) = \ell(z) = 4$

- But: $\mathcal{M} \models_{\ell} \exists y\, L(y, x)$
  because $\mathcal{M} \models_{\ell[y \mapsto 1]} L(y, x)$



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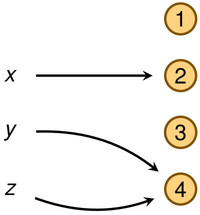

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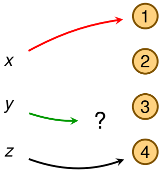
  - $\ell(x) = 2$
  - $\ell(y) = \ell(z) = 4$



- Also $\mathcal{M} \models_{\ell} \forall x \exists y\, L(y, x)$?
- No: $\mathcal{M} \not\models_{\ell[x \mapsto 1]} \exists y\, L(y, x)$

Formula Truth in a Model (Full Definition)

Truth of a formula ϕ in a model $\mathcal{M}$ with universe A *with respect to environment* ℓ is defined by induction on the structure of ϕ:

- $\mathcal{M} \vDash_\ell \neg\phi \iff \mathcal{M} \nvDash_\ell \phi$
- $\mathcal{M} \vDash_\ell \phi \wedge \psi \iff \mathcal{M} \vDash_\ell \phi$ and $\mathcal{M} \vDash_\ell \psi$
- $\mathcal{M} \vDash_\ell \phi \vee \psi \iff \mathcal{M} \vDash_\ell \phi$ or $\mathcal{M} \vDash_\ell \psi$
- $\mathcal{M} \vDash_\ell \phi \rightarrow \psi \iff$ (if $\mathcal{M} \vDash_\ell \phi$ then $\mathcal{M} \vDash_\ell \psi$)
- $\mathcal{M} \vDash_\ell \boldsymbol{P}(t_1, \ldots, t_n) \iff \langle {t_1}^{\mathcal{M},\ell}, \ldots, {t_n}^{\mathcal{M},\ell} \rangle \in \boldsymbol{P}^{\mathcal{M}}$

$$t^{\mathcal{M},\ell} = \begin{cases} \ell(\boldsymbol{x}) & \text{if } t = \boldsymbol{x} \text{ for a variable } \boldsymbol{x} \\ \boldsymbol{c}^{\mathcal{M}} & \text{if } t = \boldsymbol{c} \text{ for a constant } \boldsymbol{c} \\ \boldsymbol{f}^{\mathcal{M}}({t_1}^{\mathcal{M},\ell}, \ldots, {t_n}^{\mathcal{M},\ell}) & \text{if } t = \boldsymbol{f}(t_1, \ldots, t_n) \end{cases}$$

- $\mathcal{M} \vDash_\ell \forall x\, \phi \iff$ for all $a \in A$ it holds: $\mathcal{M} \vDash_{\ell[x \mapsto a]} \phi$
- $\mathcal{M} \vDash_\ell \exists x\, \phi \iff$ for some $a \in A$ it holds: $\mathcal{M} \vDash_{\ell[x \mapsto a]} \phi$

Well-definedness of $\models$

Proposition

If ℓ and ℓ' coincide on the free variables of ϕ, then:

$$\mathcal{M} \models_\ell \phi \iff \mathcal{M} \models_{\ell'} \phi$$

A formula ϕ is a **sentence** if ϕ does not have free variables.

Proposition

Let ϕ be a sentence. Then it holds for all enviroments ℓ and ℓ':

$$\mathcal{M} \models_\ell \phi \iff \mathcal{M} \models_{\ell'} \phi$$

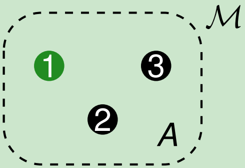
Hence for sentences ϕ, we can write

$$\mathcal{M} \models \phi \quad \text{for} \quad \mathcal{M} \models_\ell \phi,$$

since ℓ is irrelevant.

Checking Formula Satisfiability in a Model

$\mathcal{M} \models \exists x\, P(x)$ where



$A = \{\, 1, 2, 3 \,\}$

$P^{\mathcal{M}} = \{\, 1 \,\}$

$\mathcal{M} \models_{\ell} \exists x\, P(x)$ ✓

$\iff$ there is $a \in A$ such that $\mathcal{M} \models_{\ell[x \mapsto a]} P(x)$ (by def. of $\models$)

$\iff$ there is $a \in A$ s.th. $x^{\mathcal{M}, \ell[x \mapsto a]} \in P^{\mathcal{M}}$ (by def. of $\models$)

$\iff$ there is $a \in A$ such that $a \in P^{\mathcal{M}}$ (def. of $(\cdot)^{\mathcal{M}, [x \mapsto a]}$)

$\Longleftarrow$ $1 \in P^{\mathcal{M}}$ ✓ (by def. of $\mathcal{M}$)

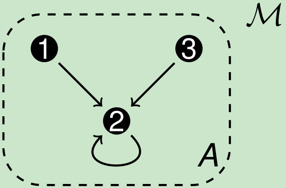
Hence we have formally established $\mathcal{M} \models \exists x\, P(x)$.

