

Logic and Modelling

— Semantics of Predicate Logic —

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Motivation of Semantics

Usefulness of Semantics

Can we prove the following by natural deduction?

$$p \vee q \rightarrow r, q \wedge r \rightarrow \neg p \vdash \neg(r \wedge \neg q)$$

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That is, find a valuation v such that

$$v(p \vee q \rightarrow r) = \text{T}$$

$$v(q \wedge r \rightarrow \neg p) = \text{T}$$

$$v(\neg(r \wedge \neg q)) = \text{F}$$

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Every valuation with $v(r) = \text{T}$ and $v(q) = \text{F}$ fulfils (1), (2), (3)

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For example, the valuation

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Thus **there is no** natural-deduction derivation!

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If we succeed here, then (*) **cannot** hold.

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Proof theory with entailment \vdash

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- ▶ gives an **existential characterisation** of the formulas that are true in a logic:

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Models

How to interpret formulas in predicate logic?

Simple formulas in predicate logic:

- ▶ $R(a, b)$
- ▶ $\forall x(P(x) \vee Q(x))$
- ▶ $\forall x(x \leq x \cdot e)$

What can we say about their **meaning** and **truth values**?

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We need a concept of **model** for the interpretation of formulas.

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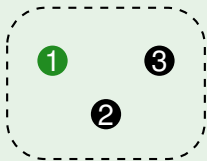
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In a picture, this looks as follows:



A **green dot** indicates that P is T.

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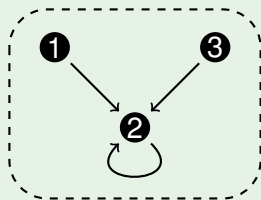
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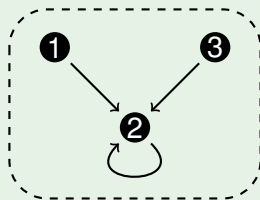
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Here the formula is true since for $y = 2$, we have: $\forall x R(x, y)$

$$R(1, 2)$$

$$R(2, 2)$$

$$R(3, 2)$$

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A **model** \mathcal{M} for $\langle \mathcal{F}, \mathcal{P} \rangle$ consists of:

- ▶ a non-empty set A , called **domain** or **universe**,
- ▶ an **interpretation operation** $(\cdot)^{\mathcal{M}}$ for the symbols in \mathcal{F}, \mathcal{P} .
 - (i) $f^{\mathcal{M}} : A^n \rightarrow A$ for every n -ary function symbol $f \in \mathcal{F}$
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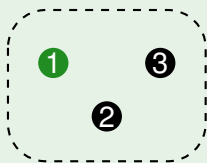
Note: concept of model is **extremely liberal**

- ▶ universe A can be **any non-empty set**
- ▶ **only one requirement** for interpretations:
 $f^{\mathcal{M}}$ and $P^{\mathcal{M}}$ have **same number of arguments** as f and P

Example (1)

$$\mathcal{F} = \emptyset, \mathcal{P} = \{ P_{/1} \}$$

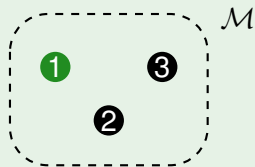
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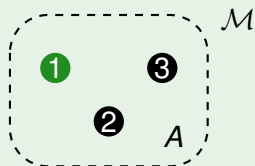


Formal definition of \mathcal{M} :

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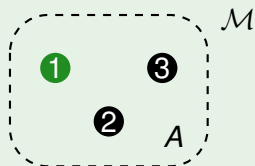
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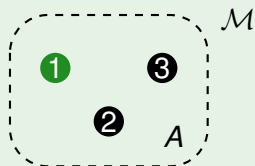
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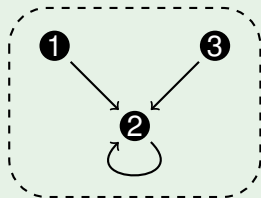
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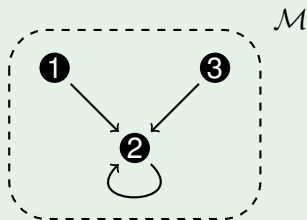
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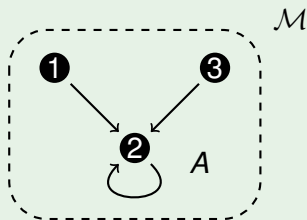


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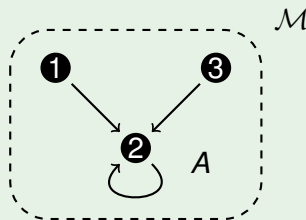
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 - ▶ R is **binary predicate**:
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- ▶ $\exists y \forall x (y \leq x)$: there is a binary string that is prefix of every binary string
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Interpretation

A Simple Model

$$\mathcal{F} = \{ c/0 \}, \mathcal{P} = \{ P/1, R/2 \}$$

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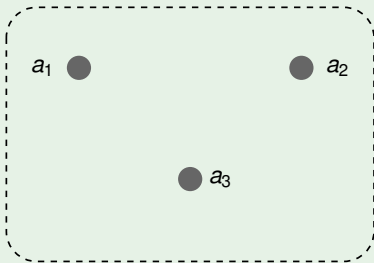
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\mathcal{M}

a_1 ●

● a_2

● a_3



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\mathcal{M}

a_1



a_2



a_3

► $c^{\mathcal{M}}$: green point

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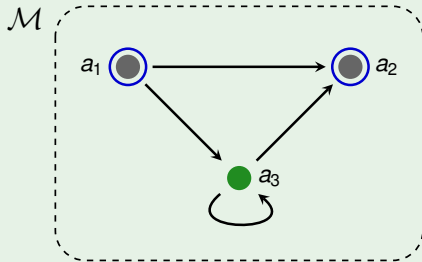


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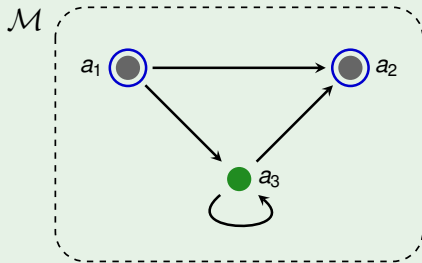


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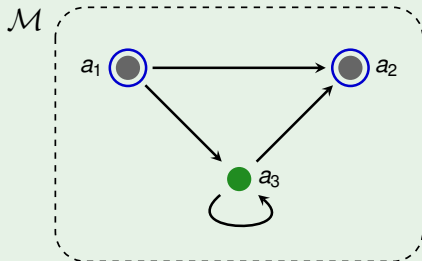
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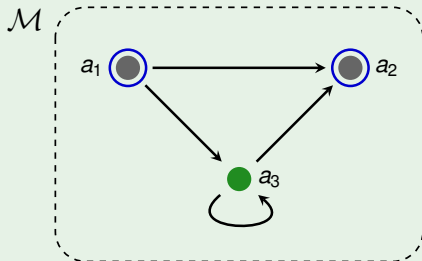
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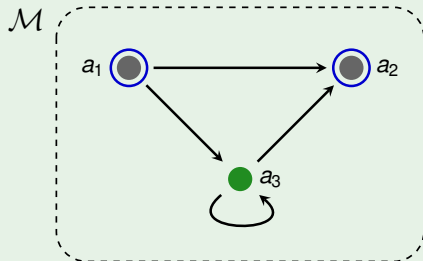
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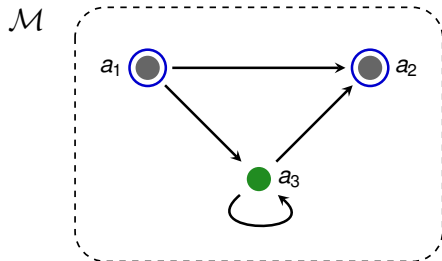
▶ $P^{\mathcal{M}} = \{ a_1, a_2 \}$

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Interpretation

of formulas **without** quantifiers and free variables

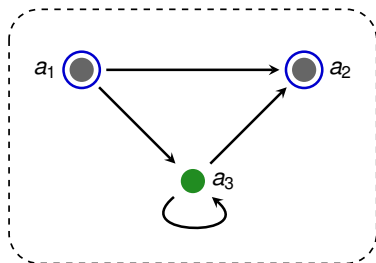
Example



- ▶ $c^{\mathcal{M}}$: green point
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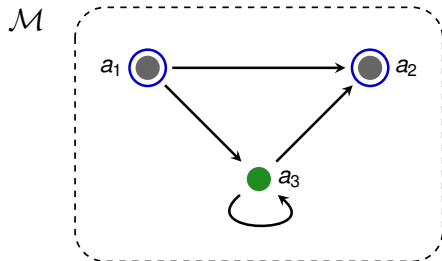
\mathcal{M}



$P(c)$

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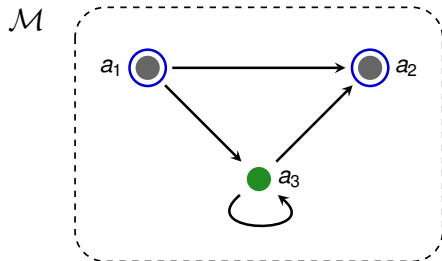
Example



► $\mathcal{M} \not\equiv P(c)$

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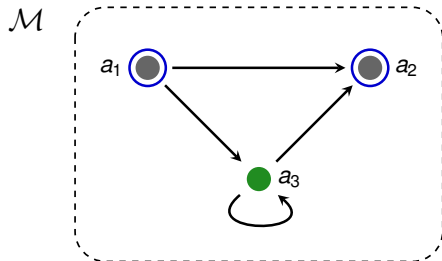
Example



