Logic and Modelling

– Natural Deduction for Predicate Logic —

Jörg Endrullis

VU University Amsterdam

Natural Deduction: Quantifiers

Introduction of Existential Quantification

$$\frac{\phi\ [t/x]}{\exists x\ \phi}\quad \exists_i$$

Condition for application of this rule: t free for x in ϕ.

$\vdash\ R(a,a) \rightarrow \exists x\ R(x,x)$

1	$R(a,a)$	assumption
2	$\exists x\ R(x,x)$	$\exists_i$ 1
3	$R(a,a) \rightarrow \exists x\ R(x,x)$	$\rightarrow_i$ 1–2

For line 2: $R(a,a) = R(x,x)[a/x]$

Natural Deduction: Quantifiers

Elimination of Universal Quantification

$$\frac{\forall x\ \phi}{\phi\ [t/x]}\ \forall_e$$

Condition for application of this rule: t free for x in ϕ.

$\forall x\ P(x)\ \vdash\ \exists x\ P(x)$

1	$\forall x\ P(x)$	premise
2	$P(z)$	$\forall_e$ 1
3	$\exists x\ P(x)$	$\exists_i$ 2

For line 2, 3: $P(z) = P(x)[z/x]$

Examples

$R(a, a) \vdash \exists x\, \exists y\, R(x, y)$

1	$R(a, a)$	premise
2	$\exists y\, R(a, y)$	$\exists_i$ 1
3	$\exists x\, \exists y\, R(x, y)$	$\exists_i$ 2

For line 2: $R(a, a) = R(a, y)[a/y]$

For line 3: $\exists y\, R(a, y) = \exists y\, R(x, y)[a/x]$

$\forall x\, \forall y\, R(x, y) \vdash \exists x\, \forall y\, R(x, y)$

1	$\forall x\, \forall y\, R(x, y)$	premise
2	$\forall y\, R(z, y)$	$\forall_e$ 1
3	$\exists x\, \forall y\, R(x, y)$	$\exists_i$ 2

For line 2, 3: $\forall y\, R(z, y) = \forall y\, R(x, y)[z/x]$

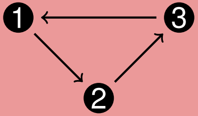
The Condition

$$\frac{\phi\ [t/x]}{\exists x\ \phi}\ \exists_i \qquad\qquad \frac{\forall x\ \phi}{\phi\ [t/x]}\ \forall_e$$

Is the following derivation correct?

1	$\forall x\ \exists y\ R(x,y)$	premise
2	$\exists y\ R(y,y)$	$\forall_e$ 1 (wrong!)
3	$\forall x\ \exists y\ R(x,y) \to \exists y\ R(y,y)$	$\to_i$ 1–2

This is clearly a wrong derivation.



This model fulfils $\forall x\ \exists y\ R(x,y)$ but not $\exists y\ R(y,y)$.

The problem: y is not free for x in $\exists y\ R(x,y)$!

Natural Deduction: Quantifiers

Introduction of Universal Quantification

$$\frac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \quad \phi\ [x_0/x] \end{array}}}{\forall x\ \phi}\ \forall_i$$

Condition for application of this rule: 'x_0 is arbitrary'.
That is: x_0 occurs only inside the box (in particular not in ϕ)

$\vdash\ \forall x\ (P(x) \to P(x))$

	x_0	
1	$\boxed{P(x_0)}$	assumption
2	$P(x_0) \to P(x_0)$	$\to_i$ 1–1
3	$\forall x\ (P(x) \to P(x))$	$\forall_i$ 1–2

Natural Deduction: Quantifiers

Elimination of Existential Quantification

$$\dfrac{\exists x\ \phi \qquad \boxed{\begin{array}{l} x_0 \\ \quad \phi[x_0/x] \\ \quad \vdots \\ \quad \psi \end{array}}}{\psi}\ \exists_e$$

Condition: x_0 nowhere outside the box (in particular not in ψ).