Checking Formula Satisfiability in a Model

$\mathcal{M} \models \exists y \forall x\, R(x,y)$ for model

$A = \{1,2,3\}$

$R^{\mathcal{M}} = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle\}$



$\mathcal{M} \models_{\ell} \exists y \forall x\, R(x,y)$ ✓

$\iff$ there is $b \in A$ such that: $\mathcal{M} \models_{\ell[y \mapsto b]} \forall x\, R(x,y)$

$\iff$ there is $b \in A$ such that for all $a \in A$:

$$\mathcal{M} \models_{\ell[y \mapsto b][x \mapsto a]} R(x,y)$$

$\iff$ there is $b \in A$ such that for all $a \in A$:

$$\langle a, b\rangle = \langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]}\rangle \in R^{\mathcal{M}}$$

$\Longleftarrow$ for all $a \in A$ it holds that $\langle a, 2\rangle \in R^{\mathcal{M}}$

$\Longleftarrow$ $\langle 1,2\rangle \in R^{\mathcal{M}}$ and $\langle 2,2\rangle \in R^{\mathcal{M}}$ and $\langle 3,2\rangle \in R^{\mathcal{M}}$ ✓

This shows $\mathcal{M} \models \exists y \forall x\, R(x,y)$.

Checking Formula Satisfiability in a Model

$\mathcal{M} \vDash \forall x\,(x \leq x \cdot e)$

- universe $B = \{\, s \mid s \text{ is binary string} \,\}$
- $e^{\mathcal{M}} = \text{empty string } \epsilon$
- $\cdot^{\mathcal{M}}$: concatenation of strings
- $\leq^{\mathcal{M}} = \{\, \langle s_1, s_2 \rangle \mid s_1 \text{ is a prefix of } s_2 \,\}$

$$
\begin{aligned}
& \mathcal{M} \vDash_{\ell} \forall x\,(x \leq x \cdot e) \quad \checkmark \\
\iff\; & \text{for all } s \in B\colon\; \mathcal{M} \vDash_{\ell[x \mapsto s]} x \leq x \cdot e \\
\iff\; & \text{for all } s \in B\colon\; \langle x^{\mathcal{M},\ell[x \mapsto s]}, (x \cdot e)^{\mathcal{M},\ell[x \mapsto s]} \rangle \in \leq^{\mathcal{M}} \\
\iff\; & \text{for all } s \in B\colon\; \langle s, x^{\mathcal{M},\ell[x \mapsto s]} \cdot^{\mathcal{M}} e^{\mathcal{M},\ell[x \mapsto s]} \rangle \in \leq^{\mathcal{M}} \\
\iff\; & \text{for all } s \in B\colon\; \langle s, s \cdot^{\mathcal{M}} e^{\mathcal{M}} \rangle \in \leq^{\mathcal{M}} \\
\iff\; & \text{for all } s \in B\colon\; \langle s, s \cdot^{\mathcal{M}} \epsilon \rangle \in \leq^{\mathcal{M}} \\
\iff\; & \text{for all } s \in B\colon\; \langle s, s \rangle \in \leq^{\mathcal{M}} \quad \checkmark
\end{aligned}
$$

Hence we have formally established: $\mathcal{M} \vDash \forall x\,(x \leq x \cdot e)$.

Semantical Entailment

Semantical Entailment in Predicate Logic

Semantic entailment in propositional logic

In **propositional logic**: $\phi_1, \ldots, \phi_n \vDash \psi$ means:

For all valuations v:

$$v(\phi_1) = \mathtt{T}, \ldots, v(\phi_n) = \mathtt{T} \implies v(\psi) = \mathtt{T}$$

Semantic entailment in predicate logic

In **predicate logic**: $\phi_1, \ldots, \phi_n \vDash \psi$ means:

For all models $\mathcal{M}$, and all environments ℓ:

$$\mathcal{M} \vDash_\ell \phi_1, \ldots, \mathcal{M} \vDash_\ell \phi_n \implies \mathcal{M} \vDash_\ell \psi$$

In words: for all models $\mathcal{M}$, and all environments ℓ such that

$$\mathcal{M} \vDash_\ell \phi_1 \quad \text{and} \quad \ldots \text{ and } \quad \mathcal{M} \vDash_\ell \phi_n \quad \text{hold,}$$

it also holds that $\mathcal{M} \vDash_\ell \psi$.