- ▶ $\mathcal{M} \not\models P(c)$
- ▶ $\neg P(c)$

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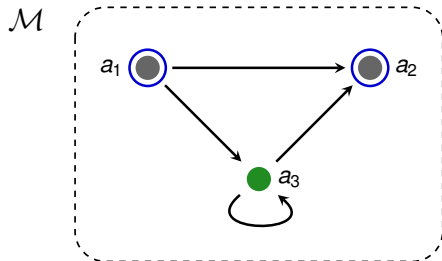
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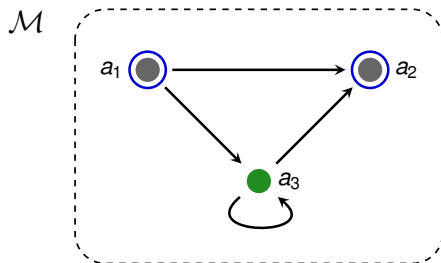
Example



- ▶ $\mathcal{M} \not\models P(c)$
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- ▶ $R(c, c)$

- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
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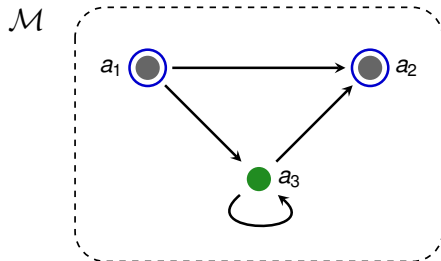
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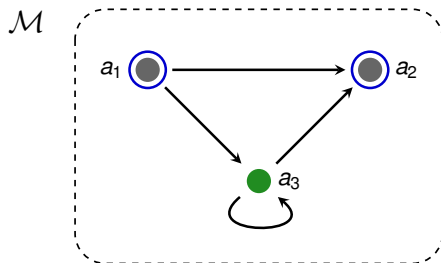
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- ▶ $\mathcal{M} \models P(t_1, \dots, t_n)$

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 $\iff (\text{if } \mathcal{M} \models \phi \text{ then } \mathcal{M} \models \psi)$
 $\iff \text{not } (\mathcal{M} \models \phi \text{ and } \mathcal{M} \not\models \psi)$
- ▶ $\mathcal{M} \models P(t_1, \dots, t_n) \iff \langle t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}} \rangle \in P^{\mathcal{M}}$

Interpreting Formulas Without Quantifiers

Truth definition for a formula ϕ without quantifiers and free variables in a model \mathcal{M} by induction on the structure of ϕ :

- ▶ $\mathcal{M} \models \neg\phi \iff \text{not: } \mathcal{M} \models \phi \iff \mathcal{M} \not\models \phi$
- ▶ $\mathcal{M} \models \phi \wedge \psi \iff \mathcal{M} \models \phi \text{ and } \mathcal{M} \models \psi$
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The last clause uses the **interpretation of terms** $t^{\mathcal{M}}$:

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The last clause uses the **interpretation of terms** $t^{\mathcal{M}}$:

- ▶ if $t = \mathbf{c}$ for a constant \mathbf{c} , then $t^{\mathcal{M}} = \mathbf{c}^{\mathcal{M}}$

Interpreting Formulas Without Quantifiers

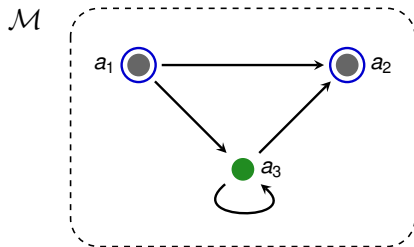
Truth definition for a formula ϕ without quantifiers and free variables in a model \mathcal{M} by induction on the structure of ϕ :

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The last clause uses the **interpretation of terms** $t^{\mathcal{M}}$:

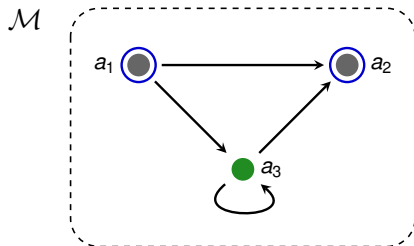
- ▶ if $t = \mathbf{c}$ for a constant \mathbf{c} , then $t^{\mathcal{M}} = \mathbf{c}^{\mathcal{M}}$
- ▶ if $t = \mathbf{f}(t_1, \dots, t_n)$, then $t^{\mathcal{M}} = \mathbf{f}^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})$

Example (1)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

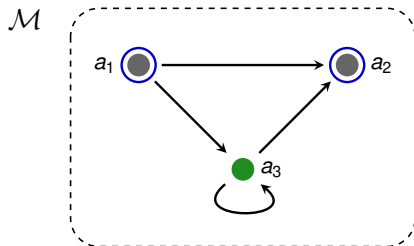
Example (1)



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We use the formal definition to check $\mathcal{M} \models P(c)$ or $\mathcal{M} \not\models P(c)$:

Example (1)

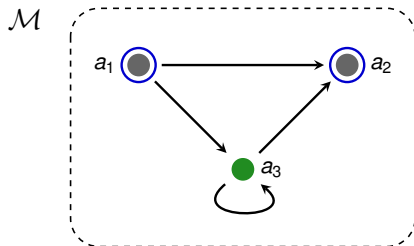


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$$\mathcal{M} \models P(c)$$

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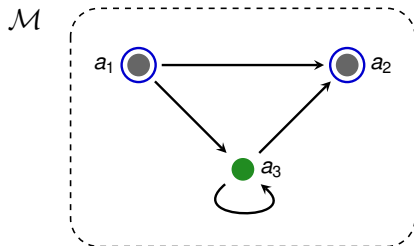
We use the formal definition to check $\mathcal{M} \models P(c)$ or $\mathcal{M} \not\models P(c)$:

$$\mathcal{M} \models P(c)$$

$$\iff c^{\mathcal{M}} \in P^{\mathcal{M}}$$

(by definition of \models)

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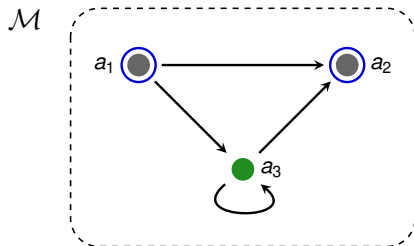
$$\iff c^{\mathcal{M}} \in P^{\mathcal{M}}$$

(by definition of \models)

$$\iff a_3 \in \{a_1, a_2\}$$

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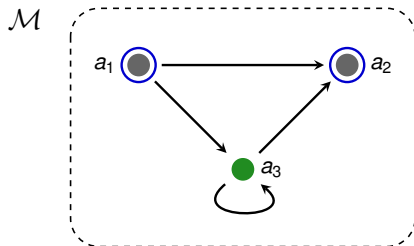
$$\iff c^{\mathcal{M}} \in P^{\mathcal{M}}$$

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$$\iff a_3 \in \{a_1, a_2\} \quad \times$$

(by definition of \mathcal{M})

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We use the formal definition to check $\mathcal{M} \models P(c)$ or $\mathcal{M} \not\models P(c)$:

$$\mathcal{M} \models P(c) \quad \times$$

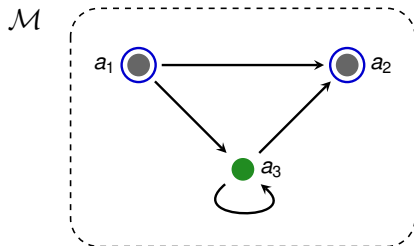
$$\iff c^{\mathcal{M}} \in P^{\mathcal{M}}$$

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We use the formal definition to check $\mathcal{M} \models P(c)$ or $\mathcal{M} \not\models P(c)$:

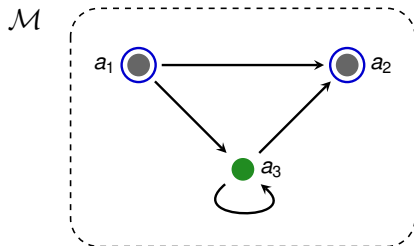
$$\mathcal{M} \models P(c) \quad \times$$

$$\iff c^{\mathcal{M}} \in P^{\mathcal{M}} \quad \text{(by definition of } \models \text{)}$$

$$\iff a_3 \in \{a_1, a_2\} \quad \times \quad \text{(by definition of } \mathcal{M} \text{)}$$

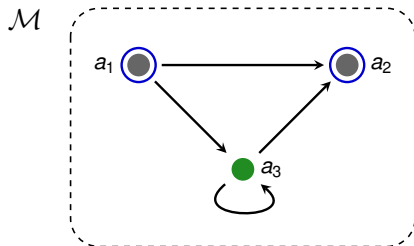
Hence we indeed conclude: $\mathcal{M} \not\models P(c)$.