$\exists x\ (P(x) \wedge Q(x)) \ \vdash\ \exists x\ P(x)$

1	$\exists x\ (P(x) \wedge Q(x))$	premise
	x_0	
2	$P(x_0) \wedge Q(x_0)$	assumption
3	$P(x_0)$	$\wedge_e$ 2
4	$\exists x\ P(x)$	$\exists_i$ 3
5	$\exists x\ P(x)$	$\exists_e$ 1,2–4

Examples

$R(a, b) \vdash \exists x\, \exists y\, R(x, y)$

1	$R(a, b)$	premise
2	$\exists y\, R(a, y)$	$\exists i$ 1
3	$\exists x\, \exists y\, R(x, y)$	$\exists i$ 2

$R(a, a) \vdash \exists x\, \exists y\, R(x, y)$

1	$R(a, a)$	premise
2	$\exists y\, R(a, y)$	$\exists i$ 1
3	$\exists x\, \exists y\, R(x, y)$	$\exists i$ 2

Examples

$R(a,a) \vdash \exists x\ R(x,x)$

1	$R(a,a)$	premise
2	$\exists x\ R(x,x)$	$\exists i$ 1

$\vdash R(a,a) \rightarrow \exists x\ R(x,x)$

1	$R(a,a)$	assumption
2	$\exists x\ R(x,x)$	$\exists i$ 1
3	$R(a,a) \rightarrow \exists x\ R(x,x)$	$\rightarrow i$ 1–2

Examples

$\forall x\, A(x) \ \vdash \ A(c) \wedge A(d)$

1	$\forall x\, A(x)$	premise
2	$A(c)$	$\forall e$ 1
3	$A(d)$	$\forall e$ 1
4	$A(c) \wedge A(d)$	$\wedge i$ 2, 3

Examples

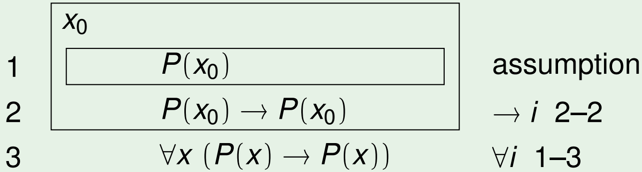
$$\forall x\ (A(x) \wedge B(x)) \ \vdash\ \forall x\ B(x)$$

1		$\forall x\ (A(x) \wedge B(x))$	premise
	x_0		
2		$A(x_0) \wedge B(x_0)$	$\forall e$ 1
3		$B(x_0)$	$\wedge e$ 2
4		$\forall x\ B(x)$	$\forall i$ 1–3

```
Theorem ex1 : all x, (A(x) /\ B(x)) -> all x, B(x).
Proof.
imp_i H.
all_i x0.
insert HAB (A(x0) /\ B(x0)).
f_all_e H.
f_con_e2 HAB.
Qed.
```

Examples

$\vdash \ \forall x\ (P(x) \rightarrow P(x))$

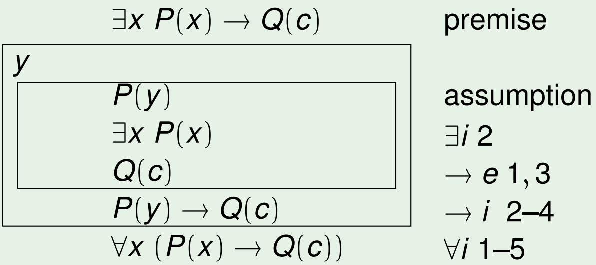


	x_0	
1	$P(x_0)$	assumption
2	$P(x_0) \rightarrow P(x_0)$	$\rightarrow i$ 2–2
3	$\forall x\ (P(x) \rightarrow P(x))$	$\forall i$ 1–3

```
Theorem ex2 : all x, ( P(x) -> P(x) ).
Proof.
all_i x0.
imp_i H.
ass H.
Qed.
```

Examples (from a final exam)

$\exists x\ P(x) \to Q(c) \ \vdash\ \forall x\ (P(x) \to Q(c))$



1	$\exists x\ P(x) \to Q(c)$	premise
	y	
2	$P(y)$	assumption
3	$\exists x\ P(x)$	$\exists i$ 2
4	$Q(c)$	$\to e$ 1, 3
5	$P(y) \to Q(c)$	$\to i$ 2–4
6	$\forall x\ (P(x) \to Q(c))$	$\forall i$ 1–5

```
Theorem ex3 : (exi x, P(x) -> Q(c)) -> all x, (P(x) -> Q(c)).

Proof.
imp_i H.
all_i x0.
imp_i Px0.
insert Hexi (exi x, P x).
f_exi_i Px0.
f_imp_e H Hexi.
Qed.
```