Proving and Disproving Semantical Entailment

Which of the following semantic implications are true?

(a)	$\forall x \exists y\, R(x, y)$	$\models$	$\exists x \forall y\, R(x, y)$	*NO*
(b)	$\forall x \exists y\, R(x, y)$	$\models$	$\exists y \forall x\, R(x, y)$	*NO*
(c)	$\exists y \forall x\, R(x, y)$	$\models$	$\forall x \exists y\, R(x, y)$	*YES*
(d)	$\exists y \forall x\, R(x, y)$	$\models$	$\forall y \exists x\, R(x, y)$	*NO*

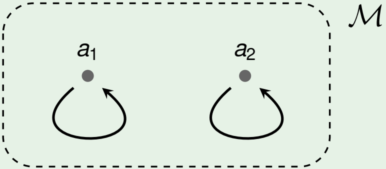
Disproving Semantic Entailment: Counter Models

$$\forall x \exists y\, R(x, y) \not\models \exists y \forall x\, R(x, y)$$

We can give a model $\mathcal{M}$ such that:

$$\mathcal{M} \models \forall x \exists y\, R(x, y)\,, \qquad \mathcal{M} \not\models \exists y \forall x\, R(x, y)\,.$$

We choose $\mathcal{M}$ as follows:



This is a **counter model**:
it satisfies the premise, but not the conclusion.

Proving Semantic Entailment

$$\exists y \forall x\, R(x,y) \models \forall x \exists y\, R(x,y)$$

For all models $\mathcal{M}$ with domain A and environments ℓ we find:

$$\begin{aligned}
&\mathcal{M} \models_{\ell} \exists y \forall x\, R(x,y) \\
&\quad \iff \text{there is } b \in A \text{ such that } \mathcal{M} \models_{\ell[y \mapsto b]} \forall x\, R(x,y) \\
&\quad \iff \text{there is } b \in A \text{ such that for all } a \in A\text{:} \\
&\qquad\qquad \mathcal{M} \models_{\ell[y \mapsto b][x \mapsto a]} R(x,y) \\
&\quad \implies \text{for all } a \in A \text{ there is } b \in A \text{ such that:} \\
&\qquad\qquad \mathcal{M} \models_{\ell[y \mapsto b][x \mapsto a]} R(x,y) \\
&\quad \iff \text{for all } a \in A \text{ there is } b \in A \text{ such that:} \\
&\qquad\qquad \mathcal{M} \models_{\ell[x \mapsto a][y \mapsto b]} R(x,y) \\
&\quad \iff \text{for all } a \in A\text{: } \mathcal{M} \models_{\ell[x \mapsto a]} \exists y\, R(x,y) \\
&\quad \iff \mathcal{M} \models_{\ell} \forall x \exists y\, R(x,y)
\end{aligned}$$

Hence we can conclude: $\exists y \forall x\, R(x,y) \models \forall x \exists y\, R(x,y)$.

Proving Semantic Entailment

$\neg\forall x\, P(x) \models \exists x\, \neg P(x)$

For all models $\mathcal{M}$ with domain A and environments ℓ we find:

$$\begin{aligned}
&\mathcal{M} \models_{\ell} \neg\forall x\, P(x) \\
&\quad\iff \text{not: } \mathcal{M} \models_{\ell} \forall x\, P(x) \\
&\quad\iff \text{not for all } a \in A\text{: } \mathcal{M} \models_{\ell[x\mapsto a]} P(x) \\
&\quad\iff \text{there exists } a \in A \text{ such that not: } \mathcal{M} \models_{\ell[x\mapsto a]} P(x) \\
&\quad\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \not\models_{\ell[x\mapsto a]} P(x) \\
&\quad\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \models_{\ell[x\mapsto a]} \neg P(x) \\
&\quad\iff \mathcal{M} \models_{\ell} \exists x\, \neg P(x)
\end{aligned}$$

Hence we can conclude: $\neg\forall x\, P(x) \models \exists x\, \neg P(x)$.