Example (2)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

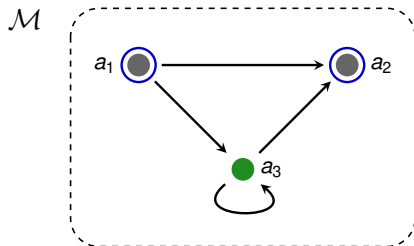
Example (2)



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We use the formal definition to check $\mathcal{M} \models \neg P(c)$:

Example (2)

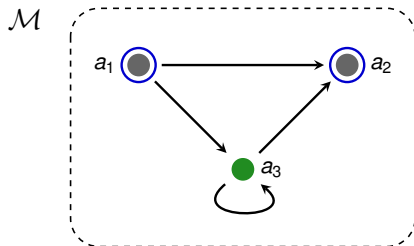


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We use the formal definition to check $\mathcal{M} \models \neg P(c)$:

$$\mathcal{M} \models \neg P(c)$$

Example (2)



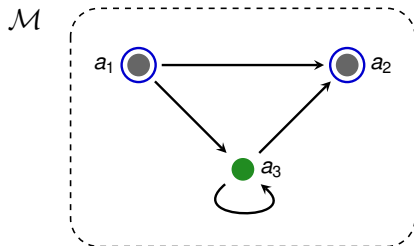
- ▶ $c^{\mathcal{M}}$: green point
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We use the formal definition to check $\mathcal{M} \models \neg P(c)$:

$$\mathcal{M} \models \neg P(c)$$

$$\iff \text{not: } \mathcal{M} \models P(c) \quad (\text{by definition of } \models)$$

Example (2)



- ▶ $c^{\mathcal{M}}$: green point
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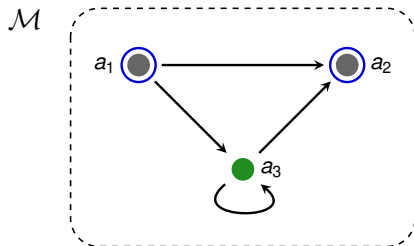
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Example (2)



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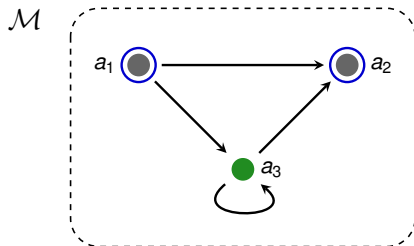
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$$\iff c^{\mathcal{M}} \notin P^{\mathcal{M}}$$

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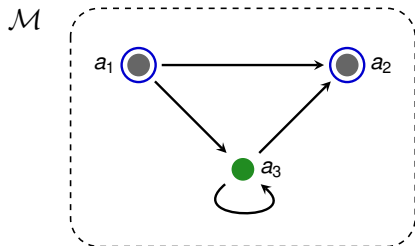
$$\iff \text{not: } \mathcal{M} \models P(c) \quad (\text{by definition of } \models)$$

$$\iff \text{not: } c^{\mathcal{M}} \in P^{\mathcal{M}} \quad (\text{by definition of } \models)$$

$$\iff c^{\mathcal{M}} \notin P^{\mathcal{M}}$$

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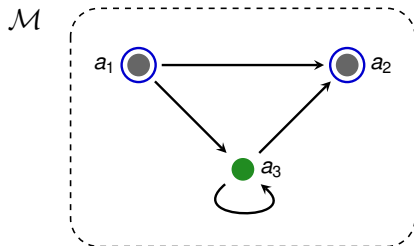
$$\iff \text{not: } \mathcal{M} \models P(c) \quad (\text{by definition of } \models)$$

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$$\iff a_3 \notin \{a_1, a_2\} \quad \checkmark \quad (\text{by definition of } \mathcal{M})$$

Example (2)



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We use the formal definition to check $\mathcal{M} \models \neg P(c)$:

$$\mathcal{M} \models \neg P(c) \quad \checkmark$$

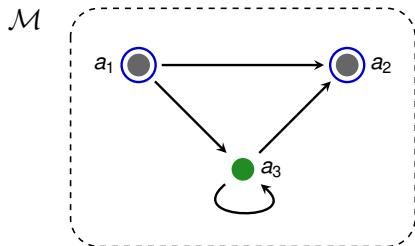
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We use the formal definition to check $\mathcal{M} \models \neg P(c)$:

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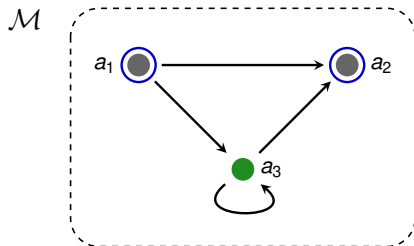
$$\iff \text{not: } c^{\mathcal{M}} \in P^{\mathcal{M}} \quad (\text{by definition of } \models)$$

$$\iff c^{\mathcal{M}} \notin P^{\mathcal{M}}$$

$$\iff a_3 \notin \{a_1, a_2\} \quad \checkmark \quad (\text{by definition of } \mathcal{M})$$

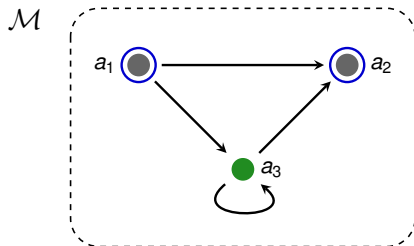
Hence we conclude: $\mathcal{M} \models \neg P(c)$.

Example (3)



- ▶ $c^{\mathcal{M}}$: green point
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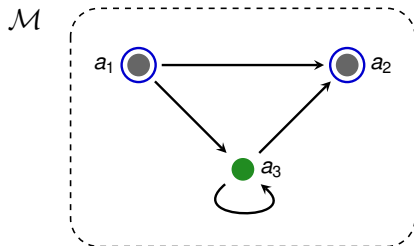
Example (3)



- ▶ $c^{\mathcal{M}}$: green point
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We use the formal definition to check $\mathcal{M} \models R(c, c)$:

Example (3)

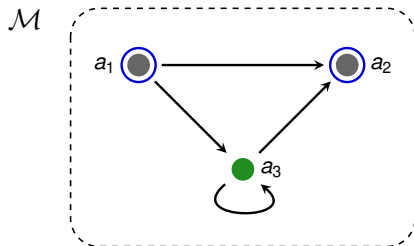


- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c)$:

$$\mathcal{M} \models R(c, c)$$

Example (3)



- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

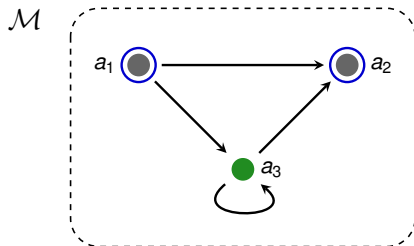
We use the formal definition to check $\mathcal{M} \models R(c, c)$:

$$\mathcal{M} \models R(c, c)$$

$$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}}$$

(by definition of \models)

Example (3)



- ▶ $c^{\mathcal{M}}$: green point
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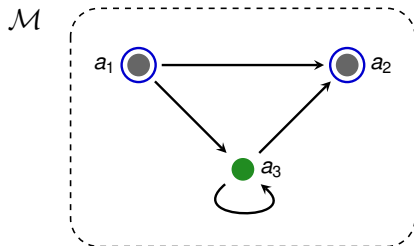
We use the formal definition to check $\mathcal{M} \models R(c, c)$:

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$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \\ (\text{by definition of } \mathcal{M})$$

Example (3)



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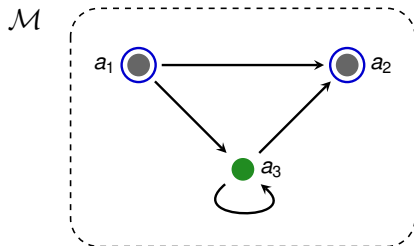
$$\mathcal{M} \models R(c, c)$$

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$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \quad \checkmark$$

(by definition of \mathcal{M})

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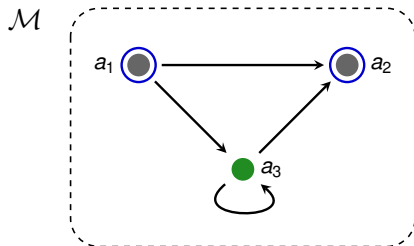
$$\mathcal{M} \models R(c, c) \quad \checkmark$$

$$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \quad (\text{by definition of } \models)$$

$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \quad \checkmark$$

(by definition of \mathcal{M})

Example (3)



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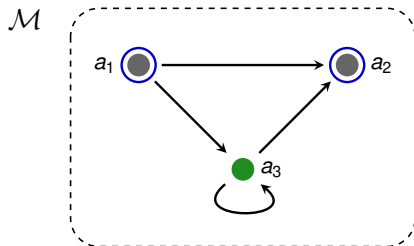
$$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \quad (\text{by definition of } \models)$$

$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \quad \checkmark$$

(by definition of \mathcal{M})

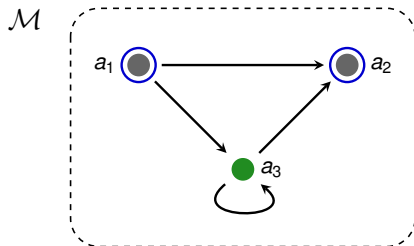
Hence we conclude: $\mathcal{M} \models R(c, c)$.