Examples

$\forall x\ (P(x) \to \neg P(x)) \ \vdash \ \forall x\ \neg P(x)$

1	$\forall x\ (P(x) \to \neg P(x))$	premise
	x_0	
2	$P(x_0)$	assumption
3	$P(x_0) \to \neg P(x_0)$	$\forall e$ 1
4	$\neg P(x_0)$	$\to_e$ 2, 3
5	$\bot$	$\neg e$ 2, 4
6	$\neg P(x_0)$	$\neg i$ 2–5
7	$\forall x\ \neg P(x)$	$\forall i$ 2–6

```
Theorem ex4 : all x, (P(x) -> ~P(x)) -> all x, ~P(x).

Proof.
imp_i H. all_i x0. neg_i HPx0.
insert Himp (P(x0) -> ~P(x0)). f_all_e H.
insert HnPx0 (~P(x0)).
f_imp_e Himp HPx0.
f_neg_e HnPx0 HPx0.
Qed.
```

Examples

$\neg\exists x\, \neg A(x) \vdash \forall x\, A(x)$

1	$\neg\exists x\, \neg A(x)$	premise
	x_0	
2	$\neg A(x_0)$	assumption
3	$\exists x\, \neg A(x)$	$\exists_i\ 2$
4	$\bot$	$\neg_e\ 1, 3$
5	$A(x_0)$	PBC 2–4
6	$\forall x\, A(x)$	$\forall_i$ 2–5

```
Theorem ex5 : ~(exi x, ~A(x)) -> all x, A(x).

Proof.
imp_i H.
all_i x0.
PBC HnAx0.
insert H2 (exi x, ~A(x)).
f_exi_i HnAx0.
f_neg_e H H2.
Qed.
```

Examples

$\exists x\ (P(x) \land Q(x)) \vdash \exists x\ P(x)$

1	$\exists x\ (P(x) \land Q(x))$	premise
	x_0	
2	$P(x_0) \land Q(x_0)$	assumption
3	$P(x_0)$	$\land_e$ 2
4	$\exists x\ P(x)$	$\exists_i$ 3
5	$\exists x\ P(x)$	$\exists_e$ 1, 2–4

```
Theorem ex6 : exi x, (P(x) /\ Q(x)) -> exi x, P(x).
Proof.
imp_i H.
f_exi_e H y Hy.
insert HPy (P(y)).
f_con_e1 Hy.
f_exi_i HPy.
Qed.
```

Examples

$\exists x\ R(x,x)\ \vdash\ \exists x\ \exists y\ R(x,y)$

1	$\exists x\ R(x,x)$	premise
	x_0	
2	$R(x_0,x_0)$	assumption
3	$\exists y\ R(x_0,y)$	$\exists_i$ 2
4	$\exists x\ \exists y\ R(x,y)$	$\exists_i$ 3
5	$\exists x\ \exists y\ R(x,y)$	$\exists_e$ 1, 2–4

```
Theorem ex7 : exi x, R(x,x) -> exi x, exi y, R(x,y).
Proof.
imp_i H.
f_exi_e H z Hz.
insert H2 (exi y, R(z,y)).
f_exi_i Hz.
f_exi_i H2.
Qed.
```

Examples

$\forall x\; A(x) \;\vdash\; \neg\exists x\; \neg A(x)$

1	$\forall x\; A(x)$	premise
2	$\exists x\; \neg A(x)$	assumption
	x_0	
3	$\neg A(x_0)$	assumption
4	$A(x_0)$	$\forall_e$ 1
5	$\bot$	$\neg_e$ 3, 4
6	$\bot$	$\exists_e$ 2, 3–5
7	$\neg\exists x\; \neg A(x)$	$\neg_i$ 2–6

```
Theorem ex8 : all x, A(x) -> ~(exi x, ~A(x)).

Proof.
imp_i H.
neg_i H2.
f_exi_e H2 y Hny.
insert Hy (A(y)). f_all_e H.
f_neg_e Hny Hy.
Qed.
```

Examples

$\forall x\ (P(x) \to \neg P(x))\ \vdash\ \neg\exists x\ P(x)$

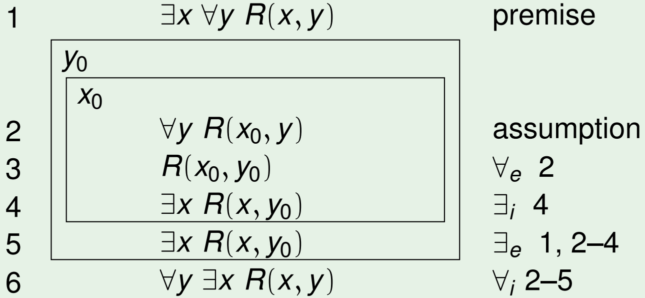
1	$\forall x\ (P(x) \to \neg P(x))$	premise
2	$\exists x\ P(x)$	assumption
	y	
3	$P(y)$	assumption
4	$P(y) \to \neg P(y)$	$\forall_e$ 1
5	$\neg P(y)$	$\to_e$ 2,3
6	$\bot$	$\neg_e$ 2,4
7	$\bot$	$\exists_e$ 2, 3–6
8	$\neg\exists x\ P(x)$	$\neg_i$ 2–7