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
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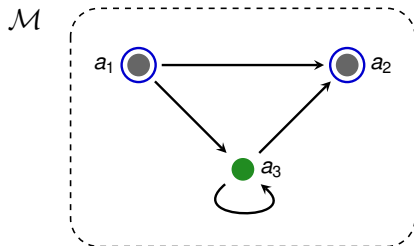
Example (4)



- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

Example (4)

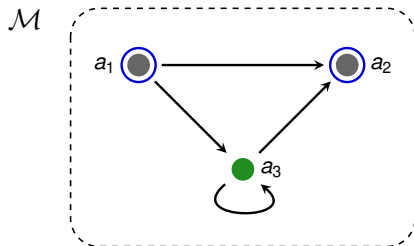


- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

$$\mathcal{M} \models R(c, c) \vee P(c)$$

Example (4)



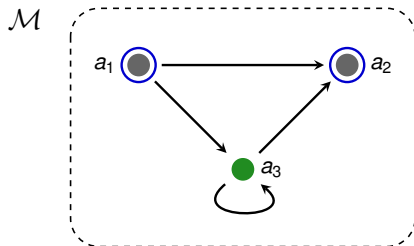
- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

$$\mathcal{M} \models R(c, c) \vee P(c)$$

$$\iff \mathcal{M} \models R(c, c) \text{ or } \mathcal{M} \models P(c) \quad (\text{by definition of } \models)$$

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

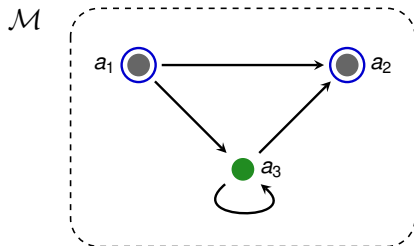
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$$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \text{ or } c^{\mathcal{M}} \in P^{\mathcal{M}} \quad (\text{by definition of } \models)$$

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
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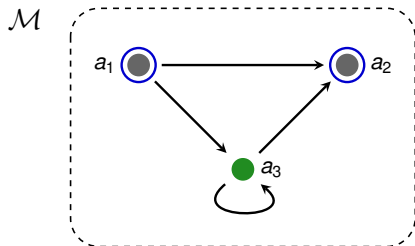
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$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \}$$

or $a_3 \in \{ a_1, a_2 \}$ (by definition of \mathcal{M})

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

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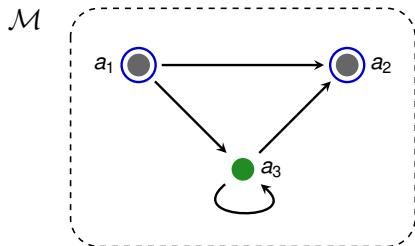
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or $a_3 \in \{ a_1, a_2 \}$ (by definition of \mathcal{M})

Example (4)



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We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

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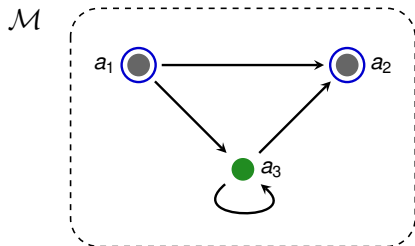
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$$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \text{ or } c^{\mathcal{M}} \in P^{\mathcal{M}} \quad (\text{by definition of } \models)$$

$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \quad \checkmark$$

or $a_3 \in \{ a_1, a_2 \} \quad \times$ (by definition of \mathcal{M})

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

$$\mathcal{M} \models R(c, c) \vee P(c)$$

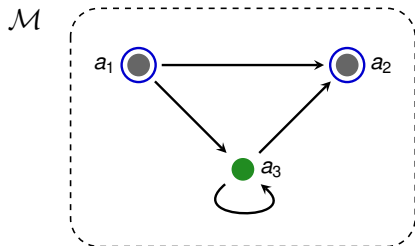
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or $a_3 \in \{ a_1, a_2 \} \quad \times \quad \checkmark \quad (\text{by definition of } \mathcal{M})$

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

$\mathcal{M} \models R(c, c) \vee P(c)$ ✓

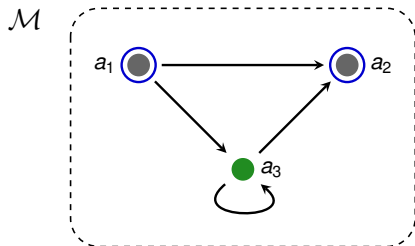
$\iff \mathcal{M} \models R(c, c)$ or $\mathcal{M} \models P(c)$ (by definition of \models)

$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}}$ or $c^{\mathcal{M}} \in P^{\mathcal{M}}$ (by definition of \models)

$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \}$ ✓

or $a_3 \in \{ a_1, a_2 \}$ ✗ ✓ (by definition of \mathcal{M})

Example (4)



- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $R^{\mathcal{M}}$: arrows

We use the formal definition to check $\mathcal{M} \models R(c, c) \vee P(c)$:

$$\mathcal{M} \models R(c, c) \vee P(c) \quad \checkmark$$

$$\iff \mathcal{M} \models R(c, c) \text{ or } \mathcal{M} \models P(c) \quad (\text{by definition of } \models)$$

$$\iff \langle c^{\mathcal{M}}, c^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \text{ or } c^{\mathcal{M}} \in P^{\mathcal{M}} \quad (\text{by definition of } \models)$$

$$\iff \langle a_3, a_3 \rangle \in \{ \langle a_1, a_2 \rangle, \langle a_1, a_3 \rangle, \langle a_3, a_2 \rangle, \langle a_3, a_3 \rangle \} \quad \checkmark$$

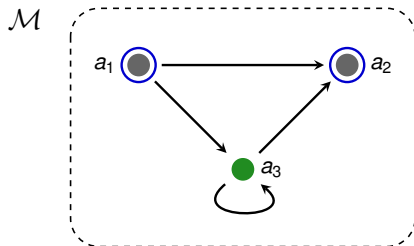
or $a_3 \in \{ a_1, a_2 \} \quad \times \quad \checkmark \quad (\text{by definition of } \mathcal{M})$

Hence we conclude: $\mathcal{M} \models R(c, c) \vee P(c)$.

Interpretation

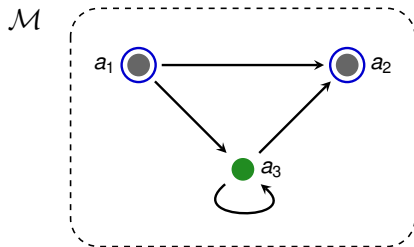
of formulas **with** quantifiers and free variables

Example (Interpretation of Quantifiers)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

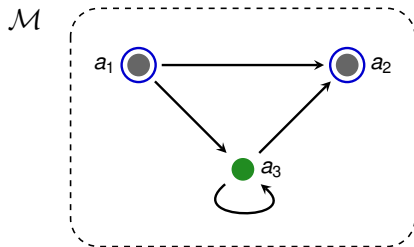
Example (Interpretation of Quantifiers)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

- ▶ $\forall x P(x)$

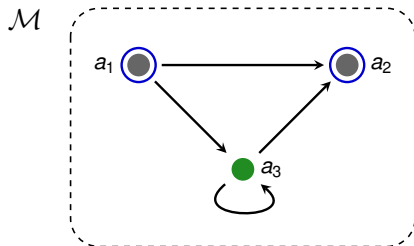
Example (Interpretation of Quantifiers)



- ▶ $c^{\mathcal{M}}$: green point
- ▶ $P^{\mathcal{M}}$: blue circles
- ▶ $R^{\mathcal{M}}$: arrows

- ▶ $\mathcal{M} \not\models \forall x P(x)$

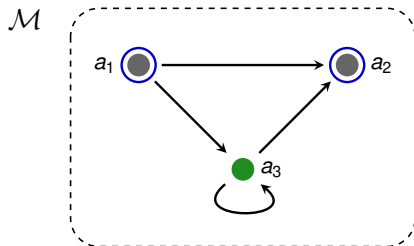
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- ▶ $c^{\mathcal{M}}$: green point
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- ▶ $\mathcal{M} \not\models \forall x P(x)$
- ▶ $\neg \forall x P(x)$

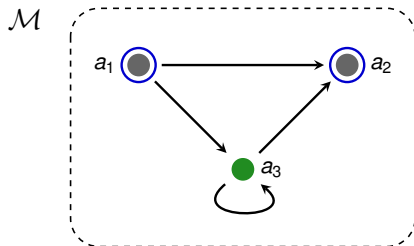
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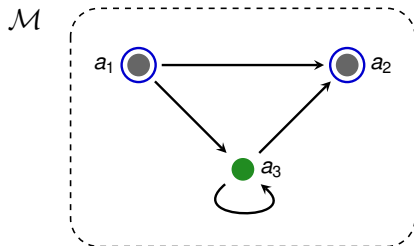
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- ▶ $\mathcal{M} \models \exists x P(x)$

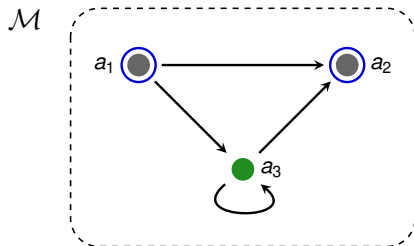
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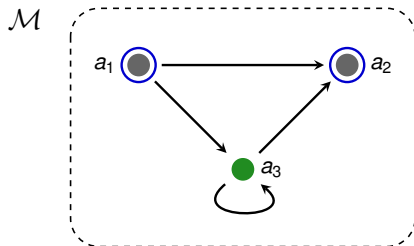
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- ▶ $\mathcal{M} \models \exists x P(x)$
- ▶ $\forall x (P(x) \rightarrow \exists y R(x, y))$

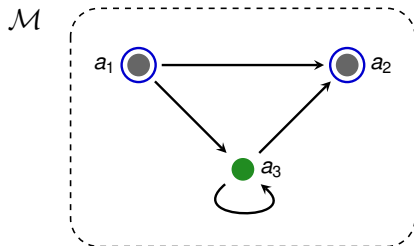
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- ▶ $\mathcal{M} \not\models \forall x (P(x) \rightarrow \exists y R(x, y))$