Similar to an example we have seen before

$$\forall x\ (P(x) \to \neg P(x))\ \vdash\ \forall x\ \neg P(x)$$

Examples

$\exists x\,\forall y\,R(x,y) \vdash \forall y\,\exists x\,R(x,y)$



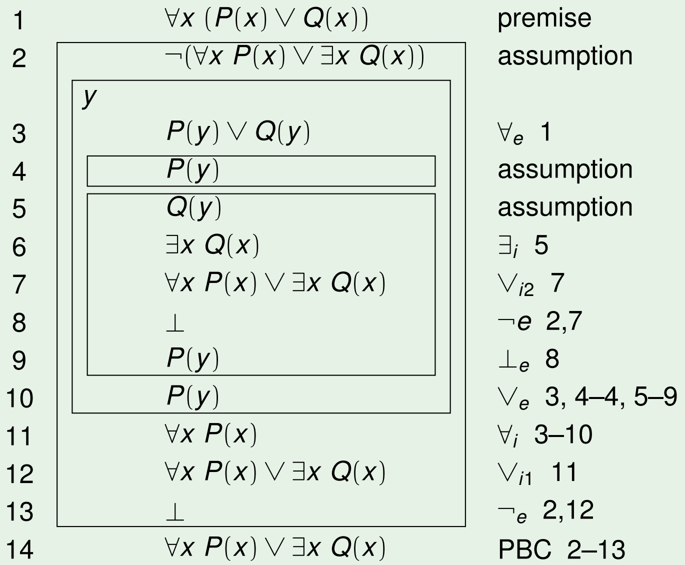
1	$\exists x\,\forall y\,R(x,y)$	premise
	y_0	
	x_0	
2	$\forall y\,R(x_0,y)$	assumption
3	$R(x_0,y_0)$	$\forall_e$ 2
4	$\exists x\,R(x,y_0)$	$\exists_i$ 4
5	$\exists x\,R(x,y_0)$	$\exists_e$ 1, 2–4
6	$\forall y\,\exists x\,R(x,y)$	$\forall_i$ 2–5

```
Theorem ex9 : exi x, all y, R(x,y) -> all y, exi x, R(x,y).

Proof.
imp_i H. all_i y0.
f_exi_e H x0 Hx0.
insert Rx0y0 (R(x0,y0)).
f_all_e Hx0.
f_exi_i Rx0y0.
Qed.
```

Examples

$\forall x\, (P(x) \vee Q(x)) \vdash \forall x\, P(x) \vee \exists x\, Q(x)$



1	$\forall x\, (P(x) \vee Q(x))$	premise
2	$\neg(\forall x\, P(x) \vee \exists x\, Q(x))$	assumption
	y	
3	$P(y) \vee Q(y)$	$\forall_e$ 1
4	$P(y)$	assumption
5	$Q(y)$	assumption
6	$\exists x\, Q(x)$	$\exists_i$ 5
7	$\forall x\, P(x) \vee \exists x\, Q(x)$	$\vee_{i2}$ 7
8	$\bot$	$\neg e$ 2,7
9	$P(y)$	$\bot_e$ 8
10	$P(y)$	$\vee_e$ 3, 4–4, 5–9
11	$\forall x\, P(x)$	$\forall_i$ 3–10
12	$\forall x\, P(x) \vee \exists x\, Q(x)$	$\vee_{i1}$ 11
13	$\bot$	$\neg_e$ 2,12
14	$\forall x\, P(x) \vee \exists x\, Q(x)$	PBC 2–13

Previous Example in ProofWeb

```
Theorem ex10 : all x, (P(x) \/ Q(x))
  -> (all x, P(x)) \/ (exi x, Q(x)).
Proof.
imp_i H.
insert Hor ((exi x, Q x) \/ ~(exi x, Q x)).
LEM.
f_dis_e Hor He Hne.
f_dis_i2 He.
dis_i1.
all_i x0.
insert Hx0 (P(x0) \/ Q(x0)).
f_all_e H.
f_dis_e Hx0 Px0 Qx0.
ass Px0.
insert He (exi x, Q x).
f_exi_i Qx0.
fls_e.
f_neg_e Hne He.
Qed.
```