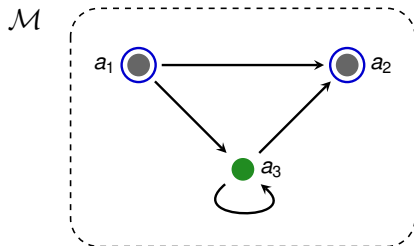
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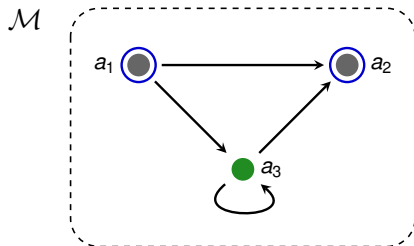
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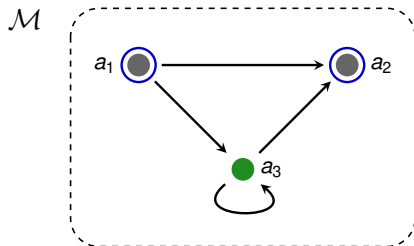
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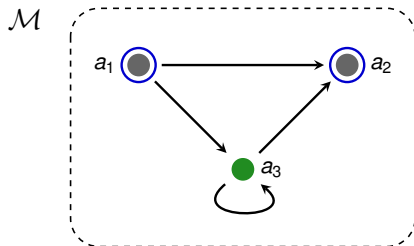
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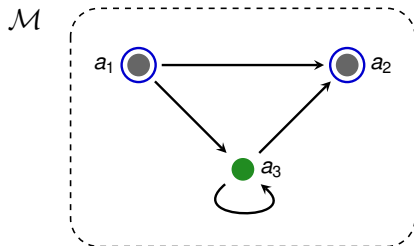
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- ▶ $\mathcal{M} \not\models \forall x (P(x) \rightarrow \exists y R(x, y))$
- ▶ $\mathcal{M} \models \forall x (P(x) \vee R(x, x))$
- ▶ $\mathcal{M} \not\models \exists x \forall y R(x, y)$
- ▶ $\forall x (\exists y R(x, y) \rightarrow \exists y (R(x, y) \wedge P(y)))$

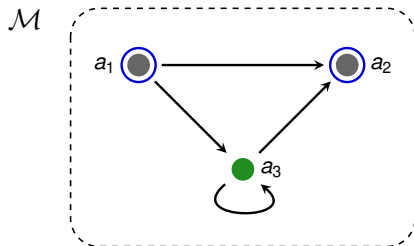
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- ▶ $\mathcal{M} \models \forall x (\exists y R(x, y) \rightarrow \exists y (R(x, y) \wedge P(y)))$

Example (Interpretation of Free Variables)

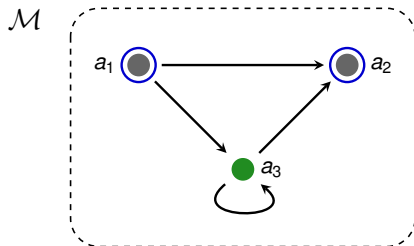


What about:

▶ $\mathcal{M} \models P(x)$

?

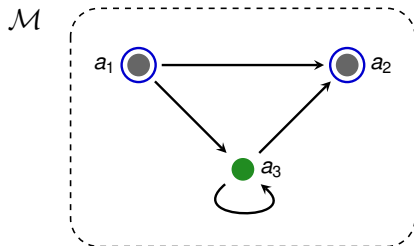
Example (Interpretation of Free Variables)



What about:

- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?

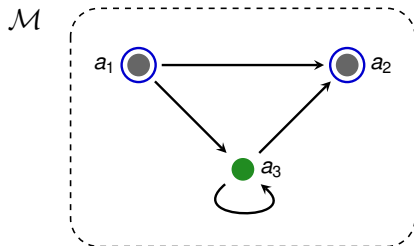
Example (Interpretation of Free Variables)



What about:

- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?
- ▶ $\mathcal{M} \models R(x, y)$?

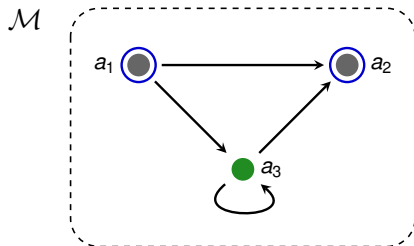
Example (Interpretation of Free Variables)



What about:

- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?
- ▶ $\mathcal{M} \models R(x, y)$?
- ▶ $\mathcal{M} \models \exists y R(x, y)$?

Example (Interpretation of Free Variables)

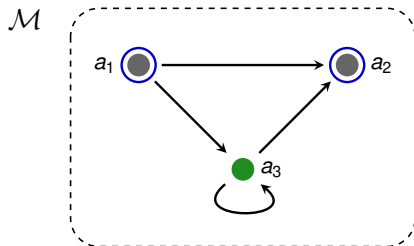


What about:

- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?
- ▶ $\mathcal{M} \models R(x, y)$?
- ▶ $\mathcal{M} \models \exists y R(x, y)$?

This depends on the interpretation of the **free variables** x and y :

Example (Interpretation of Free Variables)



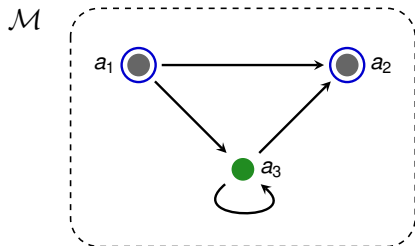
What about:

- ▶ $\mathcal{M} \models P(x)$?
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- ▶ $\mathcal{M} \models R(x, y)$?
- ▶ $\mathcal{M} \models \exists y R(x, y)$?

This depends on the interpretation of the **free variables** x and y :

- ▶ $\mathcal{M} \models_{[x \mapsto a_1]} P(x)$

Example (Interpretation of Free Variables)



What about:

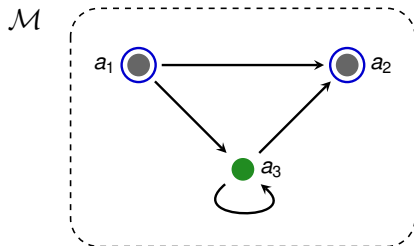
- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?
- ▶ $\mathcal{M} \models R(x, y)$?
- ▶ $\mathcal{M} \models \exists y R(x, y)$?

This depends on the interpretation of the **free variables** x and y :

▶ $\mathcal{M} \models_{[x \mapsto a_1]} P(x)$

$\mathcal{M} \not\models_{[x \mapsto a_3]} P(x)$

Example (Interpretation of Free Variables)



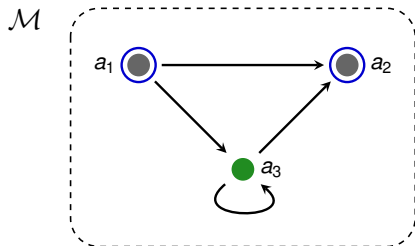
What about:

- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?
- ▶ $\mathcal{M} \models R(x, y)$?
- ▶ $\mathcal{M} \models \exists y R(x, y)$?

This depends on the interpretation of the **free variables** x and y :

- ▶ $\mathcal{M} \models_{[x \mapsto a_1]} P(x)$ $\mathcal{M} \not\models_{[x \mapsto a_3]} P(x)$
- ▶ $\mathcal{M} \not\models_{[x \mapsto a_1]} R(x, x)$

Example (Interpretation of Free Variables)



What about:

- ▶ $\mathcal{M} \models P(x)$?
- ▶ $\mathcal{M} \models R(x, x)$?
- ▶ $\mathcal{M} \models R(x, y)$?
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This depends on the interpretation of the **free variables** x and y :

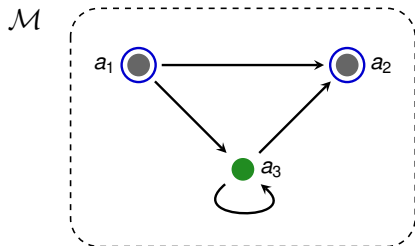
▶ $\mathcal{M} \models_{[x \mapsto a_1]} P(x)$

$\mathcal{M} \not\models_{[x \mapsto a_3]} P(x)$

▶ $\mathcal{M} \not\models_{[x \mapsto a_1]} R(x, x)$

$\mathcal{M} \models_{[x \mapsto a_3]} R(x, x)$

Example (Interpretation of Free Variables)



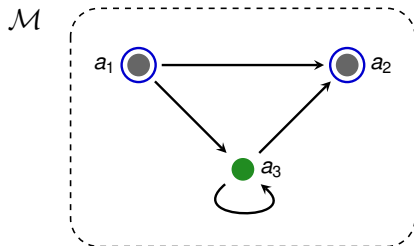
What about:

- ▶ $\mathcal{M} \models P(x)$?
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This depends on the interpretation of the **free variables** x and y :

- ▶ $\mathcal{M} \models_{[x \mapsto a_1]} P(x)$
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Example (Interpretation of Free Variables)



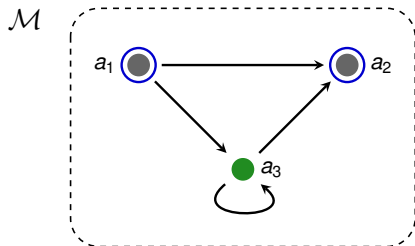
What about:

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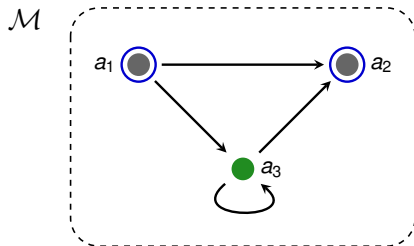
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Model plus Environments

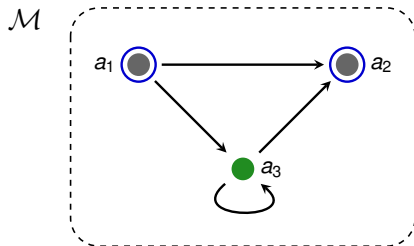


An **environment**

$$\ell : \text{var} \rightarrow A$$

(look-up function)
interprets **free** variables in
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Model plus Environments



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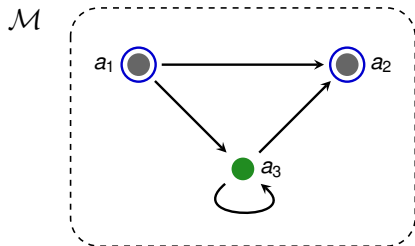
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Model plus Environments



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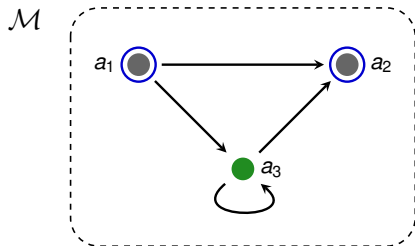
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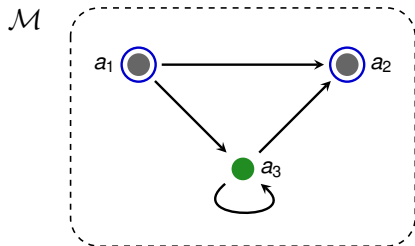
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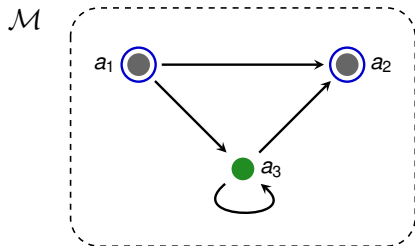
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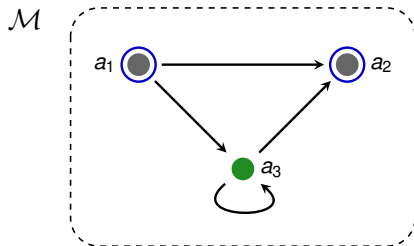
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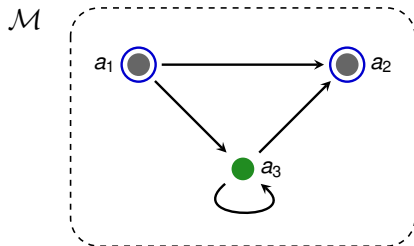
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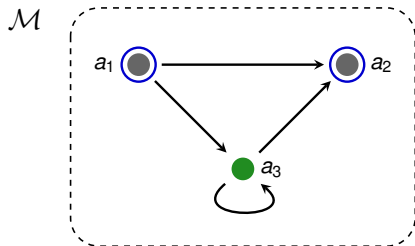
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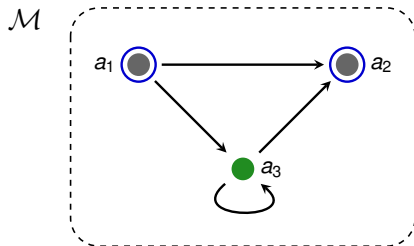
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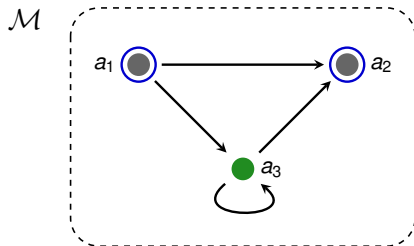
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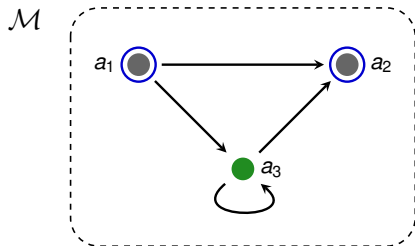
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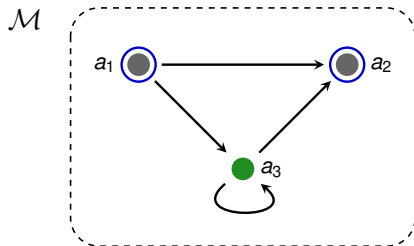
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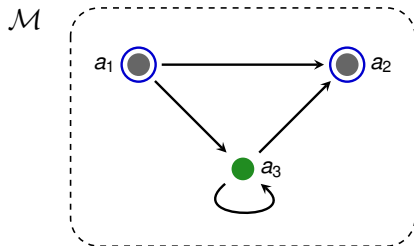
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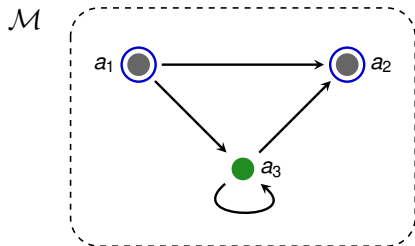
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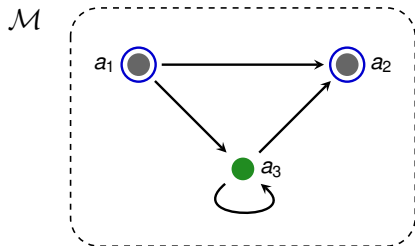
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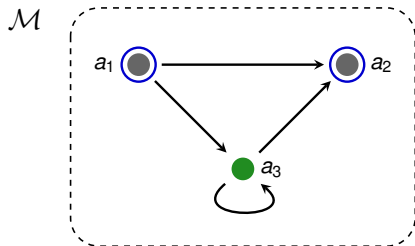
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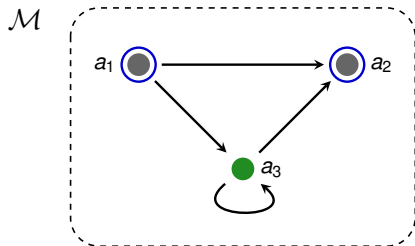
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Modification of Environments

In order to determine the truth values for $\forall x$ - en $\exists x$ -formulas we must be able to modify the interpretation of the variable x .

Definition (Modified Environment)

Let $\ell : \mathbf{var} \rightarrow A$ be an environment, and x a variable.

By $\ell[x \mapsto a]$ we denote the environment that interprets x as a , and that for all other variables acts in the same way as ℓ :

$$\ell[x \mapsto a](y) = \begin{cases} a & \text{if } y = x \\ \ell(y) & \text{if } y \neq x \end{cases}$$

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More modifications are possible.

For $\ell' = \ell[x \mapsto a][z \mapsto b]$ we have:

$$\ell'(y) = \ell[x \mapsto a][z \mapsto b](y) = \begin{cases} a & \text{if } y = x \\ b & \text{if } y = z \\ \ell(y) & \text{if } y \neq x \text{ and } y \neq z \end{cases}$$

Interpretation of Terms in Model with Environment

Terms are built from variables, constants, and function symbols:

- ▶ variables are interpreted according to the environment ℓ
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Let \mathcal{M} be a model and ℓ an environment.

Interpretation of terms

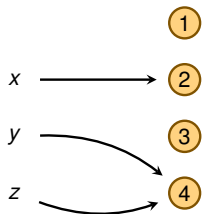
The interpretation $t^{\mathcal{M},\ell}$ of a term t is defined as:

$$t^{\mathcal{M},\ell} = \begin{cases} \ell(\mathbf{x}) & \text{if } t = \mathbf{x} \text{ for a variable } \mathbf{x} \\ \mathbf{c}^{\mathcal{M}} & \text{if } t = \mathbf{c} \text{ for a constant } \mathbf{c} \\ \mathbf{f}^{\mathcal{M}}(t_1^{\mathcal{M},\ell}, \dots, t_n^{\mathcal{M},\ell}) & \text{if } t = \mathbf{f}(t_1, \dots, t_n) \end{cases}$$

by induction on the term structure.

Quantification by Varying the Environment

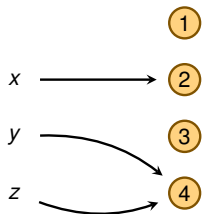
Model and environment:



- ▶ domain: $A = \{1, 2, 3, 4\}$
- ▶ $L^{\mathcal{M}} = <$
- ▶ environment ℓ :
 - ▶ $\ell(x) = 2$
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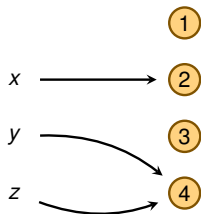
▶ $\ell(x) = 2$

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▶ Now $\mathcal{M} \not\models_e L(y, x)$

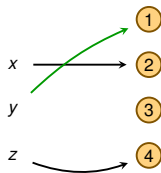
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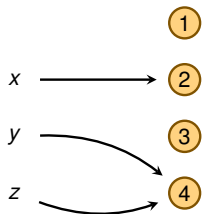
- ▶ But: $\mathcal{M} \models_{\ell} \exists y L(y, x)$
because $\mathcal{M} \models_{\ell[y \mapsto 1]} L(y, x)$

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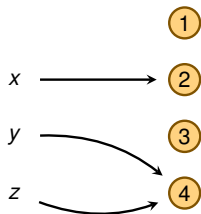
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▶ Also $\mathcal{M} \models_{\ell} \forall x \exists y L(y, x)$?

Quantification by Varying the Environment

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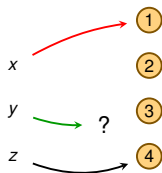
▶ environment l :

▶ $l(x) = 2$

▶ $l(y) = l(z) = 4$

▶ Also $\mathcal{M} \models_l \forall x \exists y L(y, x)$?

▶ No: $\mathcal{M} \not\models_{l[x \mapsto 1]} \exists y L(y, x)$



Formula Truth in a Model (Full Definition)

Truth of a formula ϕ in a model \mathcal{M} with universe A with respect to environment ℓ is defined by induction on the structure of ϕ :

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$$t^{\mathcal{M}, \ell} = \begin{cases} \ell(\mathbf{x}) & \text{if } t = \mathbf{x} \text{ for a variable } \mathbf{x} \\ \mathbf{c}^{\mathcal{M}} & \text{if } t = \mathbf{c} \text{ for a constant } \mathbf{c} \\ \mathbf{f}^{\mathcal{M}}(t_1^{\mathcal{M}, \ell}, \dots, t_n^{\mathcal{M}, \ell}) & \text{if } t = \mathbf{f}(t_1, \dots, t_n) \end{cases}$$

Formula Truth in a Model (Full Definition)

Truth of a formula ϕ in a model \mathcal{M} with universe A with respect to environment ℓ is defined by induction on the structure of ϕ :

- ▶ $\mathcal{M} \models_{\ell} \neg\phi \iff \mathcal{M} \not\models_{\ell} \phi$
- ▶ $\mathcal{M} \models_{\ell} \phi \wedge \psi \iff \mathcal{M} \models_{\ell} \phi$ and $\mathcal{M} \models_{\ell} \psi$
- ▶ $\mathcal{M} \models_{\ell} \phi \vee \psi \iff \mathcal{M} \models_{\ell} \phi$ or $\mathcal{M} \models_{\ell} \psi$
- ▶ $\mathcal{M} \models_{\ell} \phi \rightarrow \psi \iff$ (if $\mathcal{M} \models_{\ell} \phi$ then $\mathcal{M} \models_{\ell} \psi$)
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- ▶ $\mathcal{M} \models_{\ell} \forall \mathbf{x} \phi \iff$ for all $\mathbf{a} \in A$ it holds: $\mathcal{M} \models_{\ell[\mathbf{x} \mapsto \mathbf{a}]} \phi$

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- ▶ $\mathcal{M} \models_{\ell} \forall x \phi \iff$ for all $a \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto a]} \phi$
- ▶ $\mathcal{M} \models_{\ell} \exists x \phi \iff$ for some $a \in A$ it holds: $\mathcal{M} \models_{\ell[x \mapsto a]} \phi$

Well-definedness of \models

Proposition

If ℓ and ℓ' coincide on the free variables of ϕ , then:

$$\mathcal{M} \models_{\ell} \phi \iff \mathcal{M} \models_{\ell'} \phi$$

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A formula ϕ is a **sentence** if ϕ does not have free variables.

Proposition

Let ϕ be a sentence. Then it holds for all environments ℓ and ℓ' :

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Proposition

Let ϕ be a sentence. Then it holds for all environments ℓ and ℓ' :

$$\mathcal{M} \models_{\ell} \phi \iff \mathcal{M} \models_{\ell'} \phi$$

Hence for sentences ϕ , we can write

$$\mathcal{M} \models \phi \text{ for } \mathcal{M} \models_{\ell} \phi,$$

since ℓ is irrelevant.

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists x P(x) \quad \text{where} \quad \begin{array}{c} \text{①} \quad \text{③} \\ \text{②} \quad A \end{array}^{\mathcal{M}} \quad \begin{array}{l} A \\ \{1, 2, 3\} \\ P^{\mathcal{M}} = \{1\} \end{array} =$$

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists x P(x) \quad \text{where} \quad \begin{array}{c} \text{①} \quad \text{③} \\ \text{②} \quad A \end{array} \overset{\mathcal{M}}{=} \begin{array}{l} A \\ \{1, 2, 3\} \\ P^{\mathcal{M}} = \{1\} \end{array}$$

$$\mathcal{M} \models_{\ell} \exists x P(x)$$

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$$\mathcal{M} \models_{\ell} \exists x P(x)$$

$$\iff \text{there is } a \in A \text{ such that } \mathcal{M} \models_{\ell[x \mapsto a]} P(x) \\ \text{(by def. of } \models \text{)}$$

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$$\iff \text{there is } a \in A \text{ such that } \mathcal{M} \models_{\ell[x \mapsto a]} P(x) \quad (\text{by def. of } \models)$$

$$\iff \text{there is } a \in A \text{ s.th. } x^{\mathcal{M}, \ell[x \mapsto a]} \in P^{\mathcal{M}} \quad (\text{by def. of } \models)$$

Checking Formula Satisfiability in a Model

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Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists x P(x) \quad \text{where} \quad \begin{array}{c} \text{①} \quad \text{③} \\ \text{②} \quad A \end{array} \quad \begin{array}{l} \mathcal{M} \\ A \\ \{1, 2, 3\} \\ P^{\mathcal{M}} = \{1\} \end{array} =$$

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Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists x P(x) \quad \text{where} \quad \begin{array}{c} \text{①} \quad \text{③} \\ \text{②} \quad A \end{array} \quad \begin{array}{l} \mathcal{M} \\ A \\ \{1, 2, 3\} \\ P^{\mathcal{M}} = \{1\} \end{array} =$$

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Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists x P(x) \quad \text{where} \quad \begin{array}{c} \text{①} \quad \text{③} \\ \text{②} \quad A \end{array} \overset{\mathcal{M}}{=} \begin{array}{l} A \\ \{1, 2, 3\} \\ P^{\mathcal{M}} = \{1\} \end{array}$$

$$\mathcal{M} \models_{\ell} \exists x P(x) \quad \checkmark$$

$$\iff \text{there is } a \in A \text{ such that } \mathcal{M} \models_{\ell[x \mapsto a]} P(x) \quad \text{(by def. of } \models \text{)}$$

$$\iff \text{there is } a \in A \text{ s.th. } x^{\mathcal{M}, \ell[x \mapsto a]} \in P^{\mathcal{M}} \quad \text{(by def. of } \models \text{)}$$

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Checking Formula Satisfiability in a Model

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$$\mathcal{M} \models_{\ell} \exists x P(x) \quad \checkmark$$

$$\iff \text{there is } a \in A \text{ such that } \mathcal{M} \models_{\ell[x \mapsto a]} P(x) \quad (\text{by def. of } \models)$$

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$$\iff 1 \in P^{\mathcal{M}} \quad \checkmark \quad (\text{by def. of } \mathcal{M})$$

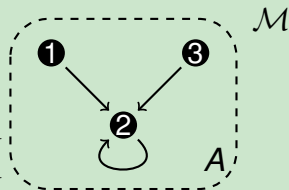
Hence we have formally established $\mathcal{M} \models \exists x P(x)$.

Checking Formula Satisfiability in a Model

$\mathcal{M} \models \exists y \forall x R(x, y)$ for model

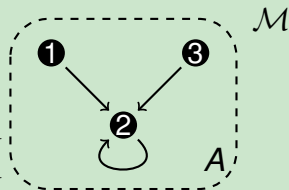
$A = \{1, 2, 3\}$

$R^{\mathcal{M}} = \{ \langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle \}$



Checking Formula Satisfiability in a Model

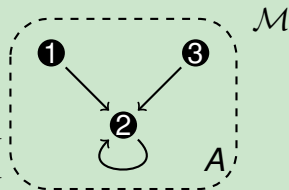
$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
$$A = \{ 1, 2, 3 \}$$
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$$\mathcal{M} \models_{\ell} \exists y \forall x R(x, y)$$

Checking Formula Satisfiability in a Model

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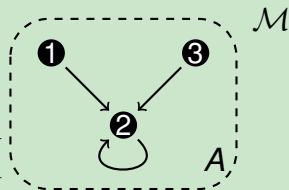


$$\mathcal{M} \models_{\ell} \exists y \forall x R(x, y)$$

\iff there is $b \in A$ such that: $\mathcal{M} \models_{\ell[y \mapsto b]} \forall x R(x, y)$

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
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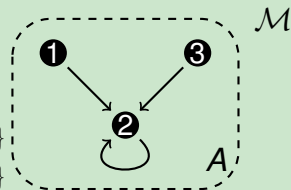
\iff there is $b \in A$ such that: $\mathcal{M} \models_{\ell[y \mapsto b]} \forall x R(x, y)$

\iff there is $b \in A$ such that for all $a \in A$:

$$\mathcal{M} \models_{\ell[y \mapsto b][x \mapsto a]} R(x, y)$$

Checking Formula Satisfiability in a Model

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$$A = \{1, 2, 3\}$$
$$R^{\mathcal{M}} = \{ \langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle \}$$



$$\mathcal{M} \models_{\ell} \exists y \forall x R(x, y)$$

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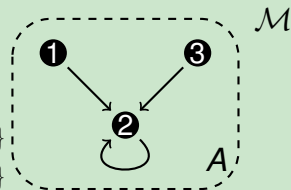
$$\langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]} \rangle \in R^{\mathcal{M}}$$

Checking Formula Satisfiability in a Model

$\mathcal{M} \models \exists y \forall x R(x, y)$ for model

$$A = \{1, 2, 3\}$$

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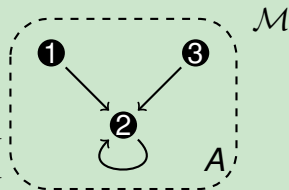
$$\mathcal{M} \models_{\ell[y \mapsto b][x \mapsto a]} R(x, y)$$

\iff there is $b \in A$ such that for all $a \in A$:

$$\langle a, b \rangle = \langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]} \rangle \in R^{\mathcal{M}}$$

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
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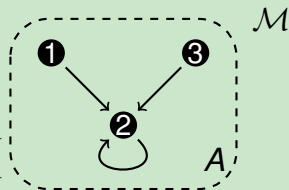
\iff there is $b \in A$ such that for all $a \in A$:

$$\langle a, b \rangle = \langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]} \rangle \in R^{\mathcal{M}}$$

\iff for all $a \in A$ it holds that $\langle a, 2 \rangle \in R^{\mathcal{M}}$

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
$$A = \{1, 2, 3\}$$
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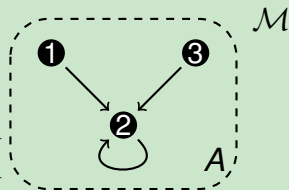
$$\langle a, b \rangle = \langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]} \rangle \in R^{\mathcal{M}}$$

$$\iff \text{for all } a \in A \text{ it holds that } \langle a, 2 \rangle \in R^{\mathcal{M}}$$

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Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
$$A = \{1, 2, 3\}$$
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$$\mathcal{M} \models_{\ell} \exists y \forall x R(x, y)$$

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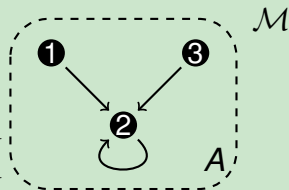
$$\langle a, b \rangle = \langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]} \rangle \in R^{\mathcal{M}}$$

\iff for all $a \in A$ it holds that $\langle a, 2 \rangle \in R^{\mathcal{M}}$

$\iff \langle 1, 2 \rangle \in R^{\mathcal{M}}$ and $\langle 2, 2 \rangle \in R^{\mathcal{M}}$ and $\langle 3, 2 \rangle \in R^{\mathcal{M}}$ \checkmark

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
$$A = \{1, 2, 3\}$$
$$R^{\mathcal{M}} = \{ \langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle \}$$



$$\mathcal{M} \models_{\ell} \exists y \forall x R(x, y) \quad \checkmark$$

$$\iff \text{there is } b \in A \text{ such that: } \mathcal{M} \models_{\ell[y \mapsto b]} \forall x R(x, y)$$

$$\iff \text{there is } b \in A \text{ such that for all } a \in A:$$

$$\mathcal{M} \models_{\ell[y \mapsto b][x \mapsto a]} R(x, y)$$

$$\iff \text{there is } b \in A \text{ such that for all } a \in A:$$

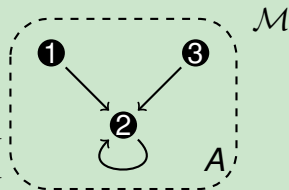
$$\langle a, b \rangle = \langle x^{\ell[y \mapsto b][x \mapsto a]}, y^{\ell[y \mapsto b][x \mapsto a]} \rangle \in R^{\mathcal{M}}$$

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$$\iff \langle 1, 2 \rangle \in R^{\mathcal{M}} \text{ and } \langle 2, 2 \rangle \in R^{\mathcal{M}} \text{ and } \langle 3, 2 \rangle \in R^{\mathcal{M}} \quad \checkmark$$

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \exists y \forall x R(x, y) \quad \text{for model}$$
$$A = \{1, 2, 3\}$$
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$$\mathcal{M} \models_{\ell} \exists y \forall x R(x, y) \quad \checkmark$$

$$\iff \text{there is } b \in A \text{ such that: } \mathcal{M} \models_{\ell[y \mapsto b]} \forall x R(x, y)$$

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$$\iff \text{for all } a \in A \text{ it holds that } \langle a, 2 \rangle \in R^{\mathcal{M}}$$

$$\iff \langle 1, 2 \rangle \in R^{\mathcal{M}} \text{ and } \langle 2, 2 \rangle \in R^{\mathcal{M}} \text{ and } \langle 3, 2 \rangle \in R^{\mathcal{M}} \quad \checkmark$$

This shows $\mathcal{M} \models \exists y \forall x R(x, y)$.

Checking Formula Satisfiability in a Model

$$\mathcal{M} \models \forall x (x \leq x \cdot e)$$

- ▶ universe $B = \{ s \mid s \text{ is binary string} \}$
- ▶ $e^{\mathcal{M}} = \text{empty string } \epsilon$
- ▶ $\cdot^{\mathcal{M}}$: concatenation of strings
- ▶ $\leq^{\mathcal{M}} = \{ \langle s_1, s_2 \rangle \mid s_1 \text{ is a prefix of } s_2 \}$

Checking Formula Satisfiability in a Model

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Hence we have formally established: $\mathcal{M} \models \forall x (x \leq x \cdot e)$.

Semantical Entailment

Semantical Entailment in Predicate Logic

Semantic entailment in propositional logic

In **propositional logic**: $\phi_1, \dots, \phi_n \models \psi$ means:

For all valuations v :

$$v(\phi_1) = \top, \dots, v(\phi_n) = \top \implies v(\psi) = \top$$

Semantical Entailment in Predicate Logic

Semantic entailment in propositional logic

In **propositional logic**: $\phi_1, \dots, \phi_n \vDash \psi$ means:

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Semantic entailment in predicate logic

In **predicate logic**: $\phi_1, \dots, \phi_n \vDash \psi$ means:

For all models \mathcal{M} , and all environments ℓ :

$$\mathcal{M} \vDash_{\ell} \phi_1, \dots, \mathcal{M} \vDash_{\ell} \phi_n \implies \mathcal{M} \vDash_{\ell} \psi$$

In words: for all models \mathcal{M} , and all environments ℓ such that

$$\mathcal{M} \vDash_{\ell} \phi_1 \text{ and } \dots \text{ and } \mathcal{M} \vDash_{\ell} \phi_n \text{ hold,}$$

it also holds that $\mathcal{M} \vDash_{\ell} \psi$.

Proving and Disproving Semantical Entailment

Which of the following semantic implications are true?

(a) $\forall x \exists y R(x, y) \models \exists x \forall y R(x, y)$ **NO**

(b) $\forall x \exists y R(x, y) \models \exists y \forall x R(x, y)$ **NO**

(c) $\exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$ **YES**

(d) $\exists y \forall x R(x, y) \models \forall y \exists x R(x, y)$ **NO**

Disproving Semantic Entailment: Counter Models

$$\forall x \exists y R(x, y) \not\models \exists y \forall x R(x, y)$$

Disproving Semantic Entailment: Counter Models

$$\forall x \exists y R(x, y) \not\models \exists y \forall x R(x, y)$$

We can give a model \mathcal{M} such that:

$$\mathcal{M} \models \forall x \exists y R(x, y), \quad \mathcal{M} \not\models \exists y \forall x R(x, y).$$

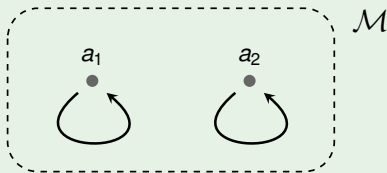
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We choose \mathcal{M} as follows:



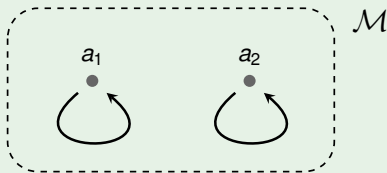
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This is a **counter model**:

it satisfies the premise, but not the conclusion.

Proving Semantic Entailment

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For all models \mathcal{M} with domain A and environments ℓ we find:

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$$\iff \mathcal{M} \models_{\ell} \forall x \exists y R(x, y)$$

Hence we can conclude: $\exists y \forall x R(x, y) \models \forall x \exists y R(x, y)$.

Proving Semantic Entailment

$$\neg \forall x P(x) \models \exists x \neg P(x)$$

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$$\neg\forall x P(x) \models \exists x \neg P(x)$$

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Proving Semantic Entailment

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For all models \mathcal{M} with domain A and environments ℓ we find:

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$$\iff \mathcal{M} \models_{\ell} \exists x \neg P(x)$$

Proving Semantic Entailment

$$\neg\forall x P(x) \models \exists x \neg P(x)$$

For all models \mathcal{M} with domain A and environments ℓ we find:

$$\mathcal{M} \models_{\ell} \neg\forall x P(x)$$

$$\iff \text{not: } \mathcal{M} \models_{\ell} \forall x P(x)$$

$$\iff \text{not for all } a \in A: \mathcal{M} \models_{\ell[x \mapsto a]} P(x)$$

$$\iff \text{there exists } a \in A \text{ such that not: } \mathcal{M} \models_{\ell[x \mapsto a]} P(x)$$

$$\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \not\models_{\ell[x \mapsto a]} P(x)$$

$$\iff \text{there exists } a \in A \text{ such that: } \mathcal{M} \models_{\ell[x \mapsto a]} \neg P(x)$$

$$\iff \mathcal{M} \models_{\ell} \exists x \neg P(x)$$

Hence we can conclude: $\neg\forall x P(x) \models \exists x \neg P(x)